Smallest Tournaments Not Realizable by $\frac{2}{3}$-Majority Voting

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Abstract

Define the *predictability* number $\alpha(G)$ of a tournament T to be the largest supermajority threshhold $\frac{1}{2} < \alpha \leq 1$ for which T could represent the pairwise voting outcomes from some population of voter preference orders. We establish that the predictability number always exists and is rational. Only acyclic tournaments have predictability 1; the Condorcet voting paradox tournament has predictability $\frac{2}{3}$; Gilboa (4) found a tournament on 54 alternatives (i.e. vertices) that has predictability less than $\frac{2}{3}$, raising the question of whether a smaller such tournament exists. We exhibit an 8-vertex tournament that has predictability $\frac{13}{20}$, and prove that it is the smallest tournament with predictability $< \frac{2}{3}$. Our methodology is to formulate the problem as a finite set of 2-person 0-sum games, employ the minimax duality and linear programming basic solution theorems, and solve using rational arithmetic.

1 Introduction

A tournament is a directed graph that represents a complete set of pairwise comparisons among a finite set of alternatives. Tournaments are pervasive in social choice. In some scenarios a given tournament must be aggregated into a ranking. (8), (10), (3), (5). In other scenarios pairwise comparisons are elicited from the population according to some protocol to produce a social choice. Many social choice rules that satisfy the Condorcet criterion are functions of the graph of pairwise simple majority voting outcomes. (9)

In this paper we solve the problem raised by Gilboa (4) of finding a smallest tournament that can not represent the pairwise outcomes of $\frac{2}{3}$-majority voting. In particular, we reduce the best known size from 54 vertices [IBID] to 8, and prove that 8 is minimal.

The problem and its solution suggest a measure which we call *predictability*. The predictability $\alpha(T)$ of a tournament T is the maximum supermajority threshhold $\frac{1}{2} < \alpha \leq 1$ for which T could represent pairwise voting outcomes for some finite population of voters. We show that predictability has a natural interpretation as the value of a 2-person game in which one player tries to predict correctly the outcome between a pair that is selected by the other player. This game will establish that the predictability value always exists and is rational. Only acyclic tournaments have predictability 1. The Condorcet paradox tournament (a directed 3-cycle) has predictability $\frac{2}{3}$, and is the smallest that has predictability < 1. Our 8-vertex tournament has predictability $\frac{13}{20}$, and is the smallest that has predictability $< \frac{2}{3}$.

In particular, we prove that all tournaments on 7 or fewer vertices have predictability $\frac{2}{3}$ or 1. All tournaments on 8 or fewer vertices have predictability at least $\frac{13}{20}$. These results are shown in Table 1, where $\alpha(T)$ denotes the predictability of T. For $n = 9$, we have found tournaments with even smaller predictability number $\frac{64}{99}$, but we have no proof that this number is the minimum.

n	2	3	4	5	6	7	8	9
$\min_{T \in \mathcal{T}_n} \alpha(T)$	1	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{13}{20}$	$\leq \frac{64}{99}$

Table 1: $\min_{T \in \mathcal{T}_n} \alpha(T)$

The technical methods that we employ are as follows. We formulate the problem of measuring predictability as a 2-person 0-sum game, and apply the minimax duality theorem. Existence and rationality of predictability then follow from basic linear programming theory. From the dual we build an integer programming formulation which we solve with software to guess the correct answer. To prove rigorously the correctness of our guess, we bound the solution values of a large finite set of linear programs, in exact arithmetic.

In related work, a simple constructive argument shows that every graph has predictability $> \frac{1}{2}$, e.g. (6). This construction is also implicit in the cyclic majority preferences Arrow used in his proof of his original possibility theorem (2). On the other hand, a classical probabilistic result, sharpened recently by Alon (1), shows that for every fixed $\epsilon > 0$, asymptotically almost all tournaments have predictability less than $\frac{1}{2} + \epsilon$. Hence, as the number of vertices grows, the smallest possible predictability decreases towards $\frac{1}{2}$, and so does the average predictability over random tournaments. The results herein imply that the asymptotic bounds are rather loose for situations with fewer than 9 alternatives.

