

# Analysis of Greedy Robot-Navigation Methods

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## Abstract

Robots often have to navigate robustly despite incomplete information about the terrain or their location in the terrain. In this case, they often use greedy methods to make planning tractable. In this paper, we analyze two such robot-navigation methods. The first method is Greedy Localization, which determines the location of a robot in known terrain by always moving it to the closest location from which it will make an observation that reduces the number of its possible locations, until it has reduced that number as much as possible. We reduce the upper bound on the number of movements of Greedy Localization from  $O(n^{\frac{3}{2}})$  to  $O(n \log n)$  on grid graphs and thus close to the known lower bound of  $\Omega(n \log n / \log \log n)$ , where  $n$  is the number of (unblocked) vertices of the graph that discretizes the terrain. The second method is Dynamic A\* (D\*), which is used on several prototypes of both urban reconnaissance and planetary robots. It moves a robot in initially unknown terrain from given start coordinates to given goal coordinates by always moving the robot on a shortest presumed unblocked path from its current coordinates to the goal coordinates, pretending that unknown terrain is unblocked, until it has reached the goal coordinates or there are no more presumed unblocked paths. We reduce the upper bound on the number of movements of D\* from  $O(n^{\frac{3}{2}})$  to  $O(n \log^2 n)$  on arbitrary graphs and  $O(n \log n)$  on planar graphs (including grid graphs) and thus close to the known lower bound of  $\Omega(n \log n / \log \log n)$ , where  $n$  is the number of (blocked and unblocked) vertices of the graph that discretizes the terrain.

## 1 Introduction

Robot-navigation problems with incomplete information are challenging because nondeterminism results in a large number of contingencies. These problems include localization (where the map is known but the location of the robot relative to the map is unknown and the objective is to move the robot until it has discovered its current location), and goal-directed

navigation in unknown terrain (where the location of the robot relative to the map is known but the map is unknown and the objective is to move the robot to a given goal location). The sensors on-board a robot can typically sense the terrain only near its current location, and the robot thus has to interleave planning with movement to sense new parts of the terrain, either to discover more about its current location or to discover more about the map.

In this paper, we analyze two robot-navigation methods that both interleave planning with movement and use greedy (that is, myopic) planning approaches to make planning tractable. The first method is Greedy Localization, which moves a robot in known terrain so that it determines its initially unknown location. The second method is Dynamic A\* (D\*), which moves a robot in initially unknown terrain from known start to known goal coordinates. Both robot-navigation methods are simple to implement, easy to integrate into complete robot architectures, and seem to result in small travel distances in practice. We analyze their travel distances to understand whether the travel distances are indeed small in any kind of terrain or whether they were small only because of properties of the terrains used to test them experimentally. We model the terrain as graph (in practice, grid graphs are commonly used) and then analyze the worst-case number of edge traversals as a function of the size of the graph, measured by the number of its vertices  $n$ . Robots move so slowly that their task-completion times are completely dominated by their travel times. Hence this criterion is at least as important as the competitive ratio [6] because it is a guarantee on the task-completion time. We reduce the upper bound on the worst-case number of edge traversals of Greedy Localization from  $O(n^{\frac{3}{2}})$  [11] to  $O(n \log n)$  on grid graphs and thus close to the best known lower bound of  $\Omega(n \log n / \log \log n)$  [11]. With a completely different analysis, we reduce the upper bound on the worst-case number of edge traversals of D\* from  $O(n^{\frac{3}{2}})$  [10] to  $O(n \log^2 n)$  on arbitrary graphs and  $O(n \log n)$  on planar graphs (including grid graphs) and thus close to the best known lower bound of  $\Omega(n \log n / \log \log n)$  [7], which holds even on grid graphs [10].

## 2 Greedy Localization

\*\*\* The notation for Greedy Localization is different from the one used for D\*, e.g. the start vertex in one case is  $x^0$  and in the other  $v_0$ . (CAT: I prefer it that way.) I also don't like that sets sometimes have superscripts and sometimes subscripts (CAT: it would be slightly nicer to be consistent, but it is not common in math journal articles to be so nice. If we were writing a book it would be more important. The truth is, I didn't want to bother. Also I am used to the notation as is so it is easier for me to proofread as is.

