An Improved Algorithm and Analysis for Finding the Simpson Point of a Convex Polygon In Competitive Location

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Abstract. We consider the problem of finding the Simpson Point, also known as the (1|1) Centroid, of a market that is uniformly distributed over a convex polygon. Diaz and O'Rourke (1994) developed an iterative search algorithm for the case when the demand distribution is uniform; they also observed that it did not appear to converge pointwise, and modified it to do so. We present an enhancement of their algorithm that improves its time complexity from $O(\log^2 1/\varepsilon + n \log 1/\varepsilon)$ to $O(n \log 1/\varepsilon)$. We also prove that both our algorithm and the original unmodified Diaz-O'Rourke algorithm, appropriately interpreted, do converge pointwise and we derive explicit bounds on convergence rates. Finally, we explore how our algorithm might be modified to find the Simpson Point when the demand distribution is non-uniform.

1 Introduction

Competitive Location models have been studied extensively since the days of Hotelling (1929) with applications in fields as diverse as economics, marketing, voting theory, operations research and regional science. Recent comprehensive surveys of the models in this area can be found in Plastria (2001), and Eiselt and Laporte (1997) and Drezner (1995). A seminal model in this area is that of Hakimi (1990) who assumes the following scenario: two competing firms vie for a common market by locating their own facilities to sell an identical product. Of the two competing firms, one is designated as the *leader* that decides to locate p of its own facilities first. The leader is aware of the fact that after it has entered the market with p facilities, the rival firm, denoted as the *follower*, will locate r of its own facilities is such a manner as to take away as much market share from the leader as possible. Given this, the decision problem facing the leader is to find optimal locations for its own p facilities such that the maximum market share that is lost to the follower is as small as possible; Hakimi refers to this as the $(r|p)$-*centroid* problem. In general, when $r,p \geq 2$, these problems are complex; for example, Hakimi (1990) has shown that when the market is given by a network, with the customers located at the nodes, the $(1|p)$-*centroid* problem, or even computing an approximate solution for it, is NP-hard. In the context of voting theory, the (1|1)-*centroid* problem is studied under a different name, namely, finding the Simpson solution (Simpson (1969)) to a voting game; it is this name that we will use throughout the paper. The Simpson solution is the point in policy space against which the fewest voters can be mustered. This special case has been extensively studied in the literature. When the market is given by a network, with the customers located at the nodes, Hansen and Labbé (1988) have given a polynomial algorithm to determine the Simpson point. The case where the market is given by a set of discrete points that represent the customers, has been studied by Carrizosa *et. al.* (1997a, 1997b), Durier (1989), Drezner (1982), Tovey (1991) and Michelot (1993). The case where the market is given by a polygon with a continuous distribution of customers has received less attention, particularly when the demand distribution is non-uniform. However, when the demand distribution is uniform, the Simpson Point of a convex polygon is identical to the Center of Area of that polygon. This problem has the flavor of computational geometry. The literature of geometry has several classical results on the Center of Area problem - see for example Yaglom and Boltyanskii (1961), Hogatt (1962), Grunbaum (1960, 1963), Singmaster and Singmaster (1967). Our point of departure in this paper is the work done by Diaz and O'Rourke (1994), where they present algorithms to compute the Center of Area of a convex polygon.

We consider a market area described by a convex polygon and explain the problem equivalences when the demand is uniformly distributed over the market area. The basic iterative algorithm enunciated by Diaz and O'Rourke (1994) for this problem computes smaller and smaller convex sets inside the polygon that are guaranteed to contain the center of area. Diaz and O'Rourke [IBID] observed that the convex sets could fail to converge to a point, and proposed a modification that pointwise converges. We present an enhancement of their algorithm that significantly reduces its time complexity from O(n log^2 1/ε) to O(n log 1/ε). We also derive explicit bounds on pointwise convergence of our or any similar algorithm. In doing so we also show that Diaz and O'Rourke's original unmodified algorithm actually does produce a series of points that converges to the Simpson point at guaranteed convergence rate. In addition, we show that its convex sets fail to converge to a point only in an easily identifiable and exactly solvable special case. Finally, we explore how our modification might be extended to find the Simpson Point of the convex polygon when the demand distribution is non-uniform – to the best of our knowledge, there is no discussion in the literature of this version of the (1|1) Centroid problem.

