

## Why this is important

This paper addresses the problem of optimal asset replacement under ongoing technological progress. Because the technological progress is ongoing, asset costs decline for challengers to later incumbent assets as well as challengers to the first asset. This greatly complicates replacement decisions.

This paper shows that:

- the timing of each replacement affects the possible replacement schedule for all later assets, and thus the discounted stream of costs;
- allowing a dynamic replacement policy for later assets can substantially affect the first replacement decision; and
- variable service lives can significantly reduce the discounted stream of costs of all future assets.

The model is deterministic, with continuous technological progress, but it can be interpreted as representing incremental technological progress with uncertain timing. In this context, the large impact of considering a series of future assets at the time of each decision is perhaps especially relevant.

We use real data for an automobile to illustrate the sub-optimality of commonly recommended methods for timing replacements. Large deviations from optimality occur for observed parameter values and are not attributable to unrealistic values. The automobile data are also consistent with the geometric cost structure.

# Abstract

This paper evaluates the cost savings achievable in the infinite horizon single asset replacement problem by considering technological progress affecting assets available after multiple future replacements and allowing variable service lives. Using a simple geometric model to specify asset costs, constant service lives are shown to be sub-optimal. It is shown by example that common replacement decision methods yield higher service lives for the first asset and substantially different discounted total costs for the series of assets. This effect is illustrated numerically using automobile cost data.

## 1 Optimal Asset Replacement Under Ongoing Technological Progress

Technological progress that continues indefinitely, reducing costs for challengers to future assets as well as to the current defender, greatly complicates replacement decisions. The decisions are further complicated when capital costs and operation and maintenance (O&M) costs are decreasing at different rates. This paper shows that:

- dynamic policies can significantly reduce the discounted stream of costs of all future assets, and a stationary policy is necessarily suboptimal when capital and O&M costs are changing at different rates;
- considering the effect of future assets can substantially affect the first replacement decision and resulting total discounted costs; and
- in particular, replacing an asset when the available challenger has lower annualized costs can cause early replacement and large deviations from optimality.

Various justifications have been used to simplify the costs of later assets for easier optimization. Some authors have assumed technological progress ceases after a finite number of replacements (Bethuyne 1998; Chand and Sethi 1982), or even immediately (Terborgh 1949; Thuesen and Fabrycky 2001). Others have restricted successive challengers to a constant service life (Grinyer 1973). Otherwise, a recursive numerical solution is recommended (Thuesen and Fabrycky 2001; Bethuyne 1998; Oakford et al. 1984).

In this paper, we present a simple model for asset costs that is a special case of the classical replacement model with deterministic and continuous technological progress. As in the classical model, technological progress is reflected exclusively in reduced capital and initial O&M costs per unit of functionality; our model is unusual in that it allows capital and O&M costs to change at different rates. The model is deterministic, with continuous technological progress, but it can be interpreted as representing technological progress with uncertain timing, with similar implications.

We examine the behavior of the optimal replacement policy, focusing on the current decision—the timing of the first replacement—as identified by dynamic programming. We also consider other methods that are commonly recommended for timing replacements. The first method minimizes the annualized cost of the current asset, and the second method calls for replacing the current asset when the minimum annualized cost of the new asset is lower than the marginal cost of keeping the existing asset. A third method, whose easy calculation depends on the geometric structure of this model, uses the best stationary policy.

Finally, we give numerical examples using parameters fitted to data for automobile costs to evaluate the performance of these decision methods. The examples illustrate the sometimes counterintuitive behavior of the optimal policy and the relative performance of alternative methods for timing the first replacement. Large deviations from optimality occur for observed values, so they are not artifacts of unrealistic parameter values. The automobile data are consistent with the geometric cost structure.

## 2 Geometric Costs Model

In our model, new-asset purchase prices, salvage values, and O&M costs for new and used assets change in each period. All cost changes are geometric: for each cost type, the cost in each period is a constant multiple of the cost for the previous period. Technological progress is reflected exclusively in new-asset purchase and O&M costs for a given level of functionality. Salvage values are a geometric decreasing function of the asset's original purchase price, and O&M costs increase geometrically for each asset as it ages.

The geometric structure is convenient, but it is consistent with most other models, except that here the dependence of O&M costs on the age of the asset is also geometric, whereas

others who specified a functional relationship (Terborgh 1949; Grinyer 1973; Oakford et al. 1984) used a linear function. Grinyer (1973) compared linear and geometric (negative exponential) forms for technological progress and recommended the geometric form.

A discrete-time formulation is used because the solution method is discrete. A continuous-time formulation better illustrates that this model is a specific case of the Bethuynne model, except that the rates of technological progress for capital and O&M costs can differ. This model is also very nearly a specific case of the exponential version of Grinyer's exponential cost model, as discussed later in this section.

## Notation

- $P$  = purchase price of a new asset at time 0
- $a$  = annual multiplier for purchase price
- $c$  = annual multiplier for salvage values ( $< a$ )<sup>1</sup>
- $b$  = multiplier for end-of-year-1 salvage value ( $\leq c$ )
- $A$  = new-asset O&M costs for asset purchased at time 0
- $q$  = annual multiplier for new-asset O&M costs
- $p$  = annual multiplier for O&M costs for a given asset as it ages ( $> q$ )
- $M$  = maximum physical service life of each asset (assumed constant)
- $k$  = index of asset
- $N_k$  = length in years of the  $k^{th}$  asset's service life,  $N_k \in 1, \dots, M$
- $T_k$  = year that  $k^{th}$  asset is purchased
  - $= \sum_{j=1}^{k-1} N_j$ , and  $T_1 = 0$
- $d$  = annual effective discount rate or minimum attractive rate of return<sup>2</sup>

To simplify present value expressions, the following are also defined:

$$x = \frac{a}{1+d} \quad y = \frac{q}{1+d} \quad w = \frac{c}{1+d} \quad z = \frac{p}{1+d}$$

The model requires  $x, y, w < 1$ . Nominal costs can increase ( $a, q > 1$  is allowed), but the rate of increase must be lower than the discount rate. However, it is usually assumed that technological progress reduces costs, at least for a given functionality. Even salvage values are allowed to increase ( $c > 1$ ), but not faster than the discount rate. On the other hand,

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<sup>1</sup>If  $c > a$ , eventually an old asset can be sold for more than the cost of a new asset.

<sup>2</sup>The discount rate is not adjusted for inflation, as all the costs and multipliers are inflation-adjusted, reflecting constant dollar values.

either  $z > 1$  or  $z < 1$  is allowed, though  $p > q \Rightarrow z > w$  is required. Typically  $z > 1$ , indicating that O&M costs grow at a rate higher than the discount rate as the asset ages.

