A finite exact algorithm for epsilon-core membership in two dimensions

Craig A. Tovey

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1 Introduction

Let V be a finite configuration of voter ideal points in the Euclidean plane. For given $\epsilon > 0$ a point $x \in \Re^2$ is in the ϵ-core if for all $y \neq x$, $||v-x|| \leq ||v-y|| + \epsilon$ for a simple majority (at least $|V|/2$) of voters $v \in V$. Let $\epsilon(x)$ denote the the least ϵ for which x is in the ϵ-core. Thus $\epsilon(x) = 0$ if and only if x is a core point. The least ϵ for which the ϵ-core is nonempty is denoted ϵ^*.

This paper provides a finite algorithm, given V, x, and ϵ, to determine whether x is in the ϵ-core. By bisection search, this yields a convergent algorithm, given V and x, to compute $\epsilon(x)$. If the function $\epsilon(x)$ were strictly convex this would lead to a convergent algorithm to compute ϵ^* and the corresponding point. However, we prove that the function $\epsilon(x)$ is not convex in general.

2 Background

In the standard 2D spatial model of social choice, points in $\Re^2$ represent policies. Each voter is associated with a point in the plane which represents his ideal point. When faced with two alternatives, i.e. points in the plane, a voter prefers the alternative that is closer to his ideal point, with respect to the Euclidean norm. This model is widely used in both theoretical and empirical studies of committees, legislative bodies, and political parties.

A core point is one such that no other point is preferred to it by the group. If a core point exists, it is the natural social choice outcome, from both normative and descriptive perspectives. However, if majority voting is used to define the group preferences, a core point usually does not exist [9, 14]. The absence of a core leads normative researchers to propose other group preference voting rules such as supermajority voting, and leads descriptive researchers to propose other solution concepts such as the ϵ-core, the finagle point, the yolk, the strong or Copeland point, and the uncovered set.

It is important to be able to compute these other solution concepts, both to understand their properties and to enable empirical work. At present, the state of the art is mixed. For the ϵ-core, a heuristic is given in [2], but little is known rigorously. The finagle point has been analytically determined for $|V| = 3$ [15] but little else is known. For the yolk there is a polynomial algorithm [11], a commercially available implementation [4], and an asymptotically accurate approximation by a linear program [5]. For the strong point there is a beautiful analytical formula [8] and a commercially available implementation [4]. For the uncovered set, there is a grid-search estimation method [1], and a similar commercially available brute force implementation [4], but little is known mathematically (see [7]) except for bounds in terms of the yolk [6].

This paper addresses the problem of computing the ϵ-core. Motivation for the concept of the ϵ-core is found in [3, 2, 13, 10, 12]; empirical support is given in [10]. The predominant idea is that an incumbent proposal receives an advantage, or "benefit of the doubt"[3] of $\epsilon > 0$ over challenger proposals, because of a cost of change, decision-making, or voting.

3 The algorithm

Geometric overview: For each voter $v \in V$ draw a circle with center v whose boundary contains x, i.e. of radius $||v - x||$. The points inside the circle are those v prefers to x. The winset of x is the set of points (excluding x itself) that are contained in more than $|V|/2$ of the circle interiors. Shrink the radius of each circle by ϵ. The point x is in the ϵ-core iff no point is contained in at least $|V|/2$ of the shrunken circles.

Suppose y defeats x when x is given an ϵ advantage. Then there exists a subset U of V, with $|U| > |V|/2$, such that for all $u \in U$, y is within the shrunk circle with center u, i.e. $||y - u|| + \epsilon < ||x - u|| \; \forall u \in U$. By Helly's theorem, and because circle interiors are convex, and we are in $\Re^2$, such a y exists iff each triplet of shrunk circles has nonempty intersection. That is,

$$\begin{aligned} \exists y: \quad & ||y - u|| + \epsilon < ||x - u|| \; \forall u \in U & (1) \\ \Leftrightarrow \quad & \forall u_1, u_2, u_3 \in U \exists y : ||y - u_j|| + \epsilon < ||x - u_j||, j = 1, 2, 3. & (2) \end{aligned}$$

This gives us a finite exact way to test whether $x \in \epsilon$-core. Build the $|V|$ shrunk circles. Enumerate all subsets U of V of cardinality $\lceil |V|/2 \rceil$. For each such U, check every triplet of voters in U for nonempty intersection. Iff for some U every triplet has nonempty intersection, x is not in the ϵ-core.

Triplets can be checked as follows. Define the shrunk circle radius of u_j as $r_j \equiv ||x - u_j|| - \epsilon$.

1. First, ensure that each pair of circle interiors intersects:

   $$\forall 1 \leq i < j \leq 3 \qquad ||u_i - u_j|| < r_i + r_j.$$

   If not, the test fails.

