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Dynamical Convergence in the Euclidean Spatial Model

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Abstract

The ϵ-core in Euclidean spatial voting is the set of points that cannot be dislodged by a point more than ϵ closer to a simple majority of voter ideal points. If ϵ is greater than the yolk radius of the set, then the ϵ-core is nonempty. If ϵ exceeds twice the yolk radius, then there are no global intransitivities and any sequence of proposals starting from x will reach the ϵ-core from x in at most $||x||/(2\epsilon - r)$ steps. An analogous result assures convergence of any supermajority voting sequence, subject to restrictions including a minimum distance between proposals. The results are valid in any dimension.

1 Introduction

A classic result of Kramer (1977) for the Euclidean spatial model shows that repeated proposals by competing vote-maximizing parties will produce sequences converging to the minmax (Simpson-Kramer) set. This "dynamical convergence" helps establish the importance of the minmax set: the solution set not only possesses attractive normative properties (Simpson (1969),Kramer (1977),Slutsky (1979)); it also possesses dynamically attractive properties in that natural forces of majority voting tend to drive a group decision towards it.

Similarly, Miller (1980) (and Miller (1983)) has shown that under a variety of agenda settings, the outcome of (strategic) majority voting will lie in the uncovered set, and this dynamical property is crucial to this solution set's importance.

Ferejohn et al. (1984) found a dynamical property for another solution set, the yolk. They showed, roughly speaking, that if proposals are made at random with majority voting, then the incumbent proposal will frequently be contained in the yolk.

All these results are of the same type: they demonstrate that some voting process, taking place over time, leads the group decision towards a particular solution set. The aim of this paper is to derive a dynamical result of this type for a new solution set, the ϵ-core. This solution set has been proposed recently by Salant and Goodstein (1990) to fit empirical data better, and by Tovey (1991b) to incorporate some "friction" into the spatial model. The idea is that the incumbent or status quo has some amount, $\epsilon > 0$, of advantage versus the alternatives. Voters will vote for an alternative only if it is at least

ϵ better (closer) than the incumbent. The parameter ϵ can be interpreted as a measure of friction, resistance to change, or incumbency advantage. This solution set is equivalent to the concept of an ϵ-equilibrium or an ϵ-core in game theory (Shubik and Wooders (1983),Wooders (1983)), and shares much of the same motivation.

In this paper the yolk is used not as a solution set, but as an asymmetry measure for configurations of voter ideal points. The larger the yolk radius, the more skewed is the configuration.

The main result (Theorem 1) states that if ϵ is sufficiently large compared with the yolk radius, then there are no global intransitivities. Moreover, any sequence of proposals starting from x will reach the ϵ-core in a number of steps which is a function of ϵ, the yolk radius, and the distance from x to the yolk center. The result holds for any number of voters in any number of dimensions.

A crucial feature of Theorem 1 is its linkage between the "skewness" of the configuration of voter ideal points (as measured by the yolk radius), and the degree of stability enjoyed by the voting process. From Theorem 1 we may infer a plausible qualitative prediction that if the frictions or resistance to change represented by ϵ increase, then the likelihood of stability, and speed of convergence, will increase. On the other hand, the more skewed the voter configuration, the slower and less likely is convergence. If we are to construct an effective predictive theory of social choice, our models should accomodate this range of observed group behavior (from rapid convergence to equilibrium, to instability), and should link these outcomes with measurable characteristics of the group (such as distributional characteristics of individual preferences).

It is worth emphasizing that Theorem 1 gives an explicit upper bound on the number of steps needed to reach equilibrium. This aspect of the result is unusually strong. As the examples in section 3 illustrate, the absence of global intransitivities is not in itself sufficient to assure finite or even infinite convergence.

On the other hand, a significant weakness of the results in this paper lies in the assumption of sincere voting. It would be very interesting to see if the results could be carried through in some form if strategic behavior were incorporated into the model.

In section 2 we formally state the definitions, assumptions, and prove the main convergence result for the ϵ-core.

In section 3 we reexamine Kramer's result. The convergence there is not fully satisfactory in two ways: first, if two parties were to follow the vote-maximizing strategy, each could make arbitrarily small steps and there might be no appreciable movement away from the initial position; second, if a point in the minmax set ever were reached, the process would jump out of the minmax set on the next step.