The following relationships describe the costs, measured in constant dollars:

$$\begin{aligned}
\text{purchase price of } k^{\text{th}} \text{ asset} &= Pa^{T_k} && \text{in year } T_k \text{ (year of purchase)} \\
\text{Salvage value for } k^{\text{th}} \text{ asset} &= Pa^{T_k}bc^{N_k-1} && \text{in year } T_k + N_k, N_k \geq 1 \\
\text{First year O\&M costs of } k^{\text{th}} \text{ asset} &= Aq^{T_k} && \text{at the end of year } T_k \\
\text{O\&M costs in } n^{\text{th}} \text{ year of } k^{\text{th}} \text{ asset} &= Aq^{T_k}p^{n-1} && \text{in year } T_k + n, 1 \leq n \leq N_k
\end{aligned}$$

Total costs for the  $k^{\text{th}}$  asset consist of the capital costs—purchase price minus the salvage value recovered after  $N_k$  periods—plus the O&M costs for  $N_k$  periods of the service life. For the  $k^{\text{th}}$  asset, the discounted value of the asset's total lifetime costs is:

$$\begin{aligned}
\text{total discounted costs} &= \frac{\overbrace{Pa^{T_k}}^{\text{purchase}}}{(1+d)^{T_k}} - \frac{\overbrace{Pa^{T_k}bc^{N_k-1}}^{\text{salvage}}}{(1+d)^{T_k+N_k}} \\
&+ \overbrace{\frac{A}{(1+d)^{T_k+1}}q^{T_k} + \frac{A}{(1+d)^{T_k+2}}q^{T_k}p + \dots + \frac{A}{(1+d)^{T_k+N_k}}q^{T_k}p^{N_k-1}}^{\text{O\&M costs}} \\
&= \underbrace{Px^{T_k} \left(1 - \frac{b}{c}w^{N_k}\right)}_{\text{capital costs}} + \underbrace{\frac{A}{1+d}y^{T_k} \frac{z^{N_k} - 1}{z - 1}}_{\text{O\&M costs}}
\end{aligned}$$

## Objective

A policy,  $\pi$ , consists of the service lives of all assets, i.e.  $\pi = (N_1, N_2, N_3, \dots)$ . The decision maker's objective is to minimize the discounted value, evaluated at her real discount rate,  $d$ , of the sum of total costs for an infinite number of successive assets,  $k = 1, 2, \dots$ :

$$\begin{aligned}
\text{objective: } & \min_{\pi=(N_1, N_2, N_3, \dots)} u(\pi) && (1) \\
\text{where } u(\pi) &= \sum_{k=1}^{\infty} \left\{ Px^{T_k} \left(1 - \frac{b}{c}w^{N_k}\right) + \frac{A}{1+d}y^{T_k} \frac{z^{N_k} - 1}{z - 1} \right\}
\end{aligned}$$

## Comparison with Grinyer’s Model

The Grinyer (1973) exponential cost model is very similar to our geometric model in both formulation and implications. Capital and O&M costs for available challengers decline as a negative exponential function of time and are allowed to change independently. This technological progress continues indefinitely at a constant exponential rate. Although he did not consider them, dynamic policies can yield significantly lower total discounted costs under Grinyer’s model as well as ours.

There are a few differences between Grinyer’s model and the geometric model presented here. Grinyer’s model allows no change in the purchase price for challengers available in the future, i.e. in Grinyer’s model,  $a$  is restricted to 1. However, by manipulating the discount rate and other parameters, problems with  $a \neq 1$  can be expressed in Grinyer’s form. Grinyer models the change in O&M costs over the life of a given asset as a linear function of the asset’s age. With our notation, the rate of O&M costs in Grinyer’s model at time  $n$  of  $k^{th}$  asset =  $q^{T_k}(A + pn)$  in period  $T_k + n$ .

Grinyer explicitly models the relationship between the rate of decrease of salvage values and the rate of technological progress affecting O&M costs but not purchase costs. In our model, this relationship is represented through the value of the parameter  $c$ , which is constrained to be less than  $a$  (but not necessarily less than  $q$ ).<sup>3</sup> As discussed in Section 6, the first replacement decision is sensitive to the choice of the value  $c$ . Its value must reflect the fact that faster technological progress—either in O&M or in purchase prices—would be expected to lead to quickly decreasing salvage values, because the more desirable available new assets are, the lower the market value for used assets.

Grinyer allows the initial capital cost, interpreted as purchase price or salvage value of the initial asset at time zero, of the incumbent asset (defender) to be different from the baseline cost of the challenger available at time zero, and likewise allows the baseline O&M costs to be different for the time zero defender and available challengers. This models an aged defender at time zero. The notation given here for our model does not include this possibility, but this difference can easily be included in a dynamic programming implementation.

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<sup>3</sup>Specifically, our parameter  $c = e^{-\beta v + k}$  in Grinyer’s notation, where Grinyer’s  $v = -\ln(q)$  in our notation, and thus  $c$  depends on the rate of technological progress in O&M. Grinyer concluded that his model’s sensitivity to  $\beta$  was small enough to make it unimportant.

Perhaps most important, Grinyer restricts the service lives for all assets following the incumbent defender to be constant. However, using Grinyer’s parameter sets, a stationary service life for later assets is not generally optimal—in fact, this constraint can lead to large increases in total cost. In addition, the optimal first service life can be different from the best first service life with the additional constraint. Using the values  $A = A_c = \$1,000$ ,  $B = B_c = \$300$ ,  $r = 10\%$ ,  $k = 0.25$ ,  $v = 0.196$ ,  $Y = 50$ ,  $\beta = 0$  and  $M = 30$  in Grinyer’s notation and linearly increasing O&M costs, the optimal policy is  $(1,1,1,1,3,7,30,30\dots)$ , resulting in a total discounted cost of \$3,700, while the best policy under the constraint that all service lives after the first must be constant is  $(3,6,6,\dots)$ , which has a total discounted cost of \$4,023, or more than 8% higher than the optimal cost.

### 3 Optimal Policy

The service life that minimizes total costs represents a compromise between long service lives that reduce annualized capital costs, and frequent replacement that reduces O&M costs. For a stationary policy with a service life of  $N$  periods, applied to an infinite horizon, the capital costs portion ( $X(N)$  defined in (2)) is a decreasing function of  $N$ , hence the longest possible service life ( $N = M$ ) minimizes capital costs. The O&M costs portion ( $Y(N)$  defined in (3)) is an increasing function of  $N$ , and is minimized by the shortest allowable service life ( $N = 1$ ). Discounted costs for infinite series of assets replaced at age  $N$ :

$$\text{capital costs} = X(N) = P \left( 1 - \frac{b}{c} w^N \right) \frac{1}{1 - x^N} \quad (2)$$

$$\text{O\&M costs} = Y(N) = \frac{A}{1 + d} \cdot \frac{z^N - 1}{z - 1} \cdot \frac{1}{1 - y^N} \quad (3)$$

To show that  $X(N)$  ( $Y(N)$ ) is decreasing (increasing) in  $N$ , we will employ the following lemma.

**Lemma 1** *The function  $\frac{x^N \ln(x)}{1 - x^N}$  is strictly decreasing in  $x$ ,  $\forall x > 0$ ,  $N \geq 1$ .*

**Proof**

$$\frac{d}{dx} \left( \frac{x^N \ln(x)}{1 - x^N} \right) = \frac{x^N}{x(1 - x^N)^2} (N \ln(x) + 1 - x^N)$$

The expression  $\frac{x^N}{x(1-x^N)^2} > 0 \quad \forall x > 0$ . The expression  $N \ln(x) + 1 - x^N$  is decreasing in  $N \forall x > 0$  and it is negative when evaluated at  $N = 1, \forall x > 0$ , so  $N \ln(x) + 1 - x^N < 0 \quad \forall x > 0, N \geq 1$ .

