The Almost Surely Shrinking Yolk*

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Abstract

The yolk, defined by McKelvey as the smallest ball intersecting all median hyperplanes, is a key concept in the Euclidean spatial model of voting. Koehler conjectured that the yolk radius of a random sample from a uniform distribution on a square tends to zero. The following sharper and more general results are proved here: Let the population be a random sample from a probability measure μ on $\Re^m$. Then the yolk of the sample does not necessarily converge to the yolk of μ. However, if μ is strictly centered, i.e. the yolk radius of μ is zero, then the radius of the sample yolk will converge to zero almost surely, and the center of the sample yolk will converge almost surely to the center of the yolk of μ. Moreover, if the yolk radius of μ is nonzero, the sample yolk radius will not converge to zero if μ contains three noncollinear mass points or if somewhere it has density bounded away from zero in some ball of positive volume. All results hold for both odd and even population sizes.

1 Introduction

In the Euclidean spatial model of voting [9, 28, e.g.], voter ideal points are located in $\Re^m$ and voters prefer policies (points) closer to their ideal points under the Euclidean norm. This is perhaps the most widely used voting model, with many applications (e.g. [26, 25, 30, 23, 24, 4, 29, 5]). Ever since the work of McKelvey and Schofield [17, 27] showed that a core would generally not exist, and in fact the space collapses into chaotic cycles, much effort has been made to find a satisfactory solution to the equilibrium problem in the spatial

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model. The yolk, established in [15, 18], has emerged as an important solution concept. Defined as the smallest ball intersecting all median hyperplanes, it is the region of policies where a voting game will tend to stabilize. The yolk is also important by virtue of its close relationships to other solution and evaluation concepts, such as the uncovered set [20, 21, 10, 18], the Pareto set [12], the win set [10], Shapley-Owen power scores [11], epsilon cores [32], and the finagle point [35]. Several researchers [12] have investigated the size of the yolk, and some have speculated that the yolk may tend to be small as the voter population increases. This would be a very desirable property, for if the yolk were very small there would (arguably) be a *de facto* equilibrium even if no single point were undominated. Koehler [16] combines geometric analysis and simulation to suggest that if voter ideal points are uniformly distributed in a square region of $\Re^2$, the yolk shrinks towards a point as the population grows. We say that the uniform distribution on a square is *strictly centered:* every halfplane that does not contain the center has total probability $< \frac{1}{2}$. In a companion paper [33] the author applies the Glivenko-Cantelli theory of uniform convergence to prove the following sufficient condition: If n ideal points are sampled at random from a strictly centered, (uniformly) continuous probability measure μ with compact support on $\Re^m$, then the yolk radius converges to 0 and the yolk center converges to the center of the distribution μ, almost surely.

This paper employs more specialized techniques to generalize and sharpen the result just cited. The main results are these: The strict centered condition is, by itself, sufficient for almost sure yolk shrinkage. The same condition comes close to being necessary as well; it is necessary for large classes of probability measures including all discrete distributions in two or more dimensions, and all measures that somewhere put positive continuous density in an open ball. The condition also implies almost sure convergence of the yolk center.

To give these results some intuitive foundation, we use the language of distributional analysis. Suppose the population is a random sample of n points from a probability measure μ on $\Re^m$, inducing empirical measure μ_n . Extend the definition of median hyperplane to distributions: hyperplane h is a median of μ if each halfspace defined by h has measure at least a half, $\mu(h^+) \geq \frac{1}{2}$ and $\mu(h^-) \geq \frac{1}{2}$. The distributional yolk $Y(\mu)$ would then be defined as the smallest ball intersecting all median hyperplanes. One would expect that the yolk of the sample $Y(\mu_n)$ would converge to $Y(\mu)$ as $n \to \infty$, and therefore the natural condition for the radius of $Y(\mu_n)$ to converge to 0 would be that $Y(\mu)$ have zero radius.

However, the yolk of a random sample may fail to converge to the distributional yolk, in both radius and location. An example is given in section 3. Nonetheless, in this paper we prove that the natural strict centeredness condition, that the radius of $Y(\mu)$ equals zero, is a sufficient and often necessary condition for the sample yolk to shrink to 0 almost surely, in any dimension. In particular, the yolk radius does not converge to zero for any distribution that violates the condition, if somewhere the distribution has density bounded from zero in some open ball, or if it contains three noncollinear mass points, or if it has singular positive density on the surface of a compact manifold with nonzero volume.

These conditions encompass all of the commonly used models of random voter population in the social choice literature, including the standard multivariate normal, uniform distributions on balls, spheres, and hyperrectangles, and all discrete distributions, except for the one-dimensional discrete. Because many populations of interest are moderate in size, the *rate* of convergence is of concern as well. We obtain an explicit but loose upper bound on the rate of convergence for the uniform distribution on the square.

Building on properties of the yolk established in the literature, our results have similar strong implications regarding the shrinkage and/or non-shrinkage of win sets, the Pareto set, and the uncovered set, and regarding the outcomes of strategic voting under voting agendas, and epsilon-cores. The win set $W(x)$ of a point x is the set of points that defeat x by majority vote. Denote the yolk center and radius by c and r, respectively. Let $B(c,k)$ denote the ball of radius k about c; if $k<0$ then $B(c,k)=\phi$. It is easy to see [22] that the winset lies between two balls about c, in particular that $W(x)\subseteq B(c,||x-c||+2r)$ and $W(x)\bigcap B(c,||x-c||-2r)=\phi$. Therefore, if $r\to 0$, we have a very good idea as to the location and small size of the winsets. If $y\in W(x)\supset W(y)$, or equivalently y defeats x and y defeats every point that x defeats, then we say *y covers x*. The uncovered set [20] is the set of all points not covered by any point. The uncovered set is contained in the ball $B(c,4r)$ [18]. Therefore, if $r\to 0$, the uncovered set shrinks towards a point. Since outcomes of many voting agendas are in the uncovered set even if voting is strategic (see [10]), a small yolk radius would greatly limit the set of possible outcomes. A point is in the ϵ-core if no other point is preferred to it by more than ϵ by a majority of voters. An amendment sequence will converge to the ϵ-core at a rate inversely proportional to the yolk radius [32]. Thus a small yolk radius greatly improves the dynamics of ϵ-cores.