The aim of section 3 is to develop an alternative dynamical convergence result for the minmax set, that does not suffer from these two problems. We attempt to find an analogous version of Theorem 1 for supermajority voting, that guarantees that any sequence of proposals will reach a supermajority core point in a finite number of steps, determined *a priori*. We show by several examples that this is not possible without making several more constraining assumptions. The main result of the section is Theorem 5: if we assume supermajority voting at a level a bit higher than the minmax level, require a minimum distance between proposals, and make one additional regularity assumption, then any sequence of proposals must reach the core within a certain number of steps.

2 Paths to the ϵ-core

First we give the necessary definitions.

The set V is any finite collection of (not necessarily distinct) points in $\Re^m$. These are the ideal or bliss points of the voters. The proposal space is assumed to be all of $\Re^m$. Throughout we will let $||y||$ denote the length or Euclidean norm of a vector $y \in \Re^m$.

We suppose that there is some *voting rule* such as simple majority by which the voters can decide between two alternatives. The voting rule must be decisive, but it need not be neutral: one of the alternatives is designated the *incumbent*, the other is the proposed alternative, and the voting rule need not treat the two alike. For example, if the two alternatives receive exactly the same number of votes, then under simple majority voting the incumbent is the winner.

Next comes the crucial definition of a sequence of proposals. Imagine any iterative process where the initial incumbent proposal is x; an alternative y is proposed and the voters decide between y and x. If y defeats x then y becomes the new incumbent proposal, and the process has completed a step or iteration. If an incumbent that can not be defeated is reached, the process terminates. Regardless of the means by which alternatives are proposed, the history of the process is summarized by the sequence of proposals $\{x, y, \ldots\}$ where each point in the sequence defeats its predecessor.

Thus, we define a *sequence of proposals* as any sequence of points $\{x^1, x^2, \ldots\}$

such that alternative x^{i+1} would defeat incumbent x^i $\forall i > 1$. We say that the sequence *starts at* x^1. If for some j the point x^j cannot be defeated then the sequence *terminates* at x^j. The point x^j is called a *core point* (with respect to the operant voting rule) because it cannot be dislodged. For example, if voting is according to simple majority rule, then a proposal sequence, if it terminates, does so at a Condorcet point.

Our assumption regarding the actions of the voters is stated next.

Assumption I: a voter with ideal point $v \in \Re^m$ *will vote for a proposal* $y \in \Re^m$ *iff the incumbent proposal* x^i *satisfies* $||x^i - v|| > ||y - v|| + \epsilon$.

The ϵ-core is the set of $x \in \Re^m$ such that if x is the incumbent, then x can not be defeated by simple majority rule when voting is according to Assumption I. When $\epsilon = 0$ a point in the ϵ-core would be a Condorcet point with respect to majority voting in the classic Euclidean model. Usually there are no such points (Plott (1967)). But, for any V the ϵ-core is nonempty for sufficiently large ϵ.

Nonemptiness is perhaps the very least one would desire of a solution concept. An additional desirable property is global transitivity, which essentially means the absence of cycles (intransitivities). We define it formally in terms of a proposal sequence: a voting rule has no global intransitivities iff there does not exist a sequence of proposals $\{x^i : i = 1, \ldots\}$ and an integer $j > 1$ such that $x^j = x^1$. In other words, it is not possible for a sequence of proposals to return to a point. This property is important because if it fails to hold then a voting process could cycle indefinitely even if the core were nonempty. As we will see in Section 3, however, global transitivity is not sufficient to guarantee even infinite convergence to the core.

We also need to define the yolk of the configuration V. Let h denote any (full-dimensional) hyperplane in $\Re^m$, and let h^+ and h^- denote the two closed halfspaces into which h divides $\Re^m$. Then h is a *median* hyperplane of V iff $|V \cap h^+| \geq |V|/2$ and $|V \cap h^-| \geq |V|/2$. That is, each closed halfspace must contain at least half the voter ideal points.