Thus  $\frac{d}{dx} \left( \frac{x^N \ln(x)}{1-x^N} \right) < 0 \quad \forall x > 0, N \geq 1$ . ■

**Proposition 1**  $\frac{dX(N)}{dN} < 0 \quad \forall N \geq 1, \quad 0 < w < x < 1, \quad 0 < b < c$ .

**Proof**

$$\frac{dX}{dN} = P \left( 1 - \frac{b}{c} w^N \right) \frac{x^N \ln(x)}{(1-x^N)^2} - \frac{1}{1-x^N} P \frac{b}{c} w^N \ln(w) \quad (4)$$

By Lemma 1, and because  $0 < w < x < 1$  and  $0 < b < c < 1$ , we have

$$\frac{x^N \ln(x)}{(1-x^N)^2} < \frac{w^N \ln(w)}{1-w^N} \leq \frac{\frac{b}{c} w^N \ln(w)}{1-\frac{b}{c} w^N} < 0. \quad (5)$$

Rearranging (5) and multiplying through by the positive terms  $\frac{P}{1-x^N}$  and  $1 - \frac{b}{c} w^N$  yields  $\frac{dX}{dN} = P \left( 1 - \frac{b}{c} w^N \right) \frac{x^N \ln(x)}{(1-x^N)^2} - \frac{1}{1-x^N} P \frac{b}{c} w^N \ln(w) < 0$ . ■

**Proposition 2**  $\frac{dY(N)}{dN} > 0 \quad \forall N \geq 1, \quad 0 < y < 1, \quad y < z, \quad A, d, y > 0$ .

**Proof**

$$\frac{dY}{dN} = \frac{A}{1+d} \cdot \frac{1}{z-1} \cdot \left[ \frac{\ln(z)z^N}{1-y^N} + \frac{\ln(y)y^N(z^N-1)}{(1-y^N)^2} \right] \quad (6)$$

By Lemma 1, and because  $z > y > 0$ ,

$$\frac{\ln(z)z^N}{1-z^N} < \frac{\ln(y)y^N}{1-y^N}. \quad (7)$$

Because  $A, d, y > 0$ , and  $y < 1$  are required by the model, (7) can be rewritten as

$$\frac{A}{1+d} \cdot \frac{\ln(z)z^N}{z-1} > \frac{A}{1+d} \cdot \frac{\ln(y)y^N(1-z^N)}{(z-1)(1-y^N)}$$

and rearranged to show that  $\frac{dY}{dN} = \frac{A}{1+d} \cdot \frac{1}{z-1} \cdot \left[ \frac{\ln(z)z^N}{1-y^N} + \frac{\ln(y)y^N(z^N-1)}{(1-y^N)^2} \right] > 0$ . ■

Thus the best stationary service life is a compromise between the two extremes, and consistent rules for behavior of the optimal policy are difficult to come by. A few general results are given below.



### 3.1 Best Stationary Policy

For a stationary policy,  $\pi_F$ ,  $N_k = N$  is constant  $\forall k$  and therefore  $T_k = N(k-1)$ . The objective (1) can be rewritten in terms of  $N$  so

$$\begin{aligned} v(N) = u(\pi_F) &= \sum_{k=1}^{\infty} \left[ x^{N(k-1)} P \left( 1 - \frac{b}{c} w^N \right) + y^{N(k-1)} \frac{A}{1+d} \cdot \frac{z^N - 1}{z-1} \right] \\ &= P \left( 1 - \frac{b}{c} w^N \right) \frac{1}{1-x^N} + \frac{A}{1+d} \cdot \frac{z^N - 1}{z-1} \cdot \frac{1}{1-y^N} = X(N) + Y(N) \end{aligned} \quad (8)$$

The best possible stationary policy  $\pi_F^* = (N_1 = N^*, N_2 = N^*, \dots)$ , where  $N^*$  minimizes  $v(N)$ . When capital and O&M costs are decreasing at the same rate, the best stationary policy is optimal over all policies.

**Proposition 3** *When  $a = q$ ,  $N_k^* = N^* \quad \forall k = 1, 2, \dots$*

**Proof** Let  $N_1^*$  be the first asset's service life in an optimal dynamic policy  $\pi_V^*$  minimizing the objective in (1). The remainder of the policy  $(N_2^*, N_3^*, \dots)$  minimizes the expression in (9), the total discounted cost for assets  $k = 2, 3, \dots$

$$(N_2^*, N_3^*, \dots) = \min_{N_2^*, N_3^*, \dots} \sum_{k=2}^{\infty} \left[ P x^{T_2} x^{T_k - T_2} \left( 1 - \frac{b}{c} w^{N_k - 1} \right) + \frac{A}{1+d} y^{T_2} y^{T_k - T_2} \frac{z^{N_k - 1} - 1}{z-1} \right] \quad (9)$$

The constant  $x^{T_2} = y^{T_2}$  can be factored out, yielding the same minimization problem as given in the original objective in (1). Therefore  $N_2^* = N_1^*$  must be optimal, and, by induction on  $k$ ,  $N_k^* = N_1^*$  is optimal. ■

This result also holds for Bethuyne (1998)'s model, though he does not show it. In Bethuyne's model, costs are a continuous exponential function of the asset's date of purchase. Bethuyne's expression for the total cost of an infinite number of successive assets is

$$\sum_{k=1}^{\infty} e^{-(t+g)T_k} \left[ \int_0^{N_k} e^{-dn} k(n, 0) dn \right]$$

where  $k(n, t) = e^{-gt} k(n, 0)$  denotes the rate of accrual of discounted costs of an asset acquired at time  $t$  and aged  $n$ . The parameter  $g$  indicates the rate of technological progress for both capital and O&M costs; in our model, Bethuyne's parameter  $g = -\ln(a) = -\ln(q)$ , implying

$a = q$ , so capital and O&M costs change at the same rate. A stationary policy is optimal, by the same argument presented above for our model.

### 3.2 Optimality of Dynamic Policy

Most of the research on the single-asset replacement problem with continuous but deterministic technological progress was published when computing power was far less available than it is now. As discussed earlier, the authors generally restricted service lives of all assets following the current defender to be constant, although this is not necessarily optimal, as some noted (Smith 1966), and as illustrated in Sections 2 and 5.2.

Unless capital and O&M costs are changing at exactly the same rate, i.e.  $a = q$ , a fixed service life policy is not necessarily optimal. If  $a \neq q$ , a fixed service life is optimal for an infinite horizon only if either O&M costs or purchase costs dominates the other for the first asset i.e.  $N^* = 1$  or  $M$ . Otherwise, it is possible to find a policy  $\pi_V$  whose service life is increasing (if O&M costs are declining faster) or decreasing (if capital costs are declining faster) that outperforms the best stationary policy.

**Theorem 1** *When both  $a > q$ , and  $N^* \neq M$ , or when both  $a < q$  and  $N^* \neq 1$ ,  $\exists \pi_V = (N_1, N_2, \dots)$  with some pair  $i \neq j$  such that  $N_i \neq N_j$ , and  $u(\pi_V) < u(\pi_F^*)$ .*

**Proof** Remember that the model requires  $0 < x, y, w < 1$ . Consider a set of policies  $\pi_1, \pi_2, \dots, \pi_J$ , denoted as follows:

$\pi_j = (K_1, K_1, \dots, K_1, K_2, K_2, \dots, K_{j-1}, K_j, K_j, \dots)$   
 $K_j =$  the  $j^{th}$  asset service life ( $K_j$  is not usually the length of the  $j^{th}$  asset's life)  
 $k_j =$  the index ( $i$ ) of the first<sup>4</sup> asset having service life  $K_j$   
 $C =$  the set of allowable service lives,

$$C = \begin{cases} \{1, 2, \dots, M\} & \text{for discrete service lives} \\ [1, M] & \text{for continuous service lives} \end{cases}$$

Each policy  $\pi_j$  starts with at least one asset life  $K_1 = N^*$  and changes service life  $j - 1$  times. Each time the service life that minimizes the total discounted cost of a policy that is stationary for the remaining assets changes, the service life is adjusted. The total discounted cost for the remaining assets after the  $i^{th}$  replacement of policy  $\pi_j$  is equal to  $f(T_i^j, N)$ , where

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<sup>4</sup>It will be shown in Corollary 1 that  $K_j$  are strictly monotonic, so  $K_j \neq K_i$ ,  $i \neq j$ .