The localization problem on grid graphs is defined as follows. The robot moves with no actuator uncertainty on a grid graph  $G$  with the usual north-south-east-west orientation. Each cell is either traversable (unblocked) or untraversable (blocked). The perimeter of  $G$  consists of blocked cells. The robot is initially situated in an unblocked cell of  $G$ . The robot has an on-board compass and a map (including orientation and traversability information). However, the robot does not know in which cell of  $G$  it is located. The robot possesses sensors that detect with certainty which of the neighboring cells (N,S,E,W) are blocked. The robot

can move from its current cell to any adjacent unblocked cell. The robot has *localized* when it has made a series of moves such that the sensory information acquired during those moves is sufficient to determine the robot’s location, or to determine that the location can not be determined (if for example the robot determined that it was within one of a pair of isomorphic connected components of  $G$ ). At each move, the robot is permitted to use all the sensory information acquired so far to help decide which direction to move next. The localization problem is localize in a minimum number of movements. We assume in the following that the sensors of the robot always detect which of its four neighboring cells are blocked and perhaps also the blockage status of additional cells.

Greedy Localization maintains the set of cells that the robot can be in given all observations that it has made so far. It always makes the robot execute a shortest sequence of movements so that all observations that it has made after the sequence of movements are guaranteed to reduce the number of cells that the robot can be in. It terminates once the robot cannot reduce the number any more. If the number is one after termination then the robot is localized, otherwise it cannot localize. Greedy Localization was pioneered by Nourbakhsh in robot programming classes where Nomad 150 mobile robots had to navigate mazes that were built with three-foot high and forty inch long cardboard walls [5]. Similar strategies are used in more complex environments where robots use probability distributions over locations rather than sets of locations to be able to deal with sensor noise, and the greedy localization methods then move the robots to decrease the entropy of the probability distribution rather than the cardinality of the set [3].

## 2.1 Worst-Case Travel Bound

Researchers often analyze localization methods using online criteria, by comparing the travel distance of a robot to the distance that would be traveled by an omniscient robot that knows its location at the outset and seeks only to verify that location. Dudek, Romanik, and Whitesides [2], for example, find a best possible online ratio of  $|H| - 1$ , where  $H$  is computed from the start location of the robot with long range sensors in a polygonal model. We, on the other hand, analyze the worst-case travel distance of Greedy Localization in the following because a small worst-case travel distance guarantees that it cannot perform very badly, which is more important than the “regret” measured by the competitive ratio. Our analysis makes heavy use of a lemma from [12].

We model the grid as a grid graph  $G = (V, E)$  whose vertices correspond to the unblocked cells. The start vertex of the robot is  $x^0$ . The observation that the robot makes in a vertex is the same as the observation that it makes in the corresponding cell. Two vertices are connected via an edge iff they correspond to neighboring cells. The number of edge traversals of the robot on the grid graph then is the same as its travel distance on the grid.

**Theorem 1** *Greedy Localization traverses at most  $|V| + 2|V| \ln |V|$  edges on grid graphs  $G = (V, E)$ .*

**Proof:** At each step in the algorithm, the robot is at a particular vertex in the graph, even

though the robot does not know which one it is. For the purposes of analysis, we follow the robot's motions as they actually occur in the graph. During iteration  $i$ , the robot follows a shortest path from vertex  $x^{i-1}$  to vertex  $x^i$  so that the observation in  $x^i$  is guaranteed to reduce the number of vertices that the robot can be in. This implies that  $x^i$  is closest to  $x^{i-1}$  among all informative vertices, where a vertex is *informative* if the observation from the vertex is guaranteed to reduce the number of vertices that the robot can be in. Note that uninformative vertices remain uninformative. Let  $d(x, x')^G$  denote the distance from vertex  $x$  to vertex  $x'$  in graph  $G$  and define  $l^i = d(x^{i-1}, x^i)^G$ . The main intuition behind the proof of the theorem is the fact that no vertex within a large distance of  $x^{i-1}$  can be informative if  $l^i$  is large. Thus, the number of large  $l^i$  and thus also the number of edge traversals of Greedy Localization must be small. We use marking sequences to formalize this intuition. A marking sequence on graph  $G = (V, E)$  is a sequence of triples  $\{v^i, r^i, M^i\}$  for  $i = 1, 2, \dots$ , whose integers  $r^i \geq 0$ , vertices  $v^i \in V$ , and sets  $M^i \subseteq V$  satisfy the following properties:

1.  $v^i \notin M^i$ ,
2.  $M^1 = \emptyset$  and  $M^i \subset M^{i+1}$ , and
3.  $d(v, v^i)^G \leq r^i$  implies  $v \in M^{i+1}$ .

