The Finagle Point and the Epsilon-Core: A Comment on Bräuninger's Proof

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The Finagle point (Wuffle et al., 1989), the yolk (McKelvey, 1986; Ferejohn et al., 1984) and the epsilon-core (Wooders, 1983; Salant and Goodstein, 1990; Tovey, 2010; Feld and Grofman, 1991) are three solution concepts that have been proposed to predict the outcome of majority voting in the Euclidean spatial model. Relationships among these three concepts are investigated in (Bräuninger, 2007). In particular, it is claimed there that the Finagle point is within the epsilon-core. The main purpose of this note is to point out that the proof of this claim has a significant logical gap. I do not know whether or not the claimed result is true, but I will show that the argument given there is not sufficient to prove it.

Definitions: We are given a odd cardinality set of voter ideal points in $\Re^2$. A line is *median* if at least half the voter ideal points lie in each closed halfplane defined by the line. For two points x and y we say $x \succeq y$ (x is majority preferred to y) if at least half the voter ideal points are not farther from x than they are from y. Similarly, for two points x and y we say $x \succeq^\epsilon y$ (x is epsilon-preferred to y) if at least half the voter ideal points are not more than ϵ farther from y than from x. For $x \in \Re^2$, the ball of radius δ around x is denoted $D(x, \delta) \equiv \{y : ||x - y|| \leq \delta\}$.

The three solution concepts can be defined in terms of three functions denoted $f(x)$, $r(x)$, and $\epsilon(x)$, respectively. For $x \in \Re^2$, the *finagle radius* of x, denoted $f(x)$, is the least $\delta \geq 0$ such that $\forall y \neq x \ \exists x' \in D(x, \delta)$ such that $x' \succeq y$. The yolk radius of x, denoted $r(x)$, is the least $\delta \geq 0$ such that every median line is within distance δ of x. The epsilon-core radius of x, denoted $\epsilon(x)$, is the least $\epsilon \geq 0$ such that $\forall y \neq x \quad x \succeq^\epsilon y$. The point x is a core point iff $f(x) = 0 \Leftrightarrow r(x) = 0 \Leftrightarrow \epsilon(x) = 0$. Usually, however, no core point exists (Plott, 1967).

The following theorem states basic relationships among the three functions. The finagle radius is sandwiched between the ϵ and yolk radii. The theorem is stated for all dimensions because its proof does not depend on the dimension being two.

Theorem 1: For any configuration of voter ideal points in $\Re^n$, for all $x \in \Re^n$, the following inequalities hold:

$\epsilon(x) \leq f(x) \leq r(x)$.

Proof: First we prove $\epsilon(x) \leq f(x)$. This has already been proved by Bräuninger (Bräuninger, 2007). Here we give a shorter proof. In words, if for any alternative $y \neq x$, x can defeat y by "finagling" to some point x', where $||x - x'|| \leq \epsilon$, then $x \succeq^\epsilon y$ for all $y \neq x$. This follows from the triangle inequality, because shifting by a distance $\leq \epsilon$ can not confer an advantage of more than ϵ. Referring to Figure 1, z is the ideal point of a voter who prefers y to x. If x finagles his location to x', however, z either prefers x' to y or, at worst, is indifferent between them. Then z can't prefer y to x by more than ϵ.

Second, we prove $f(x) \leq r(x)$. Let $y \neq x$ be arbitrary. We must show that $w \succeq y$ for some $w \in D(x, r(x))$. For y such that $||y - x|| \leq r(x)$ this follows

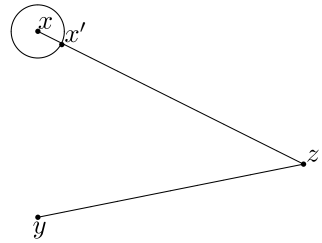

Figure 1: The circle has radius ϵ. The point y is ϵ closer to z than is x.

trivially by choosing $w = y$ because $y \succeq y$. For y such that $||y - x|| > r(x)$, let H denote the median hyperplane normal to the line $\overline{yx}$. In two dimensions H is the median line perpendicular to the line $\overline{yx}$. By the definition of $r(x)$, the median H must pass through the ball $D(x, r(x))$. Let $w = H \bigcap \overline{yx}$. Then $w \in D(x, r(x))$ because w is the closest point in H to x. Let v be any voter ideal point that is part of the majority of voters in the halfspace defined by H that does not contain y. Then $||w - v|| \leq ||y - v||$, since w is on the median H and y is on the other side of H from v by definition. This is true for a majority of the ideal points v. Therefore $w \succeq y$ as desired. This completes the proof of the theorem.

The Finagle point is defined as the point at which $f(x)$ is minimized (though, strictly speaking, it has never been shown that $\arg\min f(x)$ occurs at a single point). The corresponding Finagle radius is denoted f^*. The yolk center c^* is defined as $\arg\min r(x)$ and the yolk is the circle centered at c^* with radius $r(c^*)$. For any $\epsilon \geq 0$ the ϵ-core is defined as the set $x : \epsilon(x) \leq \epsilon$. The point or set of points $\arg\min \epsilon(x)$ is called the ϵ^*-core and ϵ^* is the corresponding value of ϵ.