$f(T, N)$  is defined below.<sup>5</sup>

$$f(T, N) = x^T P \left( 1 - \frac{b}{c} w^N \right) \frac{1}{1 - x^N} + y^T \frac{A}{1 + d} \cdot \frac{z^N - 1}{z - 1} \cdot \frac{1}{1 - y^N} \quad (10)$$

$$T_1^j = 0, \quad T_m^j = \sum_{i=1}^{m-1} N_i^j, \quad m = 2, 3, \dots$$

$$k_1 = 1, \quad k_j = \arg \min_{k \in \mathbb{N}, k \geq k_{j-1}} \arg \min_{N \in C} \{ f(T_k^{j-1}, N) \neq K_{j-1} \}, \quad j = 2, 3, \dots, J$$

$$K_j = \arg \min_N f(T_{k_j}^{j-1}, N) \quad (11)$$

The above definitions imply that a policy  $\pi_j$  exists if policies  $\pi_1$  through  $\pi_{j-1}$  exist and  $k_j < \infty$ . They also imply:

$$K_1 = N^*, \pi_1 = \pi_F^* = (K_1, K_1, \dots)$$

$$\pi_j = (N_1^j, N_2^j, \dots) \quad N_i^j = \begin{cases} N_i^{j-1}, & i < k_j \\ K_j, & i \geq k_j \end{cases}$$

We will show the following properties of the policies  $\pi_1, \pi_2, \dots, \pi_J$ :

$$\forall j > 1, \text{ if } \pi_j \text{ and } \pi_{j-1} \text{ are defined, } u(\pi_j) < u(\pi_{j-1}) \quad (12)$$

$$a < q \text{ and } K_1 \neq 1 \Rightarrow \pi_2 \text{ is defined} \quad (13)$$

$$a > q \text{ and } K_1 \neq M \Rightarrow \pi_2 \text{ is defined} \quad (14)$$

First, we will show (12). Consider any two adjacent policies  $\pi_j$  and  $\pi_{j-1}$ ,  $j \geq 2$ . By definition,

$$N_i^{j-1} = \begin{cases} N_i^j & \forall i < k_j \\ K_j = \arg \min_N f(T_{k_j}, N) & \end{cases} \quad \text{and } N_i^j = K_j \neq N_i^{j-1} = K_{j-1}, \quad \forall i \geq k_j.$$

As defined,

$$u(\pi_j) - f(T_{k_j}, K_j) = u(\pi_{j-1}) - f(T_{k_j}, K_{j-1}) = \sum_{i=1}^{k_j} P x^{T_i^{j-1}} \left( 1 - \frac{b}{c} w^{N_i^{j-1}} \right) + \frac{A}{1+d} y^{T_i^{j-1}} \frac{z^{N_i^{j-1}} - 1}{z-1}.$$

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<sup>5</sup>Note that  $f(0, N) = v(N)$ , defined in (8).

But  $f(T_{k_j}, K_j) < f(T_{k_j}, K_{j-1}) \Rightarrow u(\pi_j) < u(\pi_{j-1})$ . Therefore,  $u(\pi_1) > u(\pi_2) > \dots > u(\pi_J)$ . Since  $\pi_1 = \pi_F^*$ , if  $\pi_2$  is defined, it is a dynamic policy that outperforms the best possible stationary policy, i.e.  $u(\pi_2) < u(\pi_1) = u(\pi_F^*)$ .

It remains to be shown under what conditions the dynamic policy  $\pi_2$  exists or equivalently, under what conditions  $k_2 < \infty$ . By definition,  $k_2 < \infty$  when for some  $i > 1$ ,  $\exists T_i^1$  such that  $\operatorname{argmin}_{N \in C} f(T_i^1, N) \neq N^*$ . We will show that this is true when  $a < q$  and  $N^* \neq 1$ , and when  $a > q$  and  $N^* \neq M$ .

Because  $\frac{dX}{dN}$ , defined in (4), and  $\frac{dY}{dN}$ , defined in (6), are continuous functions on  $C$ , and because  $C$  is compact,<sup>6</sup> each attains both a finite minimum and a finite maximum in  $C$ . Therefore we can define:

$$\begin{aligned} \overline{X}' &= \max_{N \in C} \left\{ \frac{dX}{dN} \right\} < \infty, & \underline{X}' &= \min_{N \in C} \left\{ \frac{dX}{dN} \right\} > -\infty \\ \overline{Y}' &= \max_{N \in C} \left\{ \frac{dY}{dN} \right\} < \infty, & \underline{Y}' &= \min_{N \in C} \left\{ \frac{dY}{dN} \right\} > -\infty \end{aligned}$$

By Proposition 1,  $\overline{X}', \underline{X}' < 0$  and by Proposition 2,  $\overline{Y}', \underline{Y}' > 0$ . As defined in (10)  $f(T, N) = x^T X(N) + y^T Y(N)$ . Therefore,  $\frac{d}{dN} f(T, N)$  is finite and bounded as follows:

$$\frac{d}{dN} f(T, N) = x^T \frac{dX}{dN} + y^T \frac{dY}{dN} \begin{cases} \leq x^T \overline{X}' + y^T \overline{Y}' \\ \geq x^T \underline{X}' + y^T \underline{Y}' \end{cases} \quad (15)$$

Let  $\delta = \frac{x}{y}$ , so  $a < q \Rightarrow 0 < \delta < 1$ , and therefore  $\exists T^* > 0, T^* < \infty$  such that

$$\begin{aligned} \forall T > T^*, \delta^T &< \left| \frac{\underline{Y}'}{\underline{X}'} \right| \\ &\Rightarrow 0 < -x^T \underline{X}' = x^T |\underline{X}'| < y^T \underline{Y}' \Rightarrow x^T \underline{X}' + y^T \underline{Y}' > 0 \\ &\stackrel{(15)}{\Rightarrow} \frac{d}{dN} f(T, N) > 0 \quad \forall N \in C \\ &\Rightarrow \operatorname{argmin}_{N \in C} f(T, N) = \min_{N \in C} \{N\} = 1 \end{aligned} \quad (16)$$

If  $a < q$ , by (16) all service lives chosen after time  $T^*$  will equal 1; therefore, if  $N^* \neq$

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<sup>6</sup>If  $C = \{1, 2, \dots, M\}$ ,  $C$  is compact because it is finite, and if  $C = [1, M]$ ,  $C$  is compact because it is a closed and bounded interval on the real line.

1,  $k_2 < \infty$  and  $\pi_2$  exists, and (13) is shown. Similarly,  $a > q \Rightarrow \delta > 1$ , and therefore  $\exists T^* > 0, T^* < \infty$  such that

$$\begin{aligned}
\forall T > T^*, \quad \delta^T > \left| \frac{\overline{Y}'}{\overline{X}'} \right| \\
&\Rightarrow -x^T \overline{X}' = x^T |\overline{X}'| > y^T \overline{Y}' > 0 \Rightarrow x^T \overline{X}' + y^T \overline{Y}' < 0 \\
&\stackrel{(15)}{\Rightarrow} \frac{d}{dN} f(T, N) < 0 \quad \forall N \in C \\
&\Rightarrow \arg \min_{N \in C} f(T, N) = \max_{N \in C} \{N\} = M.
\end{aligned} \tag{17}$$

If  $a > q$ , by (17), all service lives chosen after time  $T^*$  will equal  $M > N^*$ , and if  $N^* \neq M$ ,  $k_2 < \infty$  and  $\pi_2$  exists, and (14) is shown. ■

The proof of Theorem 1 suggests an algorithm for generating a dynamic policy that is superior to the best stationary policy. However, this policy is not necessarily optimal.