The *cost* of the marking sequence is  $\sum_i (1 + r^i)$ . Vertices  $v$  are considered to be *marked* at step  $i$  iff  $v \in M^i$ . Our key construct then is as follows: Greedy Localization forms an associated marking sequence where  $r^i = l^i - 1$ ,  $v^i = x^{i-1}$ , and  $M^{i+1}$  is the set of uninformative vertices after the robot has reached and made an observation from  $v^i$ . The number of edge traversals of Greedy Localization equals the cost of the associated marking sequence since  $1 + r^i = l^i$ . Note that the marking sequence is less restrictive than Greedy Localization because  $v^i$  need not be at distance  $1 + r^i$  from  $v^{i+1}$ . Instead, the marking sequence consists of a sequence of choices of an unmarked vertex  $v^i$  and a radius  $r^i$ . All vertices within distance  $r^i$  of  $v^i$  (and possibly additional vertices) are marked, and the marking sequence continues.

**Lemma 1** *The cost of any marking sequence is no larger than  $|V| + 2|V| \ln |V|$  on connected graphs  $G = (V, E)$ .*

**Proof sketch:** It follows from the triangle inequality that there exists a maximum cost marking sequence that only marks one vertex, namely  $v^i$ , per step. For if another vertex  $v$  were also marked, one could replace the step with two more expensive steps that mark only  $v^i$  and  $v$ , respectively. By viewing the marking sequence as a sequence of disjoint balls of radius  $r^i$  in the metric space of graph distances, the graph's connectivity limits the number of radii that are at least  $t$  to  $2|V|/t$ . The  $O(|V| \ln |V|)$  bound follows. Full details are given in [12].

Greedy Localization constrains the movements of the robot to be in a connected component of the graph. Hence, the lemma applies and the theorem is proved. ■

Note that our proof is not specific to robots with short-range sensors that operate on two-dimensional grid graphs in which each cell's potential neighbors are located N,S,E,W. Our

proof does require that uninformative vertices remain uninformative. This is the case, for example, if the set of vertices that the robot can be in when it is in some vertex  $x$  is always included in the set of vertices that the robot could have been in when it was in the same vertex  $x$  earlier. Thus our theorem holds also, for example, for higher-dimensional or differently connected grid graphs and is completely independent of the kind of sensors used by the robot.

### 3 D\*

\*\*\* Apurva needs to confirm whether the number of vertices refers to all vertices or just the unblocked ones. (CAT: I checked with Apurva, and he confirms what I said in my email. The number of vertices refers to all vertices, whether blocked or unblocked, in the connected component of the graph, together with all adjacent blocked vertices.)

Consider again a two-dimensional grid with blocked and unblocked cells. Its perimeter consists of blocked cells. The start cell of the robot is unblocked. The robot knows its start cell and orientation and has to move to a given goal cell. It does not know initially which cells are blocked. The robot can move from its current cell north, east, south or west to any neighboring cell. The movement succeeds if the cell is unblocked. There is no actuator or sensor uncertainty. The robot can use all observations it has made so far (corresponding to the partial map that it has learned so far) to decide to which neighboring cell to move next. The goal acquisition problem is move to the goal cell in a minimum number of movements. We assume in the following that the sensors of the robot always detect only whether the cell that it attempts to move to is blocked.

D\* maintains the partial map that the robot has learned so far. It always makes the robot execute a shortest sequence of movements from its current cell to the goal cell under the optimistic assumption that cells are unblocked that have not been observed to be blocked. It terminates once the robot has reached the goal cell or no such sequences of movements exist any longer (in which case the robot cannot reach the goal cell). Whenever the robot observes a blocked cell on its current path, D\* needs to replan, which can be implemented efficiently [8] and easily [4]. D\* has been used outdoors on an autonomous high-mobility multi-wheeled vehicle that navigated 1,410 meters to the goal location in an unknown area of flat terrain with sparse mounds of slag as well as trees, bushes, rocks, and debris [9]. As a result of this demonstration, D\* is now widely used in the DARPA Unmanned Ground Vehicle (UGV) program, for example, on the UGV Demo II vehicles. D\* is also being integrated into Mars Rover prototypes, tactical mobile robot prototypes and other military robot prototypes for urban reconnaissance.