A yolk of V is a ball in $\Re^m$ of smallest radius that intersects every median hyperplane of V. The yolk was originally developed by McKelvey (1986) for $|V|$ odd and is unique in that case. Following Koehler (1990) and Stone and Tovey (1992) we permit an even number of voters as well. In this case (see Tovey (1992)) the yolk may not be unique, although its radius is unique. In the theorem that follows the statement is true for any yolk. The only property that we use is that the yolk intersects all median hyperplanes of V. As indicated in Section 1, the yolk radius r is a measure of how skewed V is. If $r = 0$ then V is symmetric in the sense of Plott (1967) or McKelvey and Schofield (1987). The greater r the more skewed is V.

We can now state the main result of the paper.

Theorem 1. *Let V be any set of voter ideal points in $\Re^m$, with yolk center and radius c and r, respectively. Suppose voting occurs according to Assumption I.*

(i) If $\epsilon > r$ then the (ϵ-) core is nonempty.

(ii) If $\epsilon > 2r$ then there are no global intransitivities and moreover, for all $x \in \Re^m$, any sequence of proposals starting at x must reach a core point in at most

$$\left\lceil \frac{||x - c|| - r}{\epsilon - 2r} \right\rceil$$

steps, where $\lceil a \rceil$ denotes the least integer greater than or equal to a.

Proof: If the dimension $m = 1$ the proof is trivial. Hereafter we take $m \geq 2$.

(i) If $\epsilon > r$ then c is in the ϵ-core. For consider any alternative y to the incumbent yolk center c. Let

$$\tilde{y} = c + r\left(\frac{y - c}{||y - c||}\right).$$

The point $\tilde{y}$ lies on the surface of the yolk and is collinear with y and c. Note that $\tilde{y}$ may lie between y and c, or y may lie between $\tilde{y}$ and c, but c is not between y and $\tilde{y}$.

The halfplane h tangent to the yolk at $\tilde{y}$ defines a closed halfspace h^+ containing the yolk. On the one hand, all voters whose ideal points are in h^+ will vote for c. But on the other hand, the halfspace h^+ contains the yolk, and therefore contains at least $|V|/2$ ideal points. (If h^+ contains the yolk of V then h^+ contains the median hyperplane of V which is parallel to h.) Thus no proposal y brought against c can muster more than $|V|/2$ votes, whence c is undefeated.

(ii) Now suppose incumbent proposal x is defeated by proposal y and $\epsilon > 2r$. Our principal claim is that y must be at least $(\epsilon - 2r)$ closer to the yolk center c than x is. That is, we claim $||y - c|| \leq ||x - c|| - \epsilon + 2r$. For readability we let $d(x, y)$ denote the distance $||x - y||$ between x and y.

Let S denote the hyperboloid defined as

$$S \equiv \{s \in \Re^m | d(x, s) = d(y, s) + \epsilon\}.$$

The hyperboloid S separates $\Re^m \setminus S$ into two open regions: let S^+ denote that open region containing the point y. Note that by Assumption I, the

Figure 1: The plane containing x, y, c, and z around here

voters who vote for y over x are precisely those whose ideal points are located in S^+.

If $c \in S^+$ or $c \in S$ then $d(y, c) \leq d(x, c) - \epsilon \leq d(x, c) - \epsilon + 2r$ and our claim would be proved. So suppose c is on the other side of S.

At any point in S there is a supporting hyperplane of S and an associated unit length normal vector. Let z be the point in S whose unit normal vector points to the yolk center c. That is, its normal vector is $(c-z)/||c-z||$. Let h denote the supporting hyperplane of S at z and let h^+ denote the closed halfspace defined by h containing y.

Since h^+ contains S^+ and y defeats x, we have $|V \cap h^+| \geq |V \cap S^+| > |V|/2$. Therefore, either h or some hyperplane parallel to h which intersects S^+ is a median of V. This fact, together with the property that $(c-z)$ is normal to h, implies that $d(z, c) \leq r$.

We now restrict our attention to the 2 dimensional plane generated (as the affine hull) by x, y, and c. It is clear that z lies on this plane as well. See Figure 2.

We have shown that $d(c, z) \leq r$. Also, by the definition of S, we have $d(z, y) = d(z, x) - \epsilon$.

By the triangle inequality and the preceding,

$$d(c, y) \leq d(c, z) + d(z, y) \leq d(z, x) - \epsilon + r.$$

Also by the triangle inequality,

$$d(z,x) \leq d(z,c) + d(c,x) \leq d(c,x) + r \Longrightarrow d(y,c) \leq d(x,c) - \epsilon + 2r.$$

This proves the claim. Every iteration of the Condorcet voting process brings us at least $(\epsilon - 2r)$ closer to the yolk center c.