The next section establishes notation and definitions. Section 3 proves the sufficiency theorem, that the yolk radius shrinks to 0 with probability 1 if the distribution is strictly centered. Section 4 improves the convergence rate for several specific distributions. Section 5 proves that strict centeredness is necessary for almost sure convergence for all distributions on $\Re^m$ with positive density in some m-dimensional ball, for all distributions containing three noncollinear mass points, and for another class of distributions that includes the uniform distribution on the unit sphere. Section 6 proves that strict centeredness also ensures convergence of the yolk center. Section 7 states conclusions and some open questions.

2 Definitions and Notation

We work in Euclidean space $\Re^m$. Let μ denote a probability measure on $\Re^m$. When n points are randomly sampled from μ, the discrete distribution that puts mass k/n on each point that occurs k times is called the *empirical distribution*, denoted μ_n. For any hyperplane $h\in\Re^m$, denote the two closed halfspaces defined by h as h^+ and h^-, respectively, where h^- contains the origin (if h contains the origin the ambiguity won't matter). A hyperplane h is a *median* of μ iff $\mu(h^+) \geq \frac{1}{2}$ and $\mu(h^-) \geq \frac{1}{2}$. In the definition of median, μ may be any probability measure, including empirical distributions. The *yolk* of μ is a smallest ball intersecting all median hyperplanes of μ. The yolk's radius and center are respectively denoted $r(\mu)$ and $c(\mu)$. A probability measure μ is *strictly centered at* 0 iff for every hyperplane h not passing through 0, $\mu(h^+) < \frac{1}{2}$. A probability measure $\mu(X)$ is *strictly centered at* z iff $\mu(X+z)$ is strictly centered at 0. It follows from these definitions that μ is strictly centered at z iff the yolk of μ is the point z.

Let V denote a sample of n (not necessarily distinct) voter ideal points in $\Re^m$. The standard definition of a median hyperplane is a hyperplane h such that $|h^+ \cap V| \geq n/2$ and $|h^- \cap V|/n \geq n/2$. We permit n to be odd or even. Now, let V_n denote the empirical distribution corresponding to V. Then for any subset $S \subseteq \Re^m$ $|S \cap V|/n = V_n(S)$. In particular, for any hyperplane h $|h^+ \cap V|/n = V_n(h^+)$. Therefore if $V_n = \mu$ the definition of median hyperplane, yolk, yolk radius, etc. which we have given for probability measures in general are equivalent to the standard definitions for finite configurations of voter ideal points.

A median hyperplane $h = \{x : c \cdot x = c_0\}$ is *strict* if no hyperplane parallel to it is median, that is, no hyperplane defined by the same normal vector c, of form $h' = \{x : c \cdot x = c_0' \neq c_0\}$, is median. The *median space* is the linear subspace spanned by the (normal) vectors that define strict median hyperplanes of μ. The *median dimension* of μ is the dimension of the median space. For example, μ on $\Re^2$ has median dimension 2 iff it has at least two distinct strict medians. As another example, μ on $\Re$ has median dimension 1 iff it is strictly centered.

Let $X_1, X_2, \ldots$ be a sequence of random variables on a common probability space Ω. If for all of Ω except a subset of probability measure zero, it is true that for all $\epsilon > 0$ there exists N such that for all $n \geq N$, $|X_n - t| < \epsilon$, then we say X_n converges to t *almost surely*. The term *almost surely* is often abbreviated as *a.s.*; it is synonymous with *with probability 1*, often abbreviated *w.p.*1, and with *almost everywhere*, often abbreviated *a.e.*. We limit our usage to the first choice, although it is tempting to claim that "the yolk converges to a point almost everywhere" in Theorem 1.

3 Shrinking the yolk

The goal of this section is to prove that the sample yolk radius $r(\mu_n)$ converges to 0 if the distributional yolk radius $r(\mu)$ equals 0. Perhaps the most natural way one might attempt a proof would be to show that in general $r(\mu_n)$ converges to $r(\mu)$. However, the more general statement is false! A simple counterexample is the distribution μ with mass $\frac{1}{4}$ at each vertex of a unit square. The lines defined by the diagonals and the sides of the square are all medians of μ; the yolk of μ is the inscribed circle of the square. Now consider a sample of n

points from μ: as $n \to \infty$ the two diagonals will almost surely be medians of the sample configuration, but the probability tends to zero that the two parallel sides of the square will be median. (See the proof of Proposition 5 for a rigorous explanation). So most of the time the yolk of the sample will be the inscribed circle of a right triangle whose legs are two sides of the square. Thus the sample yolk radius converges (in probability anyway) to $1/(2+\sqrt{2})$ while μ has yolk radius $1/2$, and the sample yolk center doesn't converge at all. Nonetheless we will prove that $r(\mu_n) \to r(\mu)$ if $r(\mu) = 0$.

In the proof of the theorem that follows, we relax a nonlinear optimization program for the yolk radius to a linear program, and apply the fundamental theory of linear programming. This kind of *theoretical* use of linear programming was pioneered by Dantzig (see e.g. [7]); a more recent example is [1]. The rest of the proof mainly uses the probabilistic method, expounded in [2].

Theorem 1: Let a population n voters be sampled independently according to any distribution μ on R^m strictly centered at 0. Then as $n \to \infty$, the radius of the sample yolk $r(\mu_n) \to 0$ almost surely.