**Corollary 1** *The policies  $\pi_2, \pi_3, \dots, \pi_J$  defined in Theorem 1 are strictly monotonic, and specifically:*

$$a < q \Rightarrow K_j < K_{j-1}, \forall j > 1, \text{ and} \tag{18}$$

$$a > q \Rightarrow K_j > K_{j-1}, \forall j > 1. \tag{19}$$

**Proof** Let  $g(T, N) = \frac{f(T, N)}{y^T}$ , which can be rewritten as

$$g(T, N) = \delta^T X(N) + Y(N) = f(0, N) - (1 - \delta^T) X(N).$$

Since  $y^T$  is constant with respect to  $N \forall T$ ,  $\arg \min_N g(T, N) = \arg \min_N f(T, N)$ . If  $k_j < \infty$ , then by the definition of  $K_j$ ,

$$\begin{aligned}
K_j &= \arg \min_N f(T_{k_j}^j, N) = \arg \min_N g(T_{k_j}^j, N) \text{ and} \\
K_{j-1} &= \arg \min_N f(T_{k_{j-1}}^j, N) = \arg \min_N g(T_{k_{j-1}}^j, N).
\end{aligned}$$

As defined,  $f(T, N), g(T, N) > 0 \quad \forall N, T$ . Therefore,

$$g(T_{k_j}^j, K_j) - g(T_{k_{j-1}}^j, K_j) \leq g(T_{k_j}^j, K_j) - g(T_{k_{j-1}}^j, K_j),$$

Plugging in the formula for  $g(T, N)$ , this can be rewritten as

$$\left( \delta^{T_{k_j}^j} - \delta^{T_{k_{j-1}}^j} \right) X(K_j) \leq \left( \delta^{T_{k_j}^j} - \delta^{T_{k_{j-1}}^j} \right) X(K_{j-1}).$$

Also, by the definition of  $T_k^j$ ,  $T_{k_j}^j > T_{k_{j-1}}^j \geq 0$ . Therefore,

$$\begin{aligned} a < q &\Leftrightarrow \delta < 1 \Rightarrow \left( \delta^{T_{k_j}^j} - \delta^{T_{k_{j-1}}^j} \right) < 0 \Rightarrow X(K_j) \geq X(K_{j-1}) \\ &\stackrel{\text{Proposition 1}}{\Rightarrow} K_j \leq K_{j-1} \end{aligned} \quad (20)$$

$$\begin{aligned} a > q &\Leftrightarrow \delta > 1 \Rightarrow \left( \delta^{T_{k_j}^j} - \delta^{T_{k_{j-1}}^j} \right) > 0 \Rightarrow X(K_j) < X(K_{j-1}) \\ &\stackrel{\text{Proposition 1}}{\Rightarrow} K_j \geq K_{j-1} \end{aligned} \quad (21)$$

The definition of  $K_j$  in (11) is required to show strict monotonicity—for  $j > 1$ ,  $K_j$  is defined only if  $\exists k > k_{j-1}$  such that  $\arg \min_{N \in C} f(T_k^j, N) \neq K_{j-1}$  and therefore if  $K_j$  is defined,  $K_j \neq K_{j-1}$ , so (20)  $\Rightarrow$  (18) and (21)  $\Rightarrow$  (19). ■

We have not shown that the optimal policy will be monotonic as described here. In fact, for finite horizons, the inverse monotonicity can be observed in the optimal policy. For example, for parameters  $P = \$15,350$ ,  $A = 140$ ,  $a = 1.05$ ,  $q = 1.0$ ,  $b = 0.83$ ,  $c = 0.86$ ,  $p = 1.55$ , and  $d = 15\%$ , with a horizon of 10 periods, the optimal service lives would be intuitively expected to increase as the relative contribution of O&M costs to the total decreased for the newer assets (because  $q < a$ ). However, the optimal service lives are  $N_1^* = 8$ ,  $N_2^* = 2$ , or, if non-integer service lives are allowed,  $N_1^* = 7.9$ ,  $N_2^* = 2.1$  for the discrete formulation and  $N_1^* = 8.4$ ,  $N_2^* = 1.6$  for the continuous formulation.

Corollary 1 also implies that, given a finite set of allowable service lives, the optimal policy will be stationary after a finite number of replacements. This makes it possible to verify the optimality (among policies using the finite set of service lives) of a policy for the infinite horizon problem, as discussed in more detail in Section 4.4.

### 3.3 Uncertain Technological Progress

Our model can also be interpreted to represent uncertain technological progress, where new-asset purchase and O&M costs decrease incrementally to known levels, but at times that are random, as in Hopp and Nair (1991). Specifically, each technological progress event decreases costs by a constant, known factor. The time elapsed between events affecting new-asset costs is exponentially distributed,<sup>7</sup> as is the time elapsed between events affecting O&M costs, with the two types of events mutually independent and occurring at different rates.<sup>8</sup> All other values are non-random.

The purchase price of a new asset available at time  $T$  is  $P\alpha_a^{Z_a(T)}$  and, based on information available at time zero, the expected purchase price for the asset available at time  $T$  is  $E\left[P\alpha_a^{Z_a(T)}\right] = Pe^{\lambda_a T(1-\alpha_a)} = a^T$ . Similarly the expected first-period O&M costs for the asset available at time  $T$  are  $Ae^{\lambda_q T(1-\alpha_q)} = q^T$ . The expected total costs, based on time zero information can therefore be written as in (1), with the new parameters defined below.

#### Notation

$Z_a(T)$  = the number of technological progress events affecting purchase price and occurring before time  $T$ ,  $Z_a(T) \sim \text{poisson}(\lambda_a T)$ ,  $Z_a(T) \in 0, 1, 2, \dots$

$Z_q(T)$  = the number of technological progress events affecting initial O&M costs and occurring before time  $T$ ,  $Z_q(T) \sim \text{poisson}(\lambda_q T)$ ,  $Z_q(T) \in 0, 1, 2, \dots$

$\alpha_a$  = purchase price reduction factor

$\alpha_q$  = new-asset O&M cost reduction factor

$\lambda_a$  = rate of occurrence of events affecting purchase price

$\lambda_q$  = rate of occurrence of events affecting first-year O&M costs

Hence,  $a = e^{\lambda_a(1-\alpha_a)}$  and  $q = e^{\lambda_q(1-\alpha_q)}$ . The optimal policy for the equivalent deterministic problem will also minimize the expected total costs in the uncertain model evaluated at time zero. The objective can be further improved if the decision-maker re-optimizes with respect to additional information after each event. Each time a technological progress event occurs, the expected purchase and first-period O&M costs for the currently available challenger and for all future challengers decrease immediately. If we assume that the decision-maker

<sup>7</sup>This is similar to Goldstein, Ladany, and Mehrez (1998) who use a geometric hazard function.

<sup>8</sup>In addition to Hopp and Nair (1991) and Goldstein, Ladany, and Mehrez (1998), a number of other recent papers on replacement focus on uncertain technological progress, though most, including Rajagopalan, Singh, and Morton (1998), do not specify the structure of the uncertain technological progress.

reassesses the replacement decision based on this new information, she will reduce the total stream of costs below that for the optimal policy for the deterministic case.

## 4 Replacement Decision Methods

Finding the optimal policy over an infinite horizon may be impossible, even for the geometric cost model. We evaluate four methods for selecting a replacement policy.

### 4.1 Economic Life, ( $EL_k$ )

The Economic Life (EL) method is given in most Engineering Economy textbooks, including (Newnan et al. 2000) and Thuesen and Fabrycky (2001), though they assume that technological progress stops at the time of the first replacement decision if not earlier.

This method evaluates the costs for the incumbent asset in isolation, and in no way takes into account changing capital and O&M costs of newer models. Instead, the EL service life,  $EL_k$ , minimizes the annualized (or periodic) total cost of the  $k^{th}$  asset, with the purchase date of the asset,  $T_k$  determined by the lengths of previous asset lives.<sup>9</sup>  $EL_k$  is the value at which the total cost for asset  $k$  expressed in (22) is minimized.

$$EL_k = \operatorname{argmin}_{1 \leq N \leq M} \left\{ \left[ \frac{d(1+d)^N}{(1+d)^N - 1} \right] \left[ Px^{T_k} \left( 1 - \frac{b}{c} w^N \right) + \frac{Aq^{T_k}}{1+d} \cdot \frac{z^N - 1}{z - 1} \right] \right\} \quad (22)$$

A dynamic EL policy can have varying service lives for successive assets. The service life for each asset would be equal to the  $EL_k$  recalculated based on the capital and O&M costs at the time of the asset's purchase. A variable policy will always yield lower total costs than a fixed policy, unless capital and O&M costs are changing at exactly the same rate ( $a = q$ ).