Since $\epsilon - 2r > 0$, we have immediately from the claim that there are no global intranstivities. Also it follows that the process will terminate in $\lceil d(x,c)/(\epsilon - 2r) \rceil$ or fewer steps.

To improve the bound to the statement of the theorem, we notice that when $\epsilon > 2r$, the entire yolk is contained in the ϵ-core. This follows from an argument almost exactly the same as in the proof of (i), above.

As a consequence, the voting process is sure to terminate once the incumbent is within distance r of c. If K is the necessary number of steps, then K integer and

$$d(x,c) - K(\epsilon - 2r) \leq r$$

implies we may take

$$K = \left\lceil \frac{||x-c|| - r}{\epsilon - 2r} \right\rceil$$

as desired. ■

3 On Kramer's result and paths to supramajority cores

One of the attractive features of Theorem 1 is the convergence to the core in a finite number of steps. In contrast, Kramer's dynamical process does not necessarily converge. For in that process, two parties alternate making proposals. Each party proposes that $y \in \Re^m$ which maximizes the number of votes y would get against incumbent proposal x. It is easy to see that if y is such a point with respect to x, then so is their midpoint, $(y+x)/2$. Indeed the latter point is a more conservative choice if for example there is any uncertainty about V. Therefore, the distance between successive proposals can be arbitrarily small and convergence may fail even in the limit (see Theorem 3). Kramer's process is also not fully satisfactory because if the incumbent were the minmax set, then the succeeding proposal would not. That is, if the process does get into the minmax set, it may pop back out in the next step.

In this section we explore the possibility of an alternative minmax set dynamical convergence result which is more satisfactory in these respects. We will eventually succeed, but only by losing much of what was desirable in Kramer's model.

The first condition we examine is intended to prohibit arbitrarily small steps. We consider convergence properties of simple majority voting under a simple condition related to Assumption I:

Assumption II: Voters vote sincerely. Any proposal x^{i+1} offered against incumbent proposal x^i must satisfy $||x^i - x^{i+1}|| > \delta$. If a voter is indifferent between x^{i+1} and x^i the vote is cast for the incumbent.

This assumption enforces a minimum distance between proposals. It is suggested informally at least as early as Tullock (1967). It can be thought of as an institutional restriction; see also Tovey (1991b) for a formal treatment.

Any point x that would be undominated under Assumption II would be in the ϵ-core with $\epsilon = \delta/2$. Clearly the converse is false in general. From the

Figure 2: Cycle with Large Steps goes around here

standpoint of core existence, Assumption II is thus weaker than Assumption I when $\epsilon = \delta/2$. The dynamical convergence properties that follows from Assumption II are unfortunately much weaker, as Theorem 2 shows.

Theorem 2. *Let V be any set of voter ideal points in $\Re^m$, $m \geq 2$, with yolk center and radius c and r, respectively. Suppose voting occurs according to Assumption II, with the winner determined by simple majority.*

(i) If $\delta > 2r$ then the core is nonempty.

(ii) For all $\delta > 0$ there may be global intransitivities. Indeed for all $r > 0$ there exists V with yolk radius r, such that for all $\delta > 0$ transitivity fails.

Proof: (i) This follows by exactly the same argument of part (i) of Theorem 1.

(ii) Let V contain three points at the vertices of an equilateral triangle (note $m \geq 2$) with center 0. In particular V consists of the points $(0, 2), (\sqrt{3}, -1)$, and $(-\sqrt{3}, -1)$. See Figure 2. Let $M > 1$ be arbitrary. Situate 3 points, a,b, and c just off the rays emanating from the origin through the triangle vertices. In particular $a = \{1, 2M\}; b = \{(M - 1/2)\sqrt{3}, -(M + 1/2)\}; c = \{-M\sqrt{3} - 1/2, \sqrt{3}/2 - M\}$.