#### 3.1 Analysis

We analyze the worst-case travel distance of D\* in the following. We model the grid as a grid graph whose vertices correspond to the cells. Thus, vertices can be blocked or unblocked.

Two vertices are connected via an edge iff they correspond to neighboring cells. However, our analysis holds for graphs in general, not just grid graphs. We therefore generalize the problem as follows:

Consider a graph  $G = (V, E)$  with blocked and unblocked vertices. We assume without loss of generality that the graph is connected (otherwise we can consider only the connected component of the graph that contains the start vertex). The start vertex  $v_0$  of the robot is unblocked. The robot always knows its current vertex – all vertices can be uniquely identified, unlike the situation in localization problems – and has to move to a given goal vertex  $t$ . It does not know initially which vertices are blocked. The robot can attempt to move from its current vertex to any neighboring vertex. If the neighboring vertex is unblocked, then the robot moves to it. If the neighboring vertex is blocked, however, then the robot remains in its current vertex. In both cases, the robot observes whether the neighboring vertex is blocked.  $D^*$  always moves the robot along a shortest presumed unblocked path from its current vertex to the goal vertex. A path is *presumed unblocked* if it does not contain vertices that the robot knows to be blocked. Whenever the robot observes that a vertex is blocked, it recalculates another shortest presumed unblocked path from its current vertex to the goal vertex and repeats the process.  $D^*$  terminates once the robot has reached the goal vertex or can no longer find a presumed unblocked path from its current vertex to the goal vertex.

\*\*\* need to add a small comment to the following that it is not a problem if  $x'_i$  does not exist because  $b_i = t$  (CAT: I don't think this is necessary)

We use the following notation: At the beginning of iteration  $i$ , the robot is at vertex  $v_{i-1}$  and  $E_i$  is the set of edges that are not incident on a vertex known to be blocked at that time. Note that  $E_1 = E$ . The robot then plans a shortest path  $P_i$  in graph  $H^i = (V, E_i)$  from vertex  $v_{i-1}$  to  $t$ , starts to follow it, and then stops at vertex  $v_i$  either because  $v_i = t$  or because the vertex  $b_i$  following  $v_i$  on  $P_i$  is blocked. In the latter case, let  $v'_i$  denote the vertex following  $b_i$  on  $P_i$ . The robot eventually stops in vertex  $v_k$  either because  $v_k = t$  or because there are no longer any presumed unblocked paths from  $v_k$  to  $t$ . Thus,  $k \leq |V|$ . Let  $d(x, x')^G$  denote the distance from vertex  $x$  to vertex  $x'$  in graph  $G$ . If  $x$  and  $x'$  are not in the same connected component of  $G$  then  $d(x, x')^G = \infty$ .

Let  $v_0, v_1, \dots, v_k$  be a run of the method (this captures a run up to ties in shortest viable paths). The total distance traveled by the robot is:

$$C = \sum_{i=1}^k d(v_{i-1}, v_i)^{H^i}.$$

## 3.2 Telescoping

**Lemma 2**  $D^*$  traverses at most  $|V| + \sum_{i=1}^{k-1} d(v_i, v'_i)^{H^{i+1}}$  edges on arbitrary connected graph  $G = (V, E)$ .

**Proof:** Since  $v_i$  lies on the shortest path  $P_i$  from  $v_{i-1}$  to  $t$  in  $H^i$ , by the principle of optimality,

$$\begin{aligned}
C &= \sum_{i=1}^k d(v_{i-1}, v_i)^{H^i} \\
&= \sum_{i=1}^k (d(v_{i-1}, t)^{H^i} - d(v_i, t)^{H^i}) \\
&= d(v_0, t)^{H^1} - d(v_k, t)^{H^k} + \sum_{i=1}^{k-1} (d(v_i, t)^{H^{i+1}} - d(v_i, t)^{H^i}) \\
&\leq |V| + \sum_{i=1}^{k-1} (d(v_i, t)^{H^{i+1}} - d(v_i, t)^{H^i}).
\end{aligned}$$

(The last inequality uses the inequality  $d(v_0, t)^{H^1} < |V|$  since  $G = (V, E) = (V, E_1) = H^1$  is connected.)