The remainder of the paper is divided as follows. The next section presents the notation used in the paper and the basic results that are to be used later. This is followed by section 3 which discusses the enhancement to the basic iterative algorithm. The last portion of that section then discusses how our iterative algorithm can be modified when the demand distribution is non-uniform. Section 4 gives the pointwise convergence analysis.

2 Preliminaries - Notation and Basic Results

In this section we present definitions, notation, and some basic results. There are two firms competing for a common market area, by locating one facility each that sells an identical product (with respect to price, quality etc.) to the customers. The market area itself is given by a convex polygon P with an inelastic demand uniformly distributed over it. Relocation of the facilities is assumed to be prohibitively expensive in the short run, thus presupposing that they are similar to heavy industries such as factories, stores, etc. One of the two firms, the *leader*, has decided to locate its facility first. The leader is aware that after it has entered the market, its competitor, which we refer to as the *follower* firm, will then locate its own facility with the aim of capturing as much market share from the leader as possible. The decision problem facing the leader is to locate its facility so as to minimize the market area that would be lost to the follower, assuming that the follower maximizes its market share; the point in the market that achieves this is referred to as the Simpson point or (1|1)-Centroid of P and we denote it by x^*. The cost of transportation is assumed to be strictly increasing in distance; thus, to minimize total costs, the customers patronize the closest facility, with all ties being broken in favor of the original entrant into the market, i.e., the leader.

The polygon P has n vertices $v_1, v_2, ..., v_n$, where the numbering is given in the clockwise order.

The edges of *P* are thus given by the line segments $\overline{v_1v_2}, \overline{v_2v_3}, \ldots\ldots, \overline{v_{n-1}v_n}$ whose union constitutes the boundary of *P*, which is denoted by *bd*(*P*). The interior of *P*, which we assume is nonempty, is denoted by *In*(*P*). The line defined by two distinct points *x* and *y* will be denoted $\overleftrightarrow{xy}$ and the halfspaces defined by that line will be denoted $\overleftrightarrow{xy}^{+}$ and $\overleftrightarrow{xy}^{-}$. Which halfspace is which will be defined as needed. In general we employ the convention that "-" faces inwards towards the center of P.