Obviously a defeats c defeats b defeats a, and these points can be made arbtrarily distant from each other by increasing M. Moreover the set V can be scaled down to have arbitrarily small yolk radius without affecting the outcomes. ■

Theorem 2 shows that while Assumption II may be strong enough to assure a nonempty core, it is too weak to eliminate global intransitivities. So we have no assurance that a sequence of proposals will ever reach or even approach the core. We next consider supermajority voting. Theorem 3 shows that supermajority voting is a little stronger than Assumption II: it assures a nonempty core and eliminates global intransitivities. However, it is not enough to guarantee infinite convergence.

Assumption III: Voters vote sincerely. Any proposal x^{i+1} offered against incumbent proposal x^i must receive more than $\alpha|V|$ votes to defeat x^i. If a voter is indifferent between x^{i+1} and x^i the vote is cast for the incumbent.

The set of points which can not be defeated when voting takes place under Assumption III is called the α-core. The smallest value α^* for which the α-core is nonempty is known as the minmax number.

Theorem 3: *Suppose voting takes place according to Assumption III with $\alpha \geq \alpha^*$. Then there are no global intransitivities, but a sequence of proposals may fail to reach the α-core, even in the limit.*

Proof: By assumption there exists at least one core point $w \in \Re^m$. Suppose y defeats x. Let h denote the hyperplane normal to and bisecting the segment $\overline{xy}$, and let h^+ denote the open halfspace defined by h which contains y. According to Assumption III, there must be more than $\alpha|V|$ ideal points in h^+. That is, $|V \cap h^+| > \alpha|V|$.

We claim that $w \in h^+$. For if not, the point $w + \tau(y - x)$, for sufficiently small $\tau > 0$, would defeat w just as y defeats x. Therefore, as in Theorem 1, $d(y, w) < d(x, w)$ and there can be no global intransitivities.

The last part of the theorem is trivial. Place all the ideal points at 0 and let the proposal sequence be a set of points colinear with 0 such that $||x^i|| = 1 + 2^{-i}$.

■

The example that frustrates hopes of convergence in Theorem 3 violates the minimum inter-proposal distance property of Assumption II. This suggests using both Assumptions II and III in the hope of achieving a “nice” convergence property as in Theorem 1. Unfortunately, combining Assumptions II and III is still not quite enough.

Theorem 4: *Suppose voting takes place according to both Assumption II and Assumption III with $\alpha \geq \alpha^*$. Then there are no global intransitivities, but for all $\delta > 0$ a sequence of proposals may fail to reach the α-core, even in the limit.*

Proof: By Theorem 3 there are no intransitivities. We modify the construction of Theorem 2 to show the failure of convergence. Place a single voter ideal point at 0, and place 2 ideal points at each of the three triangle vertices of V in figure 2. Let $\alpha = 4/7$ so a proposal needs 5 votes to defeat an incumbent. The point at 0 is undefeated. The cycle a beats c beats b beats a is no longer intact, because the voter at 0 is indifferent among a, b, and c.

For sufficiently large M let $x^0 = a(M+1)/M; x^1 = b(M+1/2)/M; x^2 = c(M+1/4)/M; x^3 = a(M+1/8)/M; \cdots$. Then $||x^{i+1}|| < ||x^i||$, and so x^i is defeated by x^{i+1}, but every point in the sequence is arbitrarily far from the α-core. ■

It is possible to add one more condition to those of Theorem 4 and finally achieve finite convergence. The necessary definition and condition follow.

Let $x, r \in \Re^m; ||r|| = 1; \tau \in \Re; \tau > 0$. Let $W(x, r, \tau) = \{y \in \Re^m | x \cdot r \leq y \cdot r \leq x \cdot r + \tau\}$, which may be visualized as a wafer of width τ wedged between two parallel hyperplanes whose normal vector is r.

Assumption IV: There exists $\tau > 0$, a point $z \in \Re^m$, and an integer β such that z is in the $(\alpha - \beta/|V|)$-core, and such that for all $||r|| = 1$, we have $|V \cap W(z, r, \tau)| \leq \beta$.

Geometrically, this assumption says that there is a point z which is a core point with respect to a smaller supramajority number $\hat{\alpha} < \alpha$, and there is a width $\tau > 0$ such that none of the wafers supported by z contain too many (a fraction $\alpha - \hat{\alpha}$) points of V.