Consider an arbitrary  $i$  with  $1 \leq i < k$ . The robot planned a shortest path  $P_i$  in  $H^i$  from  $v_{i-1}$  via  $v_i$ ,  $b_i$ , and  $v'_i$  to  $t$ . Thus, the subpath of  $P_i$  from  $v'_i$  to  $t$  is a shortest path in  $H^i$  from  $v'_i$  to  $t$ . Since it does not contain any edges incident on  $b_i$ , it is also a shortest path in  $H^{i+1}$  from  $v'_i$  to  $t$  and thus  $d(v'_i, t)^{H^{i+1}} = d(v'_i, t)^{H^i}$ . Thus, it holds by the triangle inequality that

$$d(v_i, t)^{H^{i+1}} \leq d(v_i, v'_i)^{H^{i+1}} + d(v'_i, t)^{H^{i+1}} = d(v_i, v'_i)^{H^{i+1}} + d(v'_i, t)^{H^i}.$$

According to the definition of  $v'_i$ , we have  $d(v_i, t)^{H^i} = 2 + d(v'_i, t)^{H^i}$  for  $1 \leq i < k$ . Thus,  $C \leq |V| + \sum_{i=1}^{k-1} (d(v_i, t)^{H^{i+1}} - d(v_i, t)^{H^i}) \leq |V| + \sum_{i=1}^{k-1} ((d(v_i, v'_i)^{H^{i+1}} + d(v'_i, t)^{H^i}) - (d(v'_i, t)^{H^i} + 2)) \leq |V| + \sum_{i=1}^{k-1} d(v_i, v'_i)^{H^{i+1}}$ . ■

### 3.3 Time Reversal and Weighted Edges

Consider the following function:

**CYCLE-WEIGHT(T, S):** Input: a tree  $T = (V, E)$  and an ordered list  $S = \{e_j : 1 \leq j \leq k\}$  of distinct edges from the complete graph on  $V$ , such that  $S \cap E = \phi$ . Define the weight  $w_i$  of edge  $e_i$  to be the length of a shortest cycle that contains  $e_i$  in the graph  $T_i = (V, E \cup \{e_j : i \leq j \leq k\})$ . Output:  $\sum_{i=1}^k w_i$ .

We now show that  $\sum_{i=1}^{k-1} d(v_i, v'_i)^{H^{i+1}} \leq \text{CYCLE-WEIGHT}(T, S)$  for a suitably constructed tree  $T$  and  $S = \{e_j = (v_j, b_j) : 1 \leq j < k\}$ .

\*\*\* do we need to worry about  $b_k$ ? In other words, does it matter how  $D^*$  stopped: whether at the goal, because it noticed the goal to be blocked, or because it noticed some other vertex to be blocked which reduced the number of presumed unblocked paths to zero? (CAT: we don't have to worry about it. It does not exist, that is, we don't define it and we don't use it.)

The basic idea relating the edge weights in CYCLE-WEIGHT to the  $d(v_i, v'_i)^{H^{i+1}}$  values can be understood by considering a special case. Assume that  $H^k$  is connected except for the isolated vertices  $b_1, b_2, \dots, b_{k-1}$ . Reverse the time perspective so that the robot motion adds edges, first the edges incident on  $b_{k-1}$ , then the edges incident on  $b_{k-2}$ , and so on. Pick  $T$  to be a spanning tree of the graph  $(V, E_k \cup \{(b_j, v'_j) : 1 \leq j < k\})$  and  $S$  to be  $\{e_j = (v_j, b_j) : 1 \leq j < k\}$ . Then, it holds that  $w_i \geq 2 + d(v_i, v'_i)^{H^{i+1}}$  for  $1 < i < k$  since every cycle that contains  $e_i = (v_i, b_i)$  in  $T_i$  must also contain  $(b_i, v'_i)$ . Consequently, it holds that  $\sum_{i=1}^{k-1} d(v_i, v'_i)^{H^{i+1}} \leq \sum_{i=1}^{k-1} w_i = \text{CYCLE-WEIGHT}(T, S)$ .