### 4.2 Challenger/Defender ( $CD_k$ )

This method is also recommended by Newnan, Lavelle, and Eschenbach (2000) and Thuesen and Fabrycky (2001) as well as Terborgh (1949), and given their assumptions regarding technological progress (i.e. it stops after the next challenger is acquired) this makes sense. The Challenger/Defender (CD) method does take into account the costs of the available new asset (challenger) as well as the costs of the existing asset, but it does not consider the effects

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<sup>9</sup>The annualized cost is the equal annual payment amount for the service life of the asset that is equivalent, at an interest rate of  $d$ , to the actual costs of the asset.



of the timing of the first replacement on the costs of later assets.

The CD method compares the annualized costs for a new asset available in the current period, based on its  $EL_{k+1}$  with the marginal cost of keeping the existing ( $k^{th}$ ) asset for an additional period. When the annualized cost of capital and O&M for a new asset, which decreases over time, is less than the marginal cost of keeping the existing asset for another period, including O&M costs and opportunity cost of lost salvage value, then the asset is replaced. The purchase cost of the existing asset is a sunk cost, and so is not considered.  $CD_k$  is the smallest  $N$  that satisfies the inequality in (23), where  $EL_{k+1}$  is the economic life of an asset purchased at the end of period  $T_k + N$ .

$$\frac{A}{p}y^{T_k}z^{T_k} + Pa^{T_k}\frac{b}{c}w^N(1-w) > \left[ \frac{d(1+d)^{EL_k}}{(1+d)^{EL_k}-1} \right] \left[ Px^{T_k+N} \left( 1 - \frac{b}{c}w^{EL_{k+1}} \right) + \frac{Ay^{T_k+N}z^{EL_k}-1}{1+d} \frac{z-1}{z-1} \right] \quad (23)$$

This method compares the defender's costs with annualized costs for the challenger based on its EL, though the challenger's service life may not equal its EL if the same method is used to time the next replacement.

### 4.3 Fixed Service Life ( $N^*$ )

The Fixed Service Life (FL) method does take into account the effect of the timing of each replacement on the costs of all future assets, with the constraint that the replacement policy is stationary, i.e. service life is fixed for all assets, and  $N_k = N \quad \forall k$ . In this case, the total cost, given in (1), can be rewritten, with  $T_k = N \cdot (k-1)$ , as below, and the best fixed service life ( $N^*$ ) is the value of  $N$  that minimizes  $v(N)$ .

$$\text{Total discounted costs} = \sum_{k=1}^{\infty} Px^{N(k-1)} \left( 1 - \frac{b}{c}w^N \right) + \sum_{k=1}^{\infty} \frac{A}{1+d} y^{N(k-1)} \frac{z^N - 1}{z-1} \quad (24)$$

The sums are finite and  $N^*$  exists if and only if  $a, q < 1+d$ , and under this condition, the total cost for an infinite number of assets can be written in closed form as in (8).  $N^*$  is easy to find numerically for the geometric cost structure.

## 4.4 Dynamic Programming ( $N_1^*$ )

Our model does not yield an easy analytical solution for the optimal policy, or even for the optimal first service life, while accounting for the effect on all future assets. However, the structure of the deterministic replacement problem makes it relatively easy to use a dynamic programming algorithm to find either the optimal policy for a finite horizon, or the optimal first replacement time for the infinite horizon problem.

In a dynamic programming formulation, the state can be defined as the age of the asset at the end of the period, so the total number of states in each period equals the number of discrete service lives to be considered. There are few allowed actions and transitions. The computational simplicity does not depend on the geometric structure of our cost model, but rather on characteristics of this classic replacement problem, and the fact that we are primarily interested in only the first replacement, because there will usually be an opportunity to update the problem with new information before the next decision must be implemented.

Because the problem is deterministic, the optimal first service life can be found using forward induction instead of the more common backward induction. Forward induction eliminates the need to arbitrarily choose a terminal time horizon, and then confirm that the first service life is independent of the terminal state. Instead, the computation would be carried forward only until the first service life in the optimal policy leading to each terminal state is the same.

For our model, the time horizon required to stabilize the first service life is longer than might be expected—and selected in a first pass at implementing a backward induction solution. For parameter values analyzed in Section 5.2, horizons of more than 100 periods (years), and in one case, more than 379 years, were required to terminate the algorithm. These horizons ( $H$ ) are given in Table 3. Long horizons may be characteristic of replacement problems with continuing technological progress, and may occur in models specifying other functional relationships between cost and time.

The optimal policy for an infinite horizon,  $\pi_V^* = (N_1^*, N_2^*, N_3^*, \dots)$ , can also be obtained with dynamic programming using forward induction. The time horizon required to verify optimality could be very long, and the policies leading to each terminal state would have to be stored or recalculated once the algorithm terminated. The induction must be carried

forward until the best policy,  $\pi_V = (N_1, N_2, \dots, N_J)$ , leading to each terminal state has  $N_J = M$  (for  $a > q$ ) or  $N_J = 1$  (for  $a < q$ ). Equations (16) and (17) and Corollary 1 imply that the best policy for the infinite horizon passing through a given terminal state will be  $\pi_V^* = (N_1, N_2, \dots, N_{J-1}, N_J, N_J, \dots)$ , and the objective (total discounted costs, given in (1)) for the infinite horizon problem can be calculated for this policy. The total discounted costs for the best policy would be calculated for each terminal state, and the policy with the lowest costs would be optimal among all policies (restricted by the available set of discrete service lives) for the infinite horizon problem.

## 5 Example: Automobile

With the exception of DP, none of the above methods guarantees an optimal policy. The performance of the other methods is evaluated by comparing the first asset replacement times and total cost among policies, using parameters fitted to data for an automobile.

### 5.1 Cost Data

The geometric relationships in the model were converted into linear relations by a log transformation, and the parameters were fitted to the resulting linear model using a least squares fit. In each case, the data were adjusted to 1999 dollars before fitting. Therefore, all the geometric coefficients are already inflation-adjusted, and all dollar values ( $P$  and  $A$ ) are in 1999 dollars.

For most of the geometric relationships, with the notable exception of the new asset purchase price, the available automobile data fit the model reasonably well, as detailed below. These results are consistent with the assumption of geometric cost trends.

### Capital costs

The capital parameters  $a$ ,  $b$ , and  $c$  were fitted to data for a Honda Accord DX Sedan, as obtained from the Intellichoice Car Center website ([www.intellichoice.com](http://www.intellichoice.com)). The data used were the original list price (in actual dollars for model years 1985-1999), the current wholesale value for model years 1985-1998.

The fitted values for parameters  $b$  and  $c$  and their 95% confidence intervals are given in Table 2 below. The maximum life,  $M$ , is taken as 30 years. The value for  $P$ , the purchase

Table 1: Automobile age assignment

Mileage Class	Automobile Age (years)
0-24,000	0
24,000-48,000	2
48,000-80,000	4

cost, depends on the reference period. Intellichoice gives the base list price for the 2000 model-year DX (low-end) sedan at \$15,350, available in 1999.

## O&M costs

The O&M parameters,  $A$ ,  $p$ , and  $q$ , were fitted to data obtained from the *Automotive Fleet* magazine November 1999 issue. In order to find O&M parameters best matched to the O&M parameters for a Honda Accord, the model was fitted using data for intermediate sized cars. The original data are total operating costs per mile, including gasoline, oil, tires, and maintenance/repair<sup>10</sup> for four years (1996-1999) for cars in three mileage classes. The mileage classes were associated with car ages as indicated in Table 1. The data give costs per year and costs per mile; however, the average number of miles driven varied over the periods and across the car age categories. Therefore, the per-mile costs were used, and then multiplied by 12,000 miles/year to get an annual O&M cost.