2. Second, check to see if any circle center is contained in the intersection of the other two circles:

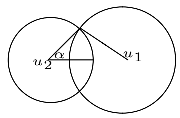

Figure 1: Rotate the radius by α to get an endpoint where the two circle boundaries intersect

$$||u_3 - u_1|| < r_1 \text{ and } ||u_3 - u_2|| < r_2;$$

and similarly for u_2 and u_1. If any of these is true, the test succeeds.

3. Third, at this stage the closest point from u_3 to the closure of the intersection of the first two circles must be on the boundary of their intersection. The boundary consists of two arcs which meet at two endpoints. Let e denote one of the endpoints. Consider the triangle with vertices u_2, u_1, and e. Denote the angle at u_2 by α. Since we know all three side lengths, we know α from the law of cosines. Therefore, we compute the two endpoints (x_2, y_2) and (x_3, y_3) of the boundary by the formulae:

$$\begin{aligned}
\cos\alpha &= (r_2^2 + ||u_1 - u_2||^2 - r_1^2)/(2r_2||u_1 - u_2||) \\
\sin\alpha &= \sqrt{1 - \cos^2\alpha} \\
(x_1, y_1) &= (u_1 - u_2)r_2/||u_1 - u_2|| \\
(x_2, y_2) &= u_2 + (\cos\alpha x_1 + \sin\alpha y_1, \cos\alpha y_1 - \sin\alpha x_1) \\
(x_3, y_3) &= u_2 + (\cos\alpha x_1 - \sin\alpha y_1, \cos\alpha y_1 + \sin\alpha x_1).
\end{aligned}$$

See Figure 1. The vector x_1, y_1 points from u_2 to u_1 and has length r_2. The two endpoints are then found by rotating x_1, y_1 by α clockwise and counterclockwise. If either endpoint is within the third circle, the three circles have an intersection:

$$||(x_2, y_2) - u_3|| < r_3 \quad \text{or } ||(x_3, y_3) - u_3|| < r_3.$$

4. Fourth and finally, if the algorithm gets to this step, the first two circles have a nonempty intersection that does not contain the center of the third circle. Also, neither endpoint of the intersection is contained in the third circle. Therefore, if the third circle has a point in common with the intersection of the two others, the closest point p from u_3, the center of

the third circle, to the closure of the intersection of the first two circles, must lie on one of the open arcs that, together with the endpoints, compose the boundary of the intersection. Hence,

$$p = u_i + r_i \frac{u_3 - u_i}{||u_3 - u_i||}$$

either for $i = 1$ or $i = 2$. If either of these possible choices for p satisfies $||p - u_3|| < r_3$ then the three circles have common intersection. If this fourth test fails, the three circles possess no common intersection.

4 Implementation

An implementation in C for 3, 5, 7, and 9 voters is available on the author's website at http://www.isye.gatech.edu/˜ctovey/. For given x and ϵ the procedure for determining membership of x in the ϵ-core is as given in the previous section. To minimize $\epsilon(x)$ the code performs local search repeatedly with decreasing step sizes. That is, the code finds a local optimum with respect to an initial step size, halves the step size, and restarts local search from the local optimum just found, using the new smaller step size. The smallest step size used is less than 10^{-8} at which point the search terminates. The starting point for the search is the center of mass of the voter ideal points, plus any offset desired by the user. The offset is useful when searching for multiple local optima.

If desired, the code also produces a snapshot of $\epsilon(x)$ values around the local optimum found, and tests for convexity amongst those values.

5 Nonconvexity of $\epsilon(x)$

Here is a counterexample to convexity, in which the function $\epsilon(x)$ has at least three local minima. There are 5 ideal points, at $(4, 8); (6, 3); (11, 14); (1, 5); (9, 2.5)$. There are local optima at $5.504, 4.998$ and at $6.065, 5.935$ and at $5.469, 4.525$ with $\epsilon()$ values 0.261 and 0.235 and 0.251 respectively. The figures below show the locations of the ideal points (Figure 2), a 3D rendering of the function (Figure 3), and a contour map of the function (Figure 4). In the contour map, the smallest level set is disconnected, which makes it clear that there are multiple local minima.

In contrast, the yolk radius function $r(x)$ is easily seen to be convex because it is the maximum of a set of convex functions, the distances from x to median lines [11].

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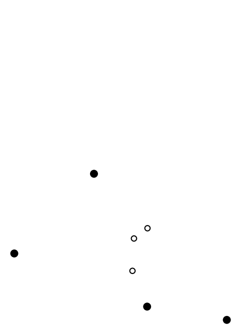

Figure 2: These 5 ideal points (solid circles) yield a nonconvex epsilon-function. Open circles are local minima.

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