Unfortunately this simple construction does not work in the general case since multiple connected components may be formed when the edges incident on a blocked vertex are removed. To get around this problem, we define the graphs  $F_k, F_{k-1}, \dots, F_1$  as follows:

- Let  $F_k$  be a spanning forest of  $H^k$ .
- For  $1 \leq i < k$ , let  $C_1^i, C_2^i, \dots, C_{k_i}^i$  be the connected components of  $H^{i+1}$  which get merged with  $b_i$  in  $H^i$ , with the restriction that  $v'_i \in C_1^i$ . Select a  $w_j^i \in C_j^i$  for  $1 \leq j \leq k_i$  such that  $w_j^i$  is a neighbor of  $b_i$  in  $G$ , with the restriction that  $w_1^i = v'_i$ . Let  $F_i$  result from  $F_{i+1}$  by adding the edges  $\{(b_i, w_j^i) : 1 \leq j \leq k_i\}$ .

The following lemma is immediate:

**Lemma 3** *For  $1 \leq i \leq k$  and all vertices  $u$  and  $v$ ,  $F_i$  is acyclic;  $d(u, v)^{F_i} < \infty$  iff  $d(u, v)^{H^i} < \infty$ ; and  $d(u, v)^{F_i} \geq d(u, v)^{H^i}$ .*

**Proof:** By induction. ■

\*\*\* CAT: I prefer the above. It would be OK to say, By downwards induction on i. But we may not say, by induction on i.

We are now ready to prove the bound.

**Lemma 4** *Let  $H^1, H^2, \dots, H^k$  be a sequence of graphs as defined above. Let  $T = F_1$  and  $S = \{e_i = (b_i, v_i) : 1 \leq i \leq k-1\}$ . Then  $\sum_{i=1}^{k-1} d(v_i, v'_i)^{H^{i+1}} \leq \text{CYCLE-WEIGHT}(T, S)$ .*

**Proof:** According to Lemma 3,  $F_{i+1}$  and  $H^{i+1}$  have the same connected components. The subgraph of  $F_1$  induced by  $C_1^i$  is connected since  $C_1^i$  is a component of  $H^{i+1}$ . The edges  $e_j$  for  $i < j < k$  are contained in  $C_1^i$  since  $v_j, v'_j, b_j \in C_1^i$  for all  $i < j < k$ . Thus, the graph obtained by contracting all vertices of  $C_1^i$  in  $T_{i+1}$  is acyclic. Since  $T_i$  is obtained from  $T_{i+1}$  by adding  $e_i$ , every cycle that contains  $e_i = (v_i, b_i)$  in  $T_i$  must also contain  $(b_i, v'_i)$ . Thus,  $w_i$  is equal to 2 plus the distance between  $v_i$  and  $v'_i$  in the subgraph  $G'$  of  $T_i$  induced by  $C_1^i$ . But  $G'$  is also a subgraph of  $H^{i+1}$  and hence it holds that  $w_i \geq 2 + d(v_i, v'_i)^{H^{i+1}}$ . Consequently, it holds that  $\sum_{i=1}^{k-1} d(v_i, v'_i)^{H^{i+1}} \leq \sum_{i=1}^{k-1} w_i = \text{CYCLE-WEIGHT}(T, S)$ . ■



### 3.4 An Extremal Problem on Graphs

We now bound  $\text{CYCLE-WEIGHT}((V, E), S)$  in terms of  $|V|$  and  $|S|$ . Let  $E_w = \{e_i; w_i \geq w\}$  be the set of edges with weight at least  $w$ . Recall that the *girth* of a graph is the length of its shortest cycle. Define  $\Gamma(n, w)$  (respectively  $\Gamma_P(n, w)$ ) to denote the maximum number of edges in a graph (respectively planar graph) with  $n$  vertices and a girth of at least  $w$ . The following lemma relates  $E_w$  and  $\Gamma(n, w)$ .

**Lemma 5**  $|E_w| \leq \Gamma(|V|, w) - |V| + 1$  for all  $\text{CYCLE-WEIGHT}((V, E), S)$  and all  $w$ .

\*\*\* We need to justify the  $=$  in  $|E \cup E_w| = |E| + |E_w|$ ! (CAT: Sven is correct. I put disjointedness into the definition of cycle-weight. We always thought it but we never wrote it!)