Proof: Let the voters be denoted $V = \{v_1, v_2, \ldots, v_n\}$ where $v_i \in \Re^m$. A *median split* of V is defined to be any pair of sets (S, T) such that:

- $S \cup T = V$.
- $|S| \geq n/2; |T| \geq n/2$.
- $|S \cap T| = 1$ (resp. 0) if n is odd (resp. even).

A hyperplane $\{x : p \cdot x = p_0\}$ is *consistent* with a median split (S, T) iff

$$p \cdot v_j \geq p_0 \quad \forall v_j \in S$$

$$p \cdot v_j \leq p_0 \quad \forall v_j \in T.$$

In other words, h is consistent with (S, T) if $S \subset h^+$ and $T \subset h^-$. Note that there may be many voters on h, i.e. $|h \cap V| \geq |S \cap T|$. It is obvious that a hyperplane is median iff it is consistent with at least one median split. For any median split (S, T) and any point $z \in \Re^m$ the nonlinear program below finds the consistent median hyperplane farthest from z.

$$\max |p \cdot z - p_0| \quad \text{subject to:} \tag{1}$$

$$\sum_{j=1}^{m} p_j^2 = 1 \tag{2}$$

$$p \cdot v_i \geq p_0 \qquad \forall v_i \in S \tag{3}$$

$$p \cdot v_i \leq p_0 \qquad \forall v_i \in T \tag{4}$$

Constraint (2) normalizes the hyperplane (p, p_0) and the other two constraints (3) and (4) ensure it is consistent with (S, T). If no solution to (2,3,4) exists we take the value of the objective function (1)to be $-\infty$.

Let $\mathcal{M}$ denote the set of all median splits. Since every median hyperplane is consistent with some $(S,T) \in \mathcal{M}$, the system below finds the radius of the smallest ball centered at z that intersects all median hyperplanes.

$$\max_{(S,T)\in\mathcal{M}} [\text{value of (1) subject to (2,3,4)}] \tag{5}$$

We call this the radius of the z-centered yolk, and denote it $r^z()$. In principle, the yolk can be found as the minimum of (5) over all z. Since the value of (5) is a supremum of convex functions, it is a convex function of z and finding the yolk would not be too difficult computationally if (5) could be solved easily. However, Bartholdi *et al.*[6] show it is co-NP-complete just to determine if the radius of the yolk is zero, when the dimension m is not fixed. So the system above probably cannot be made significantly more concise.

For any z, the radius of the z-centered yolk is at least the radius of the true yolk, $r^z(V) \geq r(V)$. We will fix $z = 0$ and denote $r^0(V)$ as the radius of the 0-centered yolk.

Summarizing what we have so far,

$$r(V) \leq r^0(V) = \max_{\mathcal{M}} \quad \begin{aligned} &\max |p_0| \; s.t. \\ &\|p\| = 1 \\ &p \cdot v_i \geq p_0 \; \forall v_i \in S \\ &p \cdot v_i \leq p_0 \; \forall v_i \in T \end{aligned} \tag{6}$$

There are two problems with (6). The inner optimization is nonlinear, and the outer maximization is taken over the exponentially large set of median splits $\mathcal{M}$. To circumvent these problems, we introduce a linear programming relaxation:

$$r^0(V) \leq \max_{\mathcal{M}} \qquad \max |p_0| \; s.t. \tag{7}$$

$$-1 \leq p_j \leq 1 \quad \forall j = 1, \ldots, m \tag{8}$$

$$p \cdot v_i \geq p_0 \; \forall v_i \in S \tag{9}$$

$$p \cdot v_i \leq p_0 \; \forall v_i \in T \tag{10}$$

(We can drop the absolute value signs on p_0 because swapping S with T reverses the sign of p_0 in the optimum solution). The set of linear programs (LPs) in (7–10) has some nice properties which we now use. The key property is that they all have the same set of basic solutions! The different LPs differ in their choice of S and T, but this only affects the direction of the inequality constraints (9–10). If a constraint is tight, it is the same regardless of S and T. Thus the exponentially large number of LPs agree on what is basic; they just disagree about what is feasible. And there are at most $\binom{n+2m}{m+1} < (n + 2m)^{m+1}$ basic solutions, a polynomial number since m is fixed.

Any basic solution may be specified by three sets of indices L (Lower), U (Upper), and T (Tight):

$U \subseteq \{1, \dots, m\}$;
$L \subseteq \{1, \dots, m\}$;
$T \subseteq \{1, \dots, n\}$;
$U \cap L = \emptyset; |L| + |U| + |T| = m + 1$,
where $j \in L \Rightarrow p_j = -1$, and $j \in U \Rightarrow p_j = 1$, and also $i \in T \Rightarrow p \cdot v_i = p_0$.

In words, L and U force the corresponding coordinates of p to their lower or upper bounds, and T forces the hyperplane (p, p_0) to pass through the corresponding voters. (If μ is smooth, then with probability 1 the only feasible basic solutions will have $1 \leq |T| \leq m$: we employ this fact in the next section to improve the upper bounds for some specific distributions μ.)

By the fundamental theory of linear programming [7], the maximum of the system (7–10) is attained at a basic feasible solution for some (S, T). This is a basic solution that is feasible with respect to the constraints (8) and is consistent with some median split. (By definition, a basic solution need not be feasible, it only must satisfy m linearly independent constraints at equality.) Thus

$$\max(7) \text{ subject to } (8, 9, 10) = \max_{\substack{\text{all basic solutions } (p, p_0) \text{ to} \\ \text{(8,9,10) that are median hyper-} \\ \text{planes and satisfy (8)}}} p_0. \tag{11}$$

Fix $\epsilon > 0$. Let the basic solutions be indexed by $k = 1, \dots, K$ where $K < (n + 2m)^{m+1}$. Thus the kth basic solution is specified by the triple (L_k, U_k, T_k) and denote the resulting solution by (p^k, p_0^k). Where appropriate we allow the $m + 1$ vector (p^k, p_0^k) to signify the associated hyperplane $\{x : p^k \cdot x = p_0^k\}$. Consider the following events:

A_k : (p^k, p_0^k) is a median hyperplane.