1.1 Definitions and Notation

A *simple graph* is a graph with no edge from any vertex to itself and no multiple edges. A simple directed graph $T = (V, E)$ on $n = |V|$ vertices is a *tournament* on V if, for every pair of distinct vertices $u, v \in V$, either $uv \in E$ or $vu \in E$, but not both. The acyclic tournaments on V correspond to the $n!$ orderings (or permutations) of the elements of V, where $uv \in E$ if vertex u precedes vertex v in the corresponding permutation. Any tournament $T = (V, E)$ can be represented by a binary *tournament vector* $\pi \in \{0, 1\}^{n(n-1)}$ with one component for each ordered pair of distinct vertices $u, v \in V$:

$$\pi_{uv} = \begin{cases} 1 & \text{if } uv \in E \\ 0 & \text{if } uv \notin E. \end{cases}$$

Let the tournament vectors $\pi^1, ..., \pi^{n!} \in \{0, 1\}^{n(n-1)}$ represent the tournaments corresponding to the $n!$ permutations of n elements. We denote the set of all tournaments on n vertices by $\mathcal{T}_n$.

A *voting profile* is a vector $x \in \mathbb{R}^{n!}$ satisfying $x_k \geq 0$ for $k \in \{1, ..., n!\}$, and $\sum_{k=1}^{n!} x_k = 1$. A voting profile represents the vote of a population where each member of the population specifies a permutation of the set V as an expression of a preference ordering of the n candidates, and x_k is the fraction of the population that specifies permutation π^k. For our purposes, the size of a voting population is treated as arbitrarily large, so that any voting profile with rational components is allowed.

A tournament $T = (V, E)$ on n vertices is said to be *realizable* for $\alpha \in (\frac{1}{2}, 1]$ (or "α-realizable") if there exists a voting profile $x \in \mathbb{R}^{n!}$ with an α-majority of the voting population preferring u to v for every edge $uv \in E$:

$$\sum_{k=1}^{n!} \pi_{uv}^k x_k \geq \alpha \qquad \Longleftrightarrow \qquad uv \in E.$$

The *predictability* of a tournament is the maximum value of α for which it is realizable.

2 2-Person Game and Linear Program Formulation

For a tournament $T = (V, E)$, the problem of finding the maximum α for which T is α-realizable can be formulated as a linear program.

$$\begin{array}{lll} \max \alpha & & \\ \text{subject to} & \sum_{k=1}^{n!} x_k = 1 & \\ & \sum_{k=1}^{n!} \pi_e^k x_k \geq \alpha, & \forall e \in E \\ & x_k \geq 0, & \forall k = 1, ..., n! \end{array} \tag{1}$$

The optimal value of (1) is the predictability $\alpha(T)$. The variable $x \in \mathbb{R}^{n!}$ is a voting profile where component x_k represents the fraction of the population voting for permutation π^k. The constraints $\sum_{k=1}^{n!} \pi_e^k x_k \geq \alpha$ enforce the α-realizability requirement that a fraction of at least α of the voting population ranks u above v for every edge uv in tournament T. From linear programming theory, we know that there exists a rational optimal solution to (1), so any tournament that is realizable can be realized by a finite voting population. The dual LP obtained from the primal formulation above is:

$$\begin{array}{lll} \min \lambda & & \\ \text{subject to} & \sum_{e \in E} \mu_e = 1 & \\ & \sum_{e \in E} \pi_e^k \mu_e \leq \lambda, & \forall k = 1, ..., n! \\ & \mu_e \geq 0, & \forall e \in E \end{array} \tag{2}$$

The symmetry between the primal (1) and the dual (2) is typical of primal/dual formulations for two-person zero-sum games. In fact, the problem of finding $\alpha(T)$ for a given tournament T *is* a two-person zero-sum game. The meaning of this game explains why $\alpha(T)$ measures the predictability of tournament T.

Definition 1 *The* **Prediction Game** *for a tournament $T = (V, E)$: To play this zero-sum game, Player I chooses a permutation π^k, which constitutes a complete consistent set of predictions of the outcomes between all vertex pairs. Player II selects a specific ordered vertex pair $uv \in E$. Player I wins 1 iff his prediction is correct for the pair selected by player II, i.e., if u precedes v in permutation π^k. Otherwise player I wins 0.*

The payoff matrix for the game on the Condorcet paradox tournament $V = \{a, b, c\}; E = \{ab, bc, ca\}$ is illustrated in (Table 2). In general, the mixed strategy available to the primal player is the variable vector x in (1), and the mixed strategy of the dual player is the variable vector μ in (2).

II I	ab	bc	ca
abc	1	1	0
acb	1	0	0
bac	0	1	0
bca	0	1	1
cab	1	0	1
cba	0	0	1

Table 2: The Prediction Game, illustrated for the Condorcet paradox tournament. The row player chooses a permutation, the column player an edge.

From the Minimax Theorem of game theory, or the Strong Duality Theorem of linear programming, the optimal value of (2) is equal to the optimal value of (1), so both linear programs have an optimal value of $\alpha(T)$.

We have established the following.