**Proof:** Consider the graph  $T_w = (V, E \cup E_w)$ . We claim that  $T_w$  has a girth of at least  $w$ . To see this, assume that it does not and thus has a cycle  $C$  of length  $w' < w$ . Since  $(V, E)$  is a tree, at least one edge of  $C$  must belong to  $E_w$ . Consider the edge  $e_j \in E_w \cap C$  with the smallest  $j$ . Then  $T_j$  contains  $C$  and thus  $w_j \leq w' < w$ . On the other hand,  $w_j \geq w$  since  $e_j \in E_w$ , which is a contradiction. Thus,  $T_w$  has a girth of at least  $w$ . This implies that  $\Gamma(|V|, w) \geq |E \cup E_w| = |E| + |E_w| = |V| - 1 + |E_w|$  and the lemma follows. ■

**Corollary 1**  $|E_w| \leq \Gamma_P(|V|, w) - |V| + 1$  for all  $\text{CYCLE-WEIGHT}((V, E), S)$  such that  $(V, E \cup S)$  is planar, and all  $w$ .

**Proof:** In the proof of lemma 5,  $T_w$  is planar because it is a subgraph of planar graph  $V, E \cup S$ . Hence  $\Gamma(|V|, w)$  may be replaced by  $\Gamma_P(|V|, w)$ . ■

We now bound  $\text{CYCLE-WEIGHT}((V, E), S)$  by making use of bounds on  $\Gamma(n, w)$ , a well studied problem in extremal combinatorics. We first consider the case that the graph  $(V, E \cup S)$  is planar.

\*\*\* The following is an absolute misuse of the terminology that we have introduced because  
 \*\*\* of the definition of gamma - we can't introduce a restriction of the graphs in the definition  
 \*\*\* of gamma afterwards. We need to clean this up. (CAT: cleaned up above. We also misuse gamma to mean both the max and a valid upper bound on the max. I fixed that too. )

**Lemma 6**  $\Gamma_P(n, w) \leq \frac{wn}{w-2}$  for all  $n$  and  $w$ .

\*\*\* I have no idea about the following (don't know what faces are etc.)

**Proof:** Since the sum of the lengths of all faces of any planar graph  $G = (V, E)$  is at most  $2|E|$  and every face has a length of at least  $w$ , the number of its faces can be at most  $2|E|/w$ . The bound of the lemma follows from substituting this relationship in Euler's formula. ■

Note that the weight of any edge in  $S$  is at most  $|V|$ . Define  $E_{w,2w} = \{e_i \in S : w \leq w < 2w\}$ . Then, by corollary 1 and lemma 6 it holds that

\*\*\* In the following, we really need a floor on the upper bound of the sum but I guess we can get away without it (CAT: yes, yes) \*\*\* Someone needs to check that  $(\frac{2^i|V|}{2^i-2} - |V| + 1) \leq 3|V|/2^i$ . (CAT: I checked. It needs  $i$  big enough. It also needs  $|V|$  big enough. I fixed it.)

$$\begin{aligned}
\text{CYCLE-WEIGHT}((V, E), S) &\leq \sum_{i=1}^{\log |V|} 2^{i+1} |E_{2^i, 2^{i+1}}| \\
&\leq O(|S|) + \sum_{i=3}^{\log |V|} 2^{i+1} |E_{2^i}| \\
&\leq O(|S|) + \sum_{i=3}^{\log |V|} 2^{i+1} (\Gamma_P(|V|, 2^i) - |V| + 1) \\
&\leq O(|S|) + \sum_{i=3}^{\log |V|} 2^{i+1} \left( \frac{2^i |V|}{2^i - 2} - |V| + 1 \right) \\
&\leq O(|S|) + \sum_{i=3}^{\log |V|} 2^{i+1} 4|V|/2^i \\
&= O(|S|) + \sum_{i=3}^{\log |V|} 8|V| \\
&= O(|S|) + O(|V| \log |V|).
\end{aligned}$$

We now repeat the analysis for general graphs. In this case, we use a recent result by Alon, Hoory and Linial that states that any graph  $G = (V, E)$  with average degree  $d > 2$  has a girth of at most  $\log_{d-1} |V| [1]$ , resulting in the following lemma.