B_k : $p_0^k > \epsilon$.

C_k : $-1 \leq p_j^k \leq 1 \; \forall j = 1, \dots, m$.

If for any k all three of these events occur, then by (11) the value of the system (7–10) will exceed ϵ. Therefore we seek upper bounds on the probability of the union over k of the intersection of these triples of events. Using a simple but extremely useful bounding technique of probabilistic combinatorics [2] followed by an equally simple application of Bayes' rule, we have

$$\begin{aligned} Prob\{\bigcup_{k=1}^{K}(A_k \cap B_k \cap C_k)\} &\leq \\ \sum_{k=1}^{K} Prob(A_k \cap B_k \cap C_k) &\leq \\ \sum_{k=1}^{K} Prob(A_k | B_k \cap C_k). \end{aligned}$$

Suppose B^k and C^k occur. The former implies $|p_0^k| > \epsilon$, the latter implies $\|p^k\| \leq \sqrt{m}$; therefore the distance from the hyperplane to the origin

$|p_0^k|/\|p^k\| > \epsilon/\sqrt{m}$, a strictly positive constant. By hypothesis, μ is strictly centered at the origin so there exists $\lambda > 0$ (depending only on $\epsilon/\sqrt{m}$) such that the total mass in the halfspace $(p^k, p_0^k)^+$ is at most $1/2 - \lambda$. (This is the second use of the "constant factor" $\sqrt{m}$.)

Each basic solution depends on only a few of the voters v_i. Consider any particular k. The hyperplane (p^k, p_0^k) is determined by $v_i \in T^k$ and the sets U^k, L^k, but is independent of v_j for all $j \notin T^k$. This is because v_j is independent of v_i for all $j \neq i$. (Another way to see this is to generate voters $v_j \notin T^k$ first, then generate the voters $v_i \in T^k$ second. At the end of the first step, nothing more is known about (p^k, p_0^k) than at the start.)

Let the random variable X be the number of voters $v_j \notin T^k$ which fall in $(p^k, p_0^k)^+$. For $(p^k, p_0^k)$to be median, X would have to be at least $n/2 - |T^k| \geq n/2 - m - 1$. Now since the v_i are independent, X simply follows a binomial distribution $B(r, q)$ with $r = n - |T^k| \geq n - m - 1$ and some $q \leq 1/2 - \lambda$, which is dominated by the distribution $B(n, \frac{1}{2} - \lambda)$. Let Y follow the latter distribution. Then the probability $(p^k, p_0^k)$is median is less than $Prob(Y \geq n/2 - m - 1)$. Applying Chernoff-type bounds [2, p.235] to $Y - E[Y]$ yields

$$Prob(Y - E[Y] \geq \tau) < e^{-2\tau^2/n}.$$

Select $\tau = \lambda n - m - 1$. Then

$$\begin{aligned} Prob((Y \geq n/2 - m - 1) &< e^{-2(\lambda n - m - 1)^2/n} \\ &= e^{-2\lambda^2 n} e^{2\lambda(m+1)} e^{-2(m+1)^2/n} \\ &< c_1 c_2^n \end{aligned}$$

for positive constants $c_1 = e^{2\lambda(m+1)}$ and $c_2 = e^{-2\lambda^2} < 1$.

Summarizing the preceding, we now have shown that for any fixed $\epsilon > 0$, $\text{Prob}(A_k | B_k \cap C_k) < c_1 (c_2)^n$ for some constants c_1 and $c_2, c_2 < 1$. Then the probability that any of the K basic solutions is feasible (in some LP) and has objective function value more than ϵ is

$$Prob\left[\bigcup_{k=1}^{K} (A_k \cap B_k \cap C_k)\right] \leq K c_1 c_2^n \leq c_1 (n + 2m)^{m+1} c_2^n. \tag{12}$$

This behaves as $n^m c_2^n$ which obviously goes to 0 as $n \to \infty$. (Notice that event B_k was defined as the objective function value of the linear program exceeding ϵ, not the hyperplane maximizing the objective being more than $\epsilon/\sqrt{m}$ from the origin. This is because the maximum to (7–10) need not occur at the same physical hyperplane as the maximum to (6).) Applying the inequalities from (6–11), we find

$$Prob\,[r(V) > \epsilon] < c_1 (n + 2m)^{m+1} c_2^n \to 0 \tag{13}$$

as $n \to \infty$. Therefore the radius of the yolk shrinks to 0 in probability.

To prove the stronger fact of almost sure convergence, we take advantage of the geometrically decreasing factor c_2^n. In 13, for sufficiently large $n \geq N$ the right hand side $c_1(n+2m)^{m+1}c_2^n < 1$. Observe that for some constants $c > 0, c_2 < 1$,

$$\sum_{n=1}^{\infty} Prob(r(V) > \epsilon) < N + \sum_{n=N}^{\infty} cn^{m+1}c_2^n < N + \sum_{n=N}^{\infty} cc_2^{n/2} < N + \frac{c}{(1-c_2^{1/2})} < \infty$$

because $c_2 < 1$. By the Borel-Cantelli lemma [14], this proves the theorem. ■

Corollary 1.1 If voters are drawn independently from any of the following distributions in $\Re^m$: multinormal (nonstandard), uniform on the unit ball, unit sphere, or hypercube — then the radius of the yolk converges to 0 almost surely.

Corollary 1.2 If μ is strictly centered at z then the radius of the sample z-centered yolk $r^z(\mu_n) \to 0$ almost surely as $n \to \infty$.
Proof: This follows because we proved Theorem 1 via the inequality (6).