## Discount rate

Rates of 10%, 15%, and 20%, which are representative of the marginal rate on consumer debt, including car loans, were used in this study.

## 5.2 Numerical Results

In order to illustrate the behavior of policies obtained by the various replacement decision methods, the policies and their costs over a horizon of 300 periods were obtained for 26 parameter sets. For each decision method, the service lives are constrained to be integer numbers of years. The results are given in Table 3. The costs found using DP are optimal

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<sup>10</sup>Warranty recovery income is subtracted from total cost in the original data, but this amount was added back in because warranty costs are not included in total costs. Total costs do not include insurance, which would be higher for newer cars, and would lead to a higher value for  $q$ . Gasoline costs, which were correlated neither with model year nor car age, were also removed from the total costs. In addition, these mileage costs are affected by the number of miles driven per year, which was not constant across years or mileage categories.

Table 2: Automobile parameter values and 95% confidence limits

Parameter	Lower Bound	Expectation	Upper Bound
$P^{11}$	NA	\$15,350	NA
$a$	0	1.00	1.00
$b$	0.75	0.83	0.90
$c$	0.85	0.86	0.87
$A$	\$60	\$91	\$140
$q$	0.97	1.04	1.12
$p$	1.29	1.41	1.55
$d$	10%	15%	20%
$M$	NA	30 years	NA

for the finite horizon, and the first asset’s service life is optimal for the infinite horizon. The ranges used for each parameter were generally kept within the 95% confidence interval for the parameters fitted to the data, except in cases S-Z. Therefore the effects observed are not degenerate nor even exceptional cases.

Within the 95% confidence intervals for the parameters, the optimal first service life in the examples ranged from seven to 15 years. Given that the model includes only financial costs, this range is reasonable: the fitted values for  $A$  and  $p$  include only financial O&M costs, and neglect non-financial costs such as time lost due to decreased reliability and decreased satisfaction with older cars, and this may underestimate O&M costs. In addition, the data only applied to cars that are relatively new ( $< 80,000$  miles), and may be a poor estimate for older cars. However, there is also evidence that the cost of repairing and maintaining a car levels off after the third or fourth year (Tom and Ray Magliozzi, *Car Talk*, p. 173).

In addition, each decision method (except the CD method) was implemented using the assumption that all decisions are made at the time of purchase of the first asset, i.e. there is no incumbent defender whose costs are sunk. In the case of the DP method and the FL method, the assumption that the purchase cost and some periods’ O&M costs are sunk does not affect the optimal solution as long as the optimal first (or best fixed) service life is greater than the age of the current defender. The EL method loses its meaning once the purchase cost is considered sunk. If new-asset purchase prices are excluded from the objective in (22), then  $EL_k^* = 1$ : both the loss of salvage value and the O&M costs are increasing the longer the asset is kept. The CD method already assumes purchase and some O&M costs are sunk

as it is based on the marginal costs of the defender.

## Economic Life

The dynamic EL method came closest to the optimal performance of the four alternatives: if the decision-maker can resist the temptation of low-cost current challengers, which would lead her to a CD-like policy, EL is likely to be the method she uses. Its worst cost was within 12% of optimal, and it was within 2.5% in all but three of the 26 cases. This method can err in either direction, by overshooting or undershooting the optimal first service life (see Cases W & X). Initially, it might seem counterintuitive that a decision method that completely ignores the benefits of technological progress in available challengers performs so well. The performance of this method illustrates the importance of allowing variable service lives when the relative cost contributions of capital and O&M costs change.

## Fixed Service Life

Though it is hard to imagine a decision maker implementing the FL method—never reevaluating the service life of an asset type—the FL usually outperformed the CD method, and was usually within a few percent of optimal. This illustrates that there is great value in accounting for the effect of ongoing technological progress when making the current replacement decision. This method performs badly when a dynamic policy is very valuable, which occurs when the rates of technological progress are very different for capital costs ( $a$ ) as compared with O&M costs ( $q$ ). In several cases the total discounted costs were substantially worse even than the CD method. As the numerical examples illustrate,  $N^*$  is longer than the optimal first service life when  $a > q$ , and shorter when  $q > a$ .<sup>12</sup>

## Challenger/Defender

The CD method is recommended by many textbooks, including Newnan, Lavelle, and Eschenbach (2000) and Thuesen and Fabrycky (2001) as well as by Terborgh (1949). However, they rely on the strong assumption that all future challengers will have the same minimum

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<sup>12</sup>The costs given in Table 3 may differ slightly however, as these are for a horizon time of 300 years, and the dynamic programming method will optimize the last few assets, while the fixed service life will use the same life until there are not enough periods remaining in the horizon.

annualized cost as the present challenger. The results given in Table 3 illustrate the vulnerability of this method, which in most cases yielded total discounted costs substantially higher than all other methods—in one case more than double the optimal costs, and more than 10% worse in over half the cases. In each period, the CD method calls for comparing the cost of keeping the existing asset for one period with the annualized cost of owning and operating the asset for the EL of a new asset. But if the new asset is kept for a shorter (or longer) period than the EL, this annualized cost is not the actual cost the asset incurs. Since the real costs of new assets are constantly decreasing, the minimum annualized costs of available challengers are constantly decreasing. When O&M costs are large relative to the capital costs, the CD method can call for replacing the asset very frequently, even every period.

As a basis for comparison, the best Fixed Service Life ( $N^*$ ) to the nearest 0.01 years was also calculated. This policy performed better than the integer-constrained fixed service life policy. However, this policy was better than optimal integer solution with variable lives only when  $a = q$ , and a fixed service life was necessarily optimal. For the parameter values tested here, the effect of discretization appears far less important than the difference among policy generation methods.

## Optimal Policy

The structure of the costs in our model is relatively simple but complex enough to generate few rules governing the optimal policy. Naturally, total horizon costs are lower, for each decision method, when prices are decreasing quickly, i.e. when  $a$  and  $q$  are small, or  $d$  is large. The effect of  $a$  and  $q$  on service life is more complicated.  $EL_1$  is unaffected by the values of  $a$  and  $q$  as it depends only on the costs for the first asset. Further, the  $EL_k$  for later assets depends only on the ratio of  $a$  to  $q$ , which can be factored out of the objective in (22).

Similarly, if the salvage value declines very fast (low  $c$  and  $b$  in our model), the total costs are high, because very little of the value of the original asset can be recovered, regardless of the policy selected. Figure 1 below illustrates the sensitivity of policy and costs to the technological progress parameters  $a$  and  $q$  and salvage value parameter  $c$ .

## 6 Conclusion

When an asset's functionality is required for a long time, myopic decision methods (EL and CD) are sub-optimal because they do not account for the effect of timing of the first replacement on future assets' costs. Stationary policies are also sub-optimal if the relative importance of capital and O&M costs is changing. Under these conditions, the discounted costs of maintaining an asset is lower for an optimal policy identified with DP. However, among the methods that are easier to implement, a policy using the EL of each asset, which does not account for the costs of available challengers, usually outperforms the other alternatives compared here.