**Lemma 7**  $\Gamma(n, w) \leq n(n^{\frac{1}{w}} + 1)/2$  for all  $n$  and  $w$ .

**Proof:** Consider any graph  $G = (V, E)$  with  $|V| = n$ ,  $|E| \geq |V| + 1$  and a girth of at least  $w$ . Then, its average degree is  $d = 2|E|/n > 2$  and thus, according to the result by Alon, Hoory and Linial,  $w \leq \log_{2|E|/n-1} n$ . Solving this inequality for  $|E|$  yields the lemma. ■

This lemma allows us to bound  $\text{CYCLE-WEIGHT}((V, E), S)$  for general graphs. Using calculus, we can show that  $w(|V|(|V|^{\frac{1}{w}} - 1)) = O(|V| \log |V|)$  for  $|V| \geq w > \log^2 |V|$ . Using this fact with lemmata 5 and 7, we have

$$\begin{aligned}
\text{CYCLE-WEIGHT}((V, E), S) &= \sum_{i:w_i \leq \log^2 |V|} w_i + \sum_{i:w_i > \log^2 |V|} w_i \\
&\leq |S| \log^2 |V| + \sum_{i=2 \log \log |V|}^{\log |V|} 2^{i+1} |E_{2^i, 2^{i+1}}|
\end{aligned}$$

$$\begin{aligned}
&\leq |S| \log^2 |V| + \sum_{i=2}^{\log |V|} 2^{i+1} |E_{2^i}| \\
&\leq |S| \log^2 |V| + \sum_{i=2}^{\log |V|} 2^{i+1} (\Gamma(|V|, 2^i) - |V| + 1) \\
&= |S| \log^2 |V| + \sum_{i=2}^{\log |V|} 2^{i+1} (|V|(|V|^{\frac{1}{2^i}} - 1)/2 + 1) \\
&= |S| \log^2 |V| + \sum_{i=2}^{\log |V|} O(|V| \log |V|) \\
&= O((|V| + |S|) \log^2 |V|).
\end{aligned}$$

We now state these results as lemma.

**Lemma 8** *CYCLE-WEIGHT* $((V, E), S) = O((|V| + |S|) \log^2 |V|)$ . If the graph  $(V, E \cup S)$  is planar, *CYCLE-WEIGHT* $((V, E), S) = O(|V| \log |V|)$ .

### 3.5 Worst-Case Travel Bound

We are now ready to prove an upper bound on the worst-case travel distance of  $D^*$ .

**Theorem 2**  $D^*$  traverses  $O(|V| \log |V|)$  edges on connected graphs  $G = (V, E)$ . It traverses  $O(|V| \log^2 |V|)$  edges on connected planar graphs  $G = (V, E)$ .

**Proof:** According to Lemmata 2 and 4,  $D^*$  traverses at most  $3|V| + \sum_{i=1}^{k-1} d(v_i, v'_i)^{H^{i+1}} \leq 3|V| + \text{CYCLE-WEIGHT}((V, E'), S)$  edges, where  $|S| < |V|$  and  $(V, E' \cup S)$  is a subgraph of  $G$ . According to Lemma 8, it holds that  $\text{CYCLE-WEIGHT}((V, E'), S) = O((|V| + |S|) \log^2 |V|) = O(|V| \log^2 |V|)$  and, if  $G$  and thus  $(V, E' \cup S)$  are planar,  $\text{CYCLE-WEIGHT}((V, E'), S) = O(|V| \log |V|)$ . The theorem follows. ■

## 4 Conclusions

The robot-navigation methods that we have analyzed in this paper, Greedy Localization and  $D^*$ , are appealingly simple and easy to implement from a robotics point of view and appealingly complicated to analyze from a mathematical point of view. Our results, likewise, are satisfying in two ways. First, our tighter upper bounds on their worst-case travel distances guarantee that they cannot perform badly at all. Second, the gaps between the best known lower and upper bounds are now quite small, namely  $O(\log \log n)$  for Greedy Localization on grid graphs and  $D^*$  on planar graphs (including grid graphs), and  $O(\log n \log \log n)$  for  $D^*$  on arbitrary graphs.

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