Recall from the introduction that the uncovered set has been shown to be within a constant factor of the radius of the yolk from the yolk center [18]. As a corollary to Theorem 1, we therefore have:

Corollary 1.3 If n voters are drawn independently from a strictly centered distribution on $\Re^m$ the size of the uncovered set converges to 0 with probability 1 as $n \to \infty$. Hence [31] the outcome of strategic voting under an amendment agenda becomes nearly known as the population increases.

Another solution concept, the Banks set, is a subset of the uncovered set. Therefore its size converges to zero too under the conditions of Corollary 1.3.

To conclude this section, we derive an explicit upper bound on the convergence rate of the yolk radius for the uniform distribution on a unit square.
Remark 1.4 For $m = 2$, if μ has zero mass on any line, the probability that the yolk radius exceeds ϵ in inequality 12 may be tightened from $c_1(n+2m)^{m+1}c_2^n$ to $c_1 2n(n+1)c_2^n$.
Proof: The probability is zero that a line passes through 3 voter ideal points, because a homogenous system in p, p_0 from (8–10) results with unique solution 0 with probability 1. Thus the number of basic solutions to be considered drops from $\sim n^{m+1}$ to $2n(n+1)$.

In the proof of Theorem 1, $\lambda > 0$ is such that the total mass in a halfplane defined by a line at distance $\epsilon/\sqrt{2}$ is at most $1/2 - \lambda$. By elementary geometry, for $\epsilon \leq 2 - \sqrt{2}$ we can take $\lambda = \epsilon/\sqrt{2}$. Applying this value of λ to Remark 1.4, the probability that the yolk radius exceeds ϵ (for $0 < \epsilon < .5858$) is less than

$$\begin{aligned} c_1 2n(n+1)c_2^n &= e^{2\lambda(m+1)} 2n(n+1)(e^{-2\lambda^2})^n & (14) \\ &= e^{3\sqrt{2}\epsilon} 2n(n+1)(e^{-\epsilon^2})^n. & (15) \end{aligned}$$

Unfortunately, this bound is too loose to be useful for $n = 100$. For $n = 600$, the probability that $\epsilon > 0.2$ is less than 0.000064. Bounds for other distributions such as uniform on a circle can be obtained similarly, but they also are too loose to be much use. Obtaining better bounds remains an open problem.

4 Not shrinking the yolk

For a simple two-dimensional example of the yolk not shrinking, place mass 1/3 on each vertex of an equilateral triangle. For large n the probability tends to 1 that each side of the triangle lies in a median hyperplane. Then the sample yolk will be the inscribed circle. Thus if each vertex is at distance 1 from the origin, the yolk radius will have expected value 1/2 in the limit as $n \to \infty$.

In the preceding example, the hyperplanes that made the distribution not strictly centered were precisely those that bounded the yolk radius away from 0. This suggests that if we weaken the condition on μ in Theorem 1 the yolk radius may not converge to 0. The following is easy to see:

Proposition 2: Fix $z \in \Re^m$ and let μ be any probability measure on $\Re^m$. The radius of the z-centered yolk of the sample μ_n converges to 0 almost surely as $n \to \infty$ if and only if μ is strictly centered at z.

Proof: If μ is not strictly centered at z then there exists a hyperplane h at some strictly positive distance δ from z, such that the mass of μ in the halfspace h^+ away from z is at least 1/2. Then the number of voters falling in h^+ is binomially distributed with success probability at least 1/2, so the probability is at least 1/2 that $\mu_n(h^+) \geq 1/2$. If this event occurs then some hyperplane parallel to h and contained in the halfspace h^+ (possibly the hyperplane h itself) is median. This means there is probability at least 1/2 that there is a median hyperplane at distance at least δ from z. Therefore for all n the radius $r^z(\mu_n)$ of the z-centered yolk has expected value $E[r^z(\mu_n)] \geq \delta/2$ which is bounded away from 0.

The "if" portion of the proof is from Corollary 1.2. ■

Proposition 2 says nothing about the true yolk radius, because the location of the yolk center might not converge. Now we come to the main result of this section, necessary conditions for convergence of the yolk radius to 0. Recall that no distinct hyperplane parallel to a strict median hyperplane can be median, and that the *median dimension* of μ is defined to be the dimension of the subspace spanned by the normal vectors to strict median hyperplanes of μ. Later we will show that large classes of probability measures on $\Re^m$ have median dimension m. Our main result is a necessary and sufficient condition for yolk shrinkage within these classes.

Theorem 3: Let μ be any probability measure on $\Re^m$ that has median dimension m. Let μ_n result from a random sample of n points according to μ. Then

$$r(\mu) = 0 \Leftrightarrow r(\mu_n) \to 0 \ a.s. \text{ as } n \to \infty.$$

Proof: If $m = 1$ then as remarked in section 2, $r(\mu) = 0$. By Theorem 1 $r(\mu_n) \to 0$ *a.s.*, and both sides of the equivalence are always true. Hereafter we assume $m \geq 2$.

Let $h_1, \ldots, h_m$ be m strict medians of μ whose normal vectors are linearly independent. Then these hyperplanes intersect at some point, say z. For notational convenience, translate μ so that $z = 0$. Since μ is not strictly centered, there exists a hyperplane h_0 not containing 0 such that the mass of μ in the halfspace h_0^+ is at least 1/2. The idea of the proof is shown geometrically in Figure 1. The hyperplanes $h_0, \ldots, h_m$ form an m-dimensional simplex. Nudge the hyperplanes $h_1, \ldots, h_m$ a little bit towards h_0. Since these hyperplanes are strict medians, when perturbed they are no longer medians. For large n, the probability tends to 1 that there is a median hyperplane parallel to each of these and outside the perturbed simplex. Meanwhile the probability is at least 1/2 that similarly there is a median hyperplane parallel to h_0 outside the perturbed simplex. Thus the probability is at least nearly 1/2 that the yolk radius is at least as large as the radius of the inscribed sphere of the perturbed simplex, so the yolk radius has expected value bounded away from 0.