The argument given in (Bräuninger, 2007) to justify the claim that the

Finagle point is within the ϵ^*-core is a proof that $f(x) \geq \epsilon(x)$. *However, just because one function lies on or above another function does not necessarily imply that the former attains its minimum at the same point as the latter.* Indeed, by that reasoning, Theorem 1 would imply that the yolk center coincides with the ϵ^*-core, something that Bräuninger demonstrates is not usually the case (Bră̈uninger, 2007).

On the other hand, the definitions of $f(x)$ and $\epsilon(x)$ are so similar that they might be equivalent. Could it be that $f(x) = \epsilon(x)$?

The potential trouble arises when a point v that is in a different direction from x is introduced. Referring to Figure 2, v is another ideal point who prefers y to x by ϵ or less. The candidate at x can finagle his location to x'' to secure v's vote. But x' and x'' are different points. *There is no one point that x can finagle to that secures both z's vote and v's vote.* Referring to Figure 2, suppose there are voter ideal points at x, v, z, y, and u. Then y would defeat x in a majority election, i.e. $y \succeq x$, because only x would vote for x. However, x is epsilon-preferred to y, i.e. $x \succeq^\epsilon y$ because x, v, and z would all vote for x. However, there is no point $x' \in D(x, \epsilon)$ such that $x' \succeq y$. That is, there is no point that x can finagle to (at distance ϵ or less) that defeats y.

A modification of this example proves that $f(x) \neq \epsilon(x)$ can occur. In fact, the ratio $f(x)/\epsilon(x)$ can be arbitrarily close to 2. Place voter ideal points at $(-M, 0), (0, 0)$, and $(M, 0)$ where M is an unspecified large number. Consider the point $x = (0, 1)$. Its finagle radius $f(x) = 1$ because $(0, 0)$ is a core point and therefore a finagle distance of 1 is both necessary and sufficient. On the other hand, a little arithmetic shows that $\epsilon(x) = \frac{1+\sqrt{M+1}-\sqrt{M}}{2}$ which tends to $\frac{1}{2}$ as $M \to \infty$.

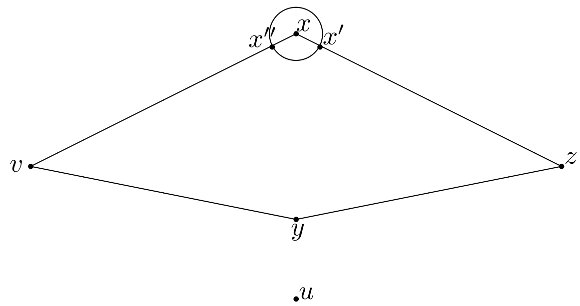

Figure 2: The circle has radius ϵ. No one point within ϵ of x wins both v and z against y.

There is another inaccuracy in (Bräuninger, 2007). There, the yolk is computed by finding the smallest circle that intersects all of the so-called "limiting median lines", i.e. median lines that pass through two (or more) voter ideal points. That circle is known as the "LP yolk" because it can be found via a linear program. However, limiting median lines do not suffice to determine the yolk (Stone and Tovey, 1992). A polynomial time algorithm to compute the true yolk is given in (Tovey, 1992).

Now let us try to evaluate the impact of the logical gaps on the main conclusions of the paper. The main conclusions are that the yolk center c^* does not in general coincide with the ϵ^*-core, that the minimal epsilon radius ϵ^* and finagle radius f^* are significantly smaller than r^*, the yolk radius, and that the finagle point is contained within the ϵ^*-core. It would follow then that the finagle point and radius are better predictors of ϵ^* and the ϵ^*-core than is the yolk.

First, the inaccurate yolk radius computation is always a *lower bound* on the true yolk radius. Since the main conclusion related to yolk radius is that it is larger than the finagle and epsilon core radii, *all the more so is that conclusion correct* for the true yolk radius. Also, it was recently proved (McKelvey and Tovey, 2010) that if points are sampled uniformly from a strictly continuous distribution on a convex region, then as the number of points increases the LP yolk converges to the true yolk in both location and radius. In the paper, between 3 and 101 points are sampled from a uniform distribution on a square, a distribution that meets the conditions stated. Therefore, especially for the larger samples, we can expect that the inaccurate yolk calculations are in fact nearly accurate. Finally, computational results of Kohler (Koehler, 1992) show empirically that the LP yolk often precisely equals the true yolk, when voter ideal points are sampled from a uniform distribution. This implies that the claim that the yolk center does not usually coincide with the ϵ^*-core is correct.

Second, suppose we know that for a particular value of ϵ the area of the epsilon-core is small. Since $f(x) \geq \epsilon(x)$, it is at least as difficult to win by finagling than to win with a benefit of the doubt. Hence, the area of the finagle equilibrium set $\{x : f(x) \leq \epsilon\}$ is less than or equal to the area of the epsilon-core $\{x : \epsilon(x) \leq \epsilon\}$. Therefore, the conclusion regarding the ratio of the yolk area to the finagle area is true all the more so. On the other hand, it also follows that the finagle radius $f^* \geq \epsilon^*$. Therefore, the conclusion regarding the ratio of the yolk radius r^* and the finagle radius f^* is not justified.

Third, the conclusion that the finagle point is in the ϵ^*-core is not justified by the arguments given.

In summary, if the notion of the finagle point is excised from the paper, then, given the results of Koehler (Koehler, 1992) and McKelvey and Tovey (McKelvey and Tovey, 2010), the other results of the paper stand. How to compute the finagle point for 5 or more voters remains a mystery.

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