Though Assumption IV looks awkward, it is essentially a regularity requirement. It can be easily satisfied if the ideal points V are in general position by setting $\beta = m$, selecting z as the minmax point, and setting $\alpha = \alpha^* + \beta/|V|$. A set of points in $\Re^m$ is in general position if no k-dimensional hyperplane contains more than k points, *i.e.,* no 2 points are coincident, no 3 points are collinear, etc., for all $1 \leq k < m$.

For example, if the U.S. Senate in 2 dimensions is in general position, and has $\alpha^* \leq 64.5$ (see Caplin and Nalebuff (1988) and Tovey (1991a)), then $\alpha = 2/3$ satisfies Assumption IV.

Theorem 5: *Suppose voting takes place according to all the Assumptions II, III, and IV. Then there are no intransitivities, and moreover, for all $x \in \Re^m$, any sequence of proposals starting at x must reach an α-core point in*

$$\left\lceil \frac{||x-z||^2}{2\delta\tau} \right\rceil$$

or fewer steps.

Proof: The first statement is a corollary to Theorem 3. As usual, suppose y defeats x, and let h denote the hyperplane bisecting the line segment between x and y. Then as proved in Theorem 3, z is in the open halfspace h^+ defined by h containing y.

But now consider the "wafer" $W(z, r, \tau)$ where $r = (x-y)/||x-y||$: it contains at most β ideal points. If the wafer intersects h, then $h^+ \setminus W(z, r, \tau)$ contains more than $\alpha|V| - \beta$ ideal points, contradicting Assumption IV that z is in the $(\alpha - \beta/|V|)$-core. Therefore $W(z, r, \tau)$ does not intersect h.

It follows that the distance from z to h is greater than τ.

We now restrict our attention to the 2 dimensional triangle defined by vertices x, y, and z. Let L denote the length of the altitude from side $\overline{xy}$ to vertex z. From the last paragraph we know that this altitude intersects the side at distance $\zeta, \zeta > \tau$ from the midpoint of the side. Then

$$||x-z||^2 = L^2 + (||x-y||/2 + \zeta)^2;$$

$$||y-z||^2 = L^2 + (||x-y||/2 - \zeta)^2.$$

Subtracting and applying $||x-y|| \geq \delta$ from Assumption 2, we get

$$||x-z||^2 - ||y-z||^2 = 4\zeta||x-y||/2 \geq 2\delta\tau$$

.

Thus the square of the distance from x to z must decrease by an amount independent of x. Therefore a core point must be reached in $||x - z||^2/2\delta\tau$ steps as claimed.

■

As with Theorem 1, the qualitative predictions of Theorem 5 are plausible. Convergence should be faster as the minimum inter-proposal distance δ increases, and as τ, which is a kind of ill-conditioning number, increases. Also note that Theorem 5 requires α somewhat greater than α^* in order to assure the stronger convergence, which again is plausible. Overall, Theorem 5 has disappointingly strong conditions, but Theorems 2–4 imply that weaker conditions along these lines would not suffice.

As stated in the introduction, these results do not take voter sophistication (strategic voting, deciding whether to vote) into account, and such extensions would be quite interesting.

Theorem 5 applies to a heavily modified version of Kramer's model of party competition: parties still make proposals to maximize their votes against the incumbent, but the voting rule is α-majority rather than simple majority, proposals must be at least δ apart, and regularity condition IV must hold. And then, we only get convergence to the α-core, not the α^*-core.

It would be interesting to see if finite convergence to the α^*-core could be obtained for a model more similar to Kramer's than that given here. One would only have to prove convergence for all the vote-maximizing proposal sequences, instead of for all proposal sequences. However, I do not know how to proceed. One problem is that the proposal sequence in the proof of Theorem 3 is vote maximizing. Therefore Assumption III does not suffice. Another problem is that if the parties maximize votes, supramajority voting would not affect the actual sequence until and unless it reaches the core. Therefore it seems impossible that Assumption II would suffice to guarantee convergence, because the vote maximizing process could begin at (or reach) a point at distance less than δ from the minmax point.

In conclusion, the ϵ-core enjoys powerful convergence properties if ϵ is large enough compared with the asymmetry of the voter configuration. Of particular interest are the explicitly bounded finite convergence, and the qualitative predictions involving (possibly) measurable quantities. With supra-majority voting, the core acquires a similar powerful convergence property, but only if several additional restrictions are enforced.

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