Formally, let $\delta > 0$ denote the distance from h_0 to the origin. Let $\tilde{h_i} : i = 1, \ldots, m$ denote the hyperplane parallel to h_i at distance δ/m^3 from 0 and selected so as to intersect the line segment of length δ between 0 and h_0. Let S denote the perturbed simplex whose facets are defined by the hyperplanes h_0 and $\tilde{h_i} : \ i = 1, \ldots, m$. Since m is fixed, and the normals to $h_i : i = 1, \ldots, m$ are linearly independent, and h_0 is not parallel to any of them (they define strict medians and there is a median hyperplane parallel to h_0 in the halfspace h_0^+), the simplex S has inscribed sphere with radius $\nu > 0$.

We now sample the set V of n points independently at random from μ. For $i = 1, \ldots, m$ let D_i denote the event: there exists a median hyperplane with respect to V, parallel to $\tilde{h_i}$, and in the halfspace $\tilde{h}_i^-$. Similarly, let D_0 denote the event: there exists a median hyperplane with respect to V, parallel to h_0, in the halfspace h_0^+. Notice that if all these events occur, there exist median hyperplanes forming a simplex containing the simplex S.

Using the same argument as in the proof of Proposition 2, in the limit as $n \to \infty$, the probability of each of the events $D_1, \ldots, D_m$ converges to 1, because the number of voters falling in the halfspace $\tilde{h}_i^-$ follows a binomial distribution with success probability strictly greater than 1/2. Therefore, for some N, $Prob[D_i] > 1 - 1/4m$ for all $n \geq N$. Similarly, $Prob[D_0] \geq 1/2$ because the number of voters falling in the halfspace h_0^+ follows a binomial distribution with success probability at least 1/2. Hence for all $n \geq N$:

$$Prob\left[\bigcap_{i=0}^{m} D_i\right] \geq 1 - 1/2 - \sum_{i=1}^{m} \frac{1}{4m} \geq 1/6.$$

Then the yolk radius does not converge,

$$\exists N, \exists \nu > 0 : Prob\left[r(V) \geq \nu\right] \geq 1/6 \ \forall n \geq N.$$

Figure 1: bounding the yolk radius away from 0

and has expected value bounded away from 0:

$$E[r(V)] \geq \nu/6 \; \forall n \geq N.$$

So if μ has median dimension m and is not strictly centered, the yolk radius does not converge to 0; if it is strictly centered it is almost sure to converge to 0 by Theorem 1. This concludes the proof of Theorem 3. ■

Under the conditions stated, Theorem 3 shows *constructively* that the yolk will not shrink to 0.

Theorem 4: Suppose μ is a probability distribution on $\Re^m$, $m \geq 2$ having positive density in some m-dimensional ball, or containing at least three non-collinear mass points. Then μ has median dimension m.
Proof: If μ has three non-collinear mass points, denote them as p_1, p_2, p_3. If $B \subset \Re^m$ is an open ball in which μ has positive density, select three non-collinear points in B (note $m \geq 2$) and denote them as p_1, p_2, p_3. We will repeatedly make use of a key observation: if h is a median hyperplane of μ passing through p_1, p_2 or p_3 then h is a strict median. This follows because any perturbation of h parallel to itself either leaves a mass point, or passes through a region of positive m-dimensional volume and positive density with respect to μ.

We proceed by constructing a set of m strict medians whose normal vectors are linearly independent. Inductively assume for $0 \leq k < m$ that there are k strict medians $h^1, \ldots, h^k$ whose unit normal vectors $v^1, \ldots, v^k$ are linearly independent. Since $k < m$ there exists a nonzero vector v in the $m-k$-dimensional subspace orthogonal to the subspace of (i.e. not spanned by) the normal vectors $v^1, \ldots, v^k$. So $v \cdot v^i = 0 : i = 1 \ldots k$. Let p be any of p_1, p_2, p_3.

For any vector u such that $u \cdot v = 0$, the three points $p, p+v, p+u$ define a 2-dimensional plane. Rotate a unit vector at p in a half circle in this plane, so for $0 \leq \theta < \pi$ the vector, denoted t^θ, equals $v \cos\theta + u \sin\theta$. For some value $\theta(u,p)$ of θ in this range, the hyperplane

$$h^{\theta(u,p)} \equiv \{x : t^{\theta(u,p)} \cdot x = t^{\theta(u,p)} \cdot p\}$$

is median. By the key observation, $h^{\theta(u,p)}$ is a strict median.

For the inductive base case $k = 0$, there exists a vector u orthogonal to v because $m \geq 2$. Choose $p = p_1$, set $v^1 = t^{\theta(u,p)}$ and $h^1 = h^{\theta(u,p)}$, and continue the induction.

For the other inductive steps, we seek some vector u and some value for p for which $\theta(u,p) \neq \pi/2$. Any value of θ other than $\pi/2$ would yield a normal vector t^θ such that

$$t^\theta \cdot v = v \cos\theta + u \sin\theta \cdot v = \cos\theta \neq 0.$$

Since $v^i \cdot v = 0$ for $1 \leq i \leq k$, t^θ would be linearly independent of the v^i and the inductive step would be complete. We will assume that no such vector u and choice of p exists and derive a contradiction. This will complete the proof.