Theorem 1 *The predictability of a tournament T is the value of the two-person zero-sum Prediction Game for T.*

Corollary 2 *The predictability of a tournament always exists and is rational.*

To find a tournament of size n with a smallest value of $\alpha(T)$, it is not practical for large n to solve a linear program for every tournament on n vertices. In the first place, for moderately large n the LP becomes unmanageably large. The primal LP has $n! + 1$ variables and $\binom{n}{2} + 1$ constraints; for $n = 10$ the primal has well over three million variables. Moreover, the number of possible tournaments is $2^{\binom{n}{2}}$; for $n = 10$ there are more than 10^{13} tournaments. Even the number of non-isomorphic tournaments grows rapidly (for $n = 10$ they number almost 10^7 (7)).

For moderate values of n, we may enumerate and evaluate all possible tournaments on n vertices implicitly by formulating and solving a single mixed-integer program (IP):

$$
\begin{array}{lll}
\min \lambda & & \\
\text{subject to} & \sum_{u,v \in V} \mu_{uv} = 1 & \\
& \sum_{u,v \in V} \pi^k_{uv} \mu_{uv} \leq \lambda, & \forall k = 1, ..., n! \\
& 0 \leq \mu_{uv} \leq z_{uv}, & \forall u, v \in V \\
& z_{uv} + z_{vu} = 1, & \forall u, v \in V \\
& z_{uv} \in \{0, 1\}, & \forall u, v \in V
\end{array}
\tag{3}
$$

In the IP (3), every assignment to the vector z of binary variables gives a tournament vector (the constraint $z_{uv} + z_{vu} = 1$ enforces the requirement that for every pair of vertices

u, v there is either an edge uv or an edge vu but not both). Since every tournament also gives a tournament vector z feasible for (3), the optimal value for (3) is the smallest possible value of $\alpha(T)$ over all tournaments T on n vertices.

The IP formulation is a convenience that we used to identify the critical value $n = 8$. It is of little formal use. It does not provide mathematically rigorous bounds because the possibility of rounding errors. It is not sure to reduce the computational requirements, either. A simple branch and bound approach to solving the IP (3) will tend to enumerate all possible binary solutions, since the LP relaxation at any stage of the branch and bound routine has an optimal solution with all unknown components of z equal to $\frac{1}{2}$. Nonetheless it was useful in identifying the critical value of n by finding an optimal value 0.667 for $n = 7$ and a feasible solution with value 0.650 for $n = 8$.

Since every tournament on n vertices has a Hamiltonian path (this is easily proven by induction on n) it is also helpful to include the requirement

$$z_{uv} = 1, \quad \forall u, v \text{ with } v = u + 1. \tag{4}$$

Fixing the edges of a Hamiltonian path as above excludes those tournaments that do not have some Hamiltonian path that traverses the vertices in order from first to last. The requirement (4) can be included without loss of generality since at least one isomorphic copy of each tournament has a Hamiltonian path that traverses the vertices in order. The number of tournament graphs (including isomorphic copies) on n vertices is $2^{\binom{n}{2}}$. Including the Hamiltonian path requirement (4) reduces the number of tournaments to be examined to $2^{\binom{n-1}{2}}$, effectively increasing by one the value of n for which it is possible in practice to solve the problem.

3 Results

For $n = 7$ we solved the LP (1) to find $\alpha(T)$ for each of the $2^{\binom{n-1}{2}} = 32,768$ tournament graphs, subject to the Hamiltonian path requirement (4). For each one we converted its solution from rational to integer and verified its correctness with exact integer calculations. For every tournament with $n = 7$, a feasible solution to (1) was found and verified using the software package CPLEX with $\alpha \geq \frac{2}{3}$, guaranteeing that every tournament on 7 vertices is $\frac{2}{3}$-realizable. By implication, every tournament on 7 or fewer vertices is $\frac{2}{3}$-realizable, since a tournament on $n-1$ vertices $\{1, ..., n-1\}$ that is not α-realizable for some $\alpha \in (\frac{1}{2}, 1]$ can be extended to a tournament on n vertices that is not α-realizable by including a vertex n and edges mn for every $m \in \{1, ..., n-1\}$. It is easy to show that every tournament which has a cycle has a 3-cycle, and that any tournament with a 3-cycle is not α-realizable for any $\alpha > \frac{2}{3}$. Also, any acyclic tournament is trivially 1-realizable. Therefore, for any tournament T on 7 or fewer vertices, either T is acyclic and $\alpha(T) = 1$, or T has a cycle and $\alpha(T) = \frac{2}{3}$.