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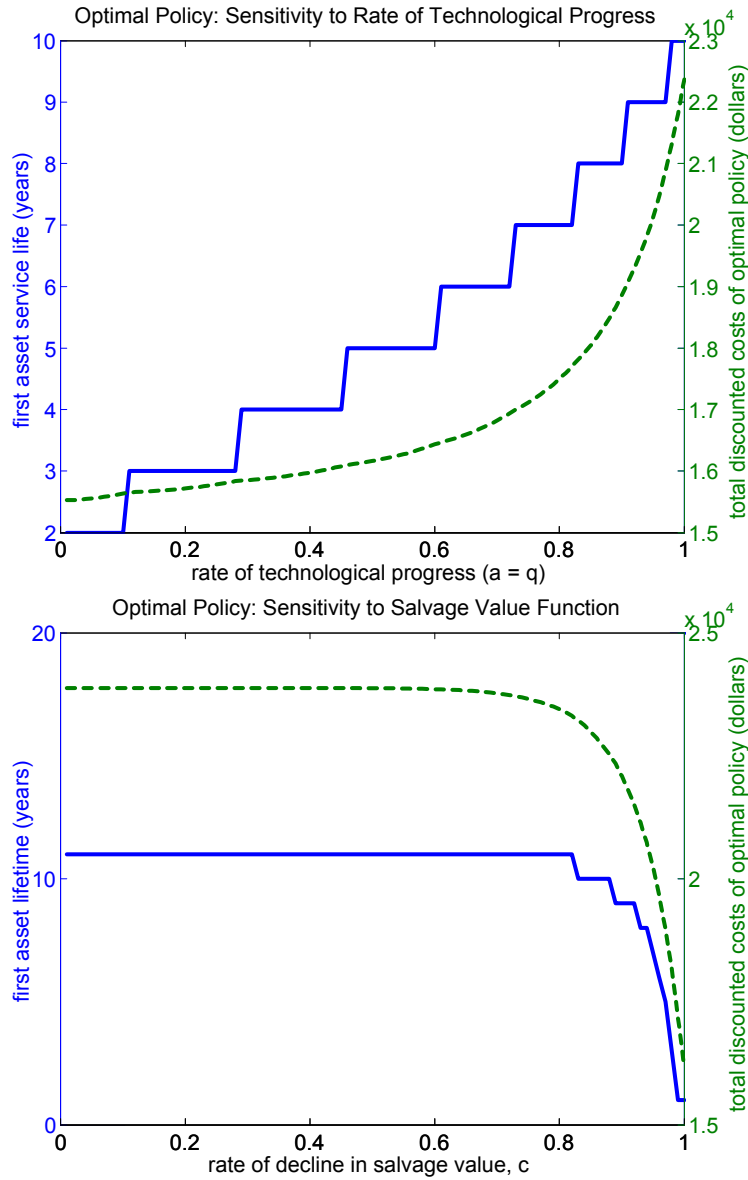


Figure 1: The dotted lines show the total discounted cost for the optimal policy, while the solid lines shows the first asset service life under the optimal variable policy,  $N_1^*$ . In the first plot, the rate of technological progress is varied, with  $a = q$ , and  $c = (0.86)a$ . In the second plot, the rate of decline in salvage values,  $c = b$ , is varied. The remaining parameter values for this example are  $P = \$15,350$ ,  $A = \$91$ ,  $a = 1.04$ ,  $q = 1.0$ ,  $c = 86$ ,  $b = 0.75$ ,  $p = 1.41$ ,  $d = 15\%$ , and  $M = 30$ .

	parameter values							$N_1$				present value of total cost			% diff from DP				
	$A$	$a$	$q$	$b$	$c$	$p$	$d$	$H$	$N_1^*$	$N^*$	$EL_1$	$CD_1$	<b>DP</b>	<b>FL</b>	<b>EL</b>	<b>CD</b>	<b>FL</b>	<b>EL</b>	<b>CD</b>
<b>A</b>	\$91	1.00	1.05	0.83	0.86	1.39	15%	50	11	10	10	11	22.8	22.9	22.8	22.8	0.58	0.08	0.13
<b>B</b>	\$60	1.00	1.05	0.83	0.86	1.39	15%	45	12	11	12	1	21.4	21.6	21.5	29.8	0.55	0.01	38.8
<b>C</b>	\$60	1.00	1.05	0.83	0.86	1.31	15%	66	15	14	14	1	20.2	20.3	20.2	32.5	0.40	0.01	60.6
<b>D</b>	\$60	1.00	1.05	0.83	0.86	1.46	15%	52	10	10	10	11	22.4	22.5	22.4	22.4	0.61	0.00	0.31
<b>E</b>	\$60	1.00	1.05	0.83	0.86	1.31	10%	55	14	12	13	1	25.6	26.2	25.7	41.4	2.08	0.12	61.6
<b>F</b>	\$60	1.00	1.05	0.83	0.86	1.39	10%	54	11	10	11	1	27.5	28.1	27.5	39.7	2.20	0.04	44.3
<b>G</b>	\$60	1.00	1.05	0.83	0.86	1.46	10%	48	10	8	10	1	28.9	29.5	28.9	36.9	2.19	0.01	27.8
<b>H</b>	\$60	1.00	1.05	0.83	0.86	1.31	20%	50	15	15	15	1	18.0	18.0	18.0	25.3	0.11	0.01	40.7
<b>I</b>	\$60	1.00	1.05	0.83	0.86	1.39	20%	56	12	12	12	13	18.9	18.9	18.9	18.9	0.13	0.01	0.03
<b>J</b>	\$60	1.00	1.05	0.83	0.86	1.46	20%	43	11	10	11	12	19.5	19.6	19.5	19.6	0.27	0.00	0.49
<b>K</b>	\$60	1.00	1.00	0.83	0.86	1.39	15%	53	12	12	12	1	21.0	21.0	21.0	33.1	0.00	0.00	57.8
<b>L</b>	\$60	1.01	1.00	0.83	0.86	1.39	15%	51	12	12	12	1	21.6	21.6	21.6	35.5	0.00	0.00	64.7
<b>M</b>	\$60	1.00	1.10	0.83	0.86	1.39	15%	55	12	10	12	1	22.3	23.3	22.3	26.8	4.17	0.08	20.2
<b>N</b>	\$140	1.00	1.05	0.83	0.86	1.39	15%	50	9	9	9	10	24.5	24.6	24.5	24.5	0.75	0.00	0.33
<b>O</b>	\$140	1.00	1.05	0.83	0.86	1.31	15%	58	12	11	11	13	23.1	23.2	23.1	23.2	0.60	0.04	0.54
<b>P</b>	\$140	1.00	1.05	0.83	0.86	1.46	15%	40	8	7	8	9	25.4	25.6	25.4	25.6	0.83	0.00	0.64
<b>Q</b>	\$91	1.00	1.05	0.75	0.86	1.39	15%	49	11	10	11	12	22.9	23.0	22.9	23.0	0.60	0.00	0.63
<b>R</b>	\$140	1.00	1.09	0.83	0.86	1.39	10%	41	9	3	8	10	43.6	50.3	43.8	43.9	15.32	0.42	0.65
<b>S</b>	\$91	1.00	1.05	0.9	0.95	1.39	15%	48	8	7	8	10	20.6	20.7	20.6	20.8	0.47	0.00	0.89
<b>T</b>	\$140	1.00	1.09	0.83	0.86	1.31	10%	52	11	4	10	1	41.5	48.5	41.7	45.8	17.0	0.54	10.3
<b>U</b>	\$140	1.09	1.00	0.83	0.86	1.31	15%	100	14	16	11	17	31.9	33.3	33.0	55.4	4.48	3.50	73.7
<b>V</b>	\$140	1.05	0.95	0.83	0.86	1.31	10%	144	11	14	10	1	36.5	39.1	36.7	83.8	7.14	0.53	129
<b>W</b>	\$140	1.05	0.95	0.83	0.86	1.15	10%	145	23	27	20	1	27.5	28.1	27.7	83.8	1.85	0.61	204
<b>X</b>	\$60	0.99	0.95	0.75	0.86	1.13	10%	88	26	27	29	1	18.4	18.5	18.5	49.2	0.07	0.27	167
<b>Y</b>	\$140	1.09	1.00	0.83	0.86	1.46	15%	101	10	11	8	11	38.4	40.5	39.2	53.0	5.44	2.02	38.1
<b>Z</b>	\$140	1.09	1.00	0.83	0.86	1.46	10%	379	9	14	7	11	105.2	143.6	108.1	347.3	36.5	2.78	230

Table 3: Numerical results for automobile replacement