The vectors $p_2 - p_1$ and $p_3 - p_1$ must be linearly independent, because p_1, p_2, p_3 are not collinear (i.e. they are affinely independent). Hence at least one of these two vectors is not a multiple of e_m. Without loss of generality assume $\forall \alpha p_2 - p_1 \neq \alpha e_m$. Now define the vector w' which is the projection of $p_2 - p_1$ onto the space orthogonal to v, $w' \equiv (p_2 - p_1) - ((p_2 - p_1) \cdot v)v$. Let $w = w'/||w'||$ be the vector scaled to be a unit vector. Note $w \cdot v = 0$ and $w \neq 0$. By assumption, $\theta(w, p_1) = \theta(w, p_2) = \pi/2$. Hence the hyperplanes $h^1 = \{x : w \cdot x = w \cdot p_1\}$ and $h^2 = \{x : w \cdot x = w \cdot p_2\}$ are both strict medians. The hyperplanes have the same normal vector w. However,

$$w \cdot p_2 - w \cdot p_1 = \frac{1}{||w'||}(w' \cdot (w' + v((p_2 - p_1) \cdot v))) = \frac{1}{||w'||}(w' \cdot w' \ + 0) \neq 0.$$

Therefore the two hyperplanes are parallel and distinct, which contradicts their being strict medians. This completes the proof of Theorem 4. ■

If we place weight 1/4 at each vertex of a rectangle, we get a good illustration of Theorem 4. Notice that in the degenerate case of 4 collinear points, there is only one strict median.

Corollary 4.1 Suppose μ is a probability distribution on $\Re^m$, $m \geq 2$ having positive density in some m-dimensional ball, or containing at least three non-collinear mass points. Then the radius of the sample yolk converges to 0 if and only if μ is strictly centered.

Corollary 4.2 Let μ be any continuous probability distribution on $\Re^m$, $m \geq 2$. Then the radius of the sample yolk converges to 0 if and only if μ is strictly centered.

Corollary 4.3 Let μ be any not strictly centered distribution on $\Re^m : m \geq 2$ with positive density in some m-dimensional ball. Then the size of the Pareto set does not converge to 0.

Proof: Feld *et al.*[12] have shown that the yolk is contained in the Pareto set. The result then follows from Theorem 4.

Notice that the proof of Theorem 4 fails for $m = 1$. It should, since 1-dimensional distributions can have median dimension 0 but positive density on an interval. For example, a uniform density on $[0, 1/2] \cup [3/2, 2]$ has positive density on a 1-dimensional ball but has median dimension 0 since it has no strict medians.

That Theorem 4 is false for $m = 1$ suggests that Corollary 4.1 would be similarly false. And indeed, it is easy to see that for the distribution given above, the yolk radius converges to 0 in probability. Yet part of Corollary 4.1 remains true: in one dimension, strict centeredness is necessary and sufficient for almost sure convergence. The next proposition clarifies the situation:

Proposition 5 Let μ be any distribution on $\Re$. Then the radius of the sample yolk converges to 0 almost surely iff μ is strictly centered.
Proof: The "if" follows from Theorem 1. So suppose μ is not strictly centered. Then there exist distinct points $x, x + \delta$ $(\delta > 0)$ which are both bisectors of μ. Thus μ has no mass in the open interval $(x, x + \delta)$, and mass 1/2 in each closed ray $(-\infty, x]$ and $[x + \delta, \infty)$. Denote these rays L and R respectively. For n even, if n voters are drawn according to μ, the yolk radius is zero iff exactly $n/2$ voters fall in L (and R). This occurs with probability $\binom{n}{n/2}2^{-n} \sim 1/\sqrt{n}$, and is precisely equivalent to a random (drunkard's) walk returning to its starting point. It is well known that a random walk almost surely returns infinitely often to its starting point, see e.g. [14]. Therefore the radius $r(V)$ does not converge to 0 almost surely. On the contrary, it is almost sure that for all N there exists $n \geq N$ such that $r(\mu_n) \geq \delta/2$. Notice that in general when μ has finite second moment, $r(V)$ will converge to 0 in probability because $1/\sqrt{n} \to 0$ as $n \to \infty$.
$\blacksquare$

The next proposition extends theorem 4 to account for population models such as the uniform distribution on the sphere.
Proposition 6: Suppose μ is a distribution on $\Re^m$ containing a singular component with (strictly) positive density on the surface of a closed compact $m-1$-dimensional manifold embedded in $\Re^m$ with interior of positive m-dimensional volume. Then μ has median dimension m.
Proof: Observe that if h is any bisecting hyperplane of μ passing through any point x in the interior of the manifold, then h is a strict median. Now the proof is exactly as in Theorem 4.

5 The Location of the Yolk

The aim of this section is to prove that if μ is strictly centered, then the yolk center converges almost surely to the center of μ.

Theorem 7: Let μ be any strictly centered distribution on $\Re^m$. Let a set V of n voter ideal points be independently distributed according to μ and let $c(V)$ denote the center of the yolk of V. Then $||z - c(V)||^\infty \to 0$ almost surely as $n \to \infty$, where z is the center of μ.

Proof: Select arbitrary $\delta > 0$ and let μ be strictly centered around $z = 0$ without loss of generality. Our goal then is to show that Prob $\{||c(V)|| > \delta\}$ decreases to 0 at geometric rate.