For $n = 8$ graphs were found for which $\alpha(T) = \frac{13}{20}$. The smallest tournament that is not $\frac{2}{3}$-realizable, then, is a tournament on 8 vertices. An example of a tournament $\widetilde{T}$ for which $\alpha(\widetilde{T}) = \frac{13}{20}$ is presented below as an adjacency matrix:

$$\widetilde{T} = \begin{bmatrix} * & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & * & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & * & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & * & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & * & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & * & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & * & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & * \end{bmatrix}$$

An optimal solution to the dual LP (2) for tournament $\widetilde{T}$ has $\lambda = \frac{13}{20}$ and the optimal dual variables μ_{uv} as given below in matrix form:

$$\mu = \frac{1}{20} \begin{bmatrix} * & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & * & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & * & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & * & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & * & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & * & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & * \end{bmatrix}. \quad (5)$$

The nonzero dual variables define a subgraph $H = (V, E)$ of tournament $\widetilde{T}$, where $uv \in E$ if and only if $\mu_{uv} > 0$. The adjacency matrix for H is 20μ where μ is given in 5. The corresponding subgraph is depicted in figure 1.

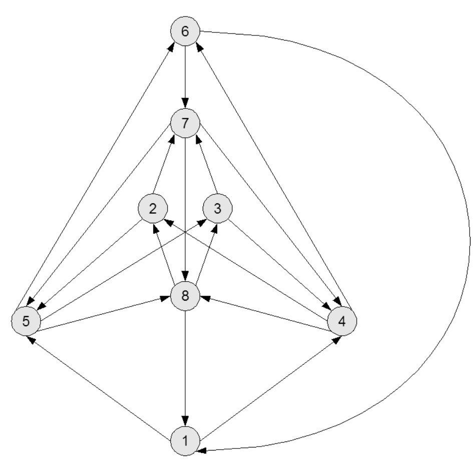

Figure 1: The subgraph H. Any tournament containing this subgraph has predictability $\leq \frac{13}{20}$

H has the property that no permutation of the vertices of V agrees with more than 13 of the 20 edges in H, so any tournament containing H as a subgraph will not be realizable for $\alpha > \frac{13}{20}$, and the family of tournaments on 8 vertices containing H as a subgraph is a family of smallest tournaments not realizable for $\alpha = \frac{2}{3}$.

Solving the linear program for one representative of each of the 6880 isomorphism classes for tournaments with $n = 8$ (7) reveals that every tournament on 8 vertices has $\alpha(T) \geq \frac{13}{20}$. For the case $n = 7$ we performed exact arithmetic calculations on the primal linear program (1), because our goal was to prove that every Prediction Game has value at least $\frac{2}{3}$. For the case $n = 8$, once we verified a single dual value of $\frac{13}{20}$, our goal was to prove that every Prediction Game has value at least $\frac{13}{20}$. Therefore, we verified these values with exact integer arithmetic on the primal linear program (1), scaled by a factor of 60 to convert rationals to integers. In terms of the Prediction Game, we verified by exact integer computation a mixed strategy for player I with value at least $\frac{13}{20}$ for each tournament. This completes the proof of the following result.

Theorem 3 *For all tournaments on 7 or fewer vertices, the predictability is either 1 or $\frac{2}{3}$. For tournaments on 8 vertices, the minimum predictability is $\frac{13}{20}$.*

Partial exploration of the set $\mathcal{T}_9$ gives several tournaments with $\alpha(T) < \frac{13}{20}$. The smallest observed value of $\alpha(T)$ for a tournament on 9 vertices was $\frac{64}{99}$. The optimal variables μ_{uv} for the dual LP (2) for a 9-vertex tournament with predictability $\frac{64}{99}$ are given below in matrix form:

$$\mu = \frac{1}{99}\begin{bmatrix} * & 2 & 4 & 3 & 0 & 0 & 0 & 0 & 2 \\ 0 & * & 0 & 4 & 4 & 4 & 0 & 0 & 0 \\ 0 & 0 & * & 2 & 2 & 2 & 6 & 0 & 0 \\ 0 & 0 & 0 & * & 0 & 0 & 3 & 4 & 2 \\ 4 & 0 & 0 & 0 & * & 0 & 3 & 2 & 0 \\ 2 & 0 & 0 & 0 & 0 & * & 3 & 0 & 4 \\ 3 & 6 & 0 & 0 & 0 & 0 & * & 3 & 3 \\ 2 & 2 & 4 & 0 & 0 & 3 & 0 & * & 0 \\ 0 & 2 & 4 & 0 & 3 & 0 & 0 & 2 & * \end{bmatrix}. \quad (6)$$

The nonzero dual variables again define a subgraph H' with the property that any tournament containing H' is not realizable for $\alpha > \frac{64}{99}$.

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