Construct $2m$ hyperplanes $h_i : i = 1, \ldots, 2m$ defined as

$$\begin{aligned} h_i &= \{x \in \Re^m | x_i = \delta/2\} \qquad i = 1, \ldots, m \\ h_{i+m} &= \{x \in \Re^m | x_i = -\delta/2\} \qquad i = 1, \ldots, m \end{aligned}$$

that form a hypercube $\mathcal{H}$ of side δ centered at 0. Since μ is strictly centered at 0, the amount of mass in each of the $2m$ halfspaces h_i^- containing $\mathcal{H}$ is strictly greater than 1/2. Let $\tau > 1/2$ be the minimum of these amounts. Thus if v is distributed according to μ,

$$Prob\{v \in h_i^-\} \geq \tau > \frac{1}{2} \; \forall i = 1, \ldots, 2m. \tag{16}$$

Let A_i denote the event that at least half the voters in V are in the halfspace h_i^-. The key idea is that if A_j occurs, then (as in the proof of Proposition 2) there is a median hyperplane parallel to h_j in the halfspace h_j^-. Suppose Y were a binomially distributed variable with parameters (τ, n). Since each voter falls in the halfspace h_i^- with probability at least τ, we have for all i,

$$Prob(A_i) \geq P(Y \geq n/2). \tag{17}$$

By Chernoff type bounds on the binomial distribution [2, p. 233],

$$P(Y \geq n/2) > 1 - e^{-2n(1/2-\tau)^2} \tag{18}$$

Combining (15) and (16) and applying the same bounding technique of [2] used in the proof of Theorem 1, we have

$$P\left(\bigcap_{i=1}^{2m} A_i\right) \geq 1 - 2me^{-2n(1/2-\tau)^2}. \tag{19}$$

Therefore the probability that all the events occur converges rapidly to 1.

Now for any $w \in \Re^m$, $||w||^\infty > \delta$ there is a hyperplane h_i at distance greater than $\delta/2$ from w that separates w from the hypercube $\mathcal{H}$ (in the coordinate that exceeds $\pm\delta$). If the event A_i occurs, then $r^w(V) > \delta/2$ because there is a median hyperplane parallel to h_i in the halfspace h_i^- which is necessarily further than $\delta/2$ from w.

We see that if all the events A_i occur, all the w-centered yolks (for all $||w|| > \delta$) have radii $r^w(V)$ exceeding $\delta/2$:

$$\bigcap_{i=1}^{2m} A_i \Rightarrow r^w(V) > \delta/2 \quad \forall w : ||w||^\infty > \delta. \tag{20}$$

Therefore, *if the true yolk radius $r(V)$ is less than $\delta/2$, and the events A_i occur, then the center of the yolk cannot be further than δ from the origin.* Using $\epsilon = \delta/2$ in (13) from the proof of Theorem 1, there exist constants $c_1 > 0, c_2 < 1$ such that:

$$P\left[r(V) \leq \delta/2\right] > 1 - c_1(n+2m)^{m+1}c_2^{n/2}.$$

Therefore, the probability that the yolk center is far from the origin is

$$P\left[||c(V)||^\infty > \delta||\right] < 2me^{-2n(\tau-1/2)^2} + c_1(n+2m)^{m+1}c_2^{n/2} \tag{21}$$

which converges to 0 as $n \to \infty$. As usual, the stronger convergence with probability 1 follows because the sum (over n) of the terms on the right hand side of (19) is bounded. This proves Theorem 7. ■

The status of a converse to Theorem 7 is a bit odd. Recall that in 1 dimension, strict centeredness is not necessary for the yolk radius $r(\mu_n)$ to converge to 0 (at least in probability—see the discussion preceding Proposition 5). In general for higher dimensions, however, strict centeredness is necessary (Corollary 4.1). The situation is the opposite in regard to the convergence of the yolk center $c(\mu_n)$. For in one dimension, the proof of Proposition 5 shows that $c(\mu_n)$ does not converge if μ is not strictly centered (because the random walk is recurrent). But in two or more dimensions, $c(V)$ can converge even if μ is not strictly centered. To re-use an example, suppose μ has mass 1/3 at each vertex of an equilateral triangle with center 0. Then μ is not strictly centered yet obviously $||c(\mu_n)||^\infty \to 0$ *a.s.* as $n \to \infty$.

6 Conclusions and Open Questions

The main contribution of this paper is to give a nearly definitive answer to the question of when the yolk radius of a population sample tends to zero. It provides a rigorous explanation for simulation results. Theorem 1 shows that for any strictly centered distribution the size of the yolk will shrink to 0 almost surely. Corollary 4.1 and Proposition 6 give us necessary and sufficient conditions for yolk shrinkage and hence *de facto* equilibrium over an enormous class of distributions, including all that are commonly used.

Since all the common models of voter population in the literature (as well as some potentially useful ones such as the nonstandard normal) are strictly centered , it may be tempting to conclude that *de facto* equilibrium will occur and so the nightmare of chaos is figuratively dispelled by the (almost sure) light of probability. However, I think that this is too optimistic a conclusion.

First, the asymptotic results given here do not hold for the small committee-sized populations that so often make real decisions. Second, empirical data can suggest noncentered distributions, such as the triangular type observed in the U.S.A. Congress in the 1960s.

Instead, I think that the results here suggest that the yolk radius is a good measure of how uncentered, and how potentially unstable, is a voting population.

This paper leaves several open questions. Chief among them is whether strict centeredness is a completely general necessary and sufficient condition for almost sure yolk convergence to a point. The other main open question is to determine tighter bounds on the rate of convergence than we were able to obtain in section 3. Some tighter bounds are found in the companion paper [33], albeit for a less general class of distributions than is treated here.

Another question arises if we compare the conditions for yolk shrinkage with conditions for equilibrium in configurations and continuous distributions [19, 8, e.g.]. The condition employed there might here be termed "weak centeredness": a configuration is *weakly centered* at x iff the open halfspaces defined by each hyperplane passing through x contain mass at most $1/2$. Thus the conditions are quite similar though not equivalent. Another striking similarity with results in [8] is the following: the sufficient conditions for total ordering given there are weak centeredness and that all median hyperplanes are strict (to use the terminology from this paper.) This similarity suggests a question of whether the total ordering result in [8] holds under the weaker conditions of weak centeredness and full median dimension.

The results on non-shrinkage suggest some extremal problems: given m and n, for what configuration of n points within a unit ball in $\Re^m$ is the yolk radius maximal? Given m, for what probability distribution μ, restricted to have support within a unit ball in $\Re^m$, is the expected sample yolk radius $r(\mu_n)$ maximal as $n \to \infty$?

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