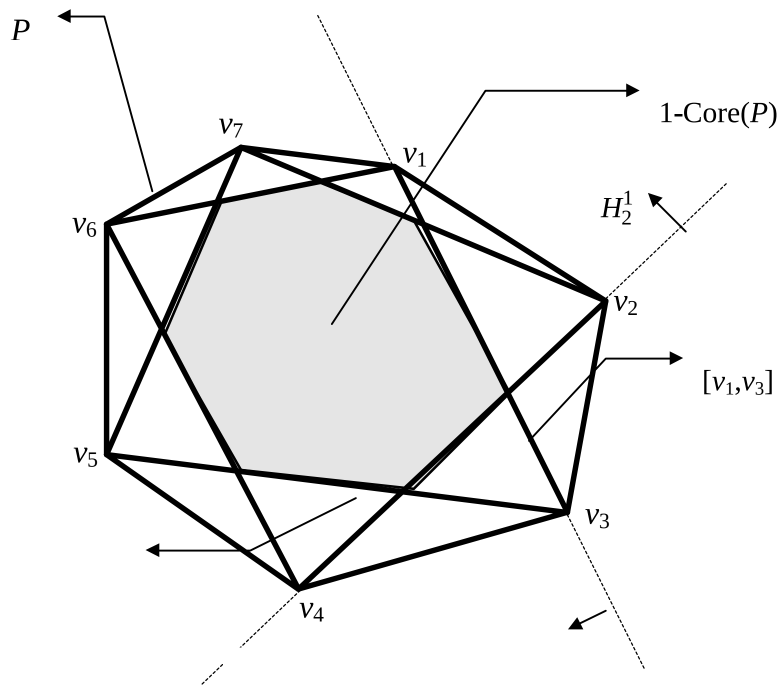

Figure 1

Consider any two distinct points $y,z \in bd(P)$. The line segment $\overline{yz}$ is referred to as a cut of *P*, since it divides *P* into two smaller convex polygons. In order to define these two smaller polygons, *left-arc*[*y*,*z*] (respectively, *right-arc*[*y*,*z*]) is defined as the section of *bd*(*P*) that consists of all points on *bd*(*P*) that are encountered in a clockwise (respectively, counterclockwise) traversal of *bd*(*P*) from *y* to *z*, including these two points themselves. Given this, the convex polygon whose boundary is the union of $\overline{yz}$ and left-arc[*y*,*z*] (respectively, right-arc[*y*,*z*]) is denoted as *Left[y,z]* (respectively, *Right[y,z]*). For example, in Figure 1, *Left* [v_7, v_2] (respectively, *Right*[v_7, v_2]) has vertices v_7, v_1 and v_2 (respectively, v_2, v_3, v_4, v_5, v_6 and v_7).

The area of any polygon S will be denoted by *A*(S). Since the demand distribution is uniform, the demand generated in *S* is equal to *A*(*S*). The term "larger of two given polygons" will refer to the one with the greater area. Consider now a point $x \in P$ and the any cut $\overline{yz}$ of *P* that passes through *x*. The point *z* is determined by *P*, *x* and *y* and is denoted *z*(*x*,*y*,*P*). This set of all such cuts $\overline{yz(x,y,P)}$ is parameterized by *y* as *y* ranges over the set *bd(P*). Let *y'* denote the value of *y* in *bd*(*P*) that maximizes *A*(*Left*[*y*,*z*(*x*,*y*,*P*)]) (note that the maximum is attained because *bd*(*P*) is compact and the area function is continuous). Define *H*(*x*) = *A*(*Left*[*y'*,*z*(*x*,*y'*,*P*)]). Thus *H*(*x*) denotes the larger of the two most unequal (in terms of area) pair of smaller polygons that any cut through *x* can partition *P* into. (Along similar lines, when the demand distribution in non-uniform, *H*(*x*) will be defined by considering all the sets that any cut through *x* can partition *P* into and choosing the one with the maximum demand.)

With *H*(*x*) defined as above, then following Hotelling's (1929) "Principle of Minimum Differentiation", if the leader locates at the point $x \in P$, the follower can be expected to locate at a distinct but arbitrarily close point *q* in *H*(*x*) such that $\overleftrightarrow{qx}$ is normal to $\overleftrightarrow{y'z(x,y',P)}$. Hence the leader must be prepared to lose the entire set *In*(*H*(*x*)) to the follower and be left with the market region *P**In*(*H*(*x*)). Since *A*(*bd*(*H*(*x*)))=0, an alternate definition of the Simpson point *x** could be:

$$x^* = \operatorname{argmin}_{x \in P} \{A(H(x))\}$$

When defined as above, the Simpson point *x** is referred to as the *Center of Area* of *P* (as distinct from the center of mass) in the geometry literature. Diaz and O'Rourke (1994) study the center of area problem for a convex polygon and begin by giving an *O*(*n*) time algorithm to compute *H*(*x*) for any given point $x \in P$. They also give an $O(n^6)$ combinatorial algorithm to compute *x** - however, given the procedural and time complexity of this algorithm they also describe an iterative search algorithm; it is this algorithm that is the basis of our paper. We need three facts to describe their algorithm. The first two lemmas are immediate; the third is a classical result.

Lemma 2.1 *Diaz and O'Rourke (1994)***:** *For any convex polygon P with A(P)>0,* $x^* \in In(P)$.

Lemma 2.2 *Diaz and O'Rourke (1994)*: *For any convex polygon P*, $x^* \in H(x)\ \forall\ x \hat{I}\ P$.

Lemma 2.3 *Grunbaum* (1960), *Hammer* (1960)): *Given a polytope P' in d-dimensions with a volume of unity, let x denote its centroid (center of mass). Then any hyperplane through x divides P' into two polytopes, the larger of which has a volume of no more than* $[\ 1 - (d/(d+1))^d\ ]$.

In terms of our function *H()*, Lemma 2.3 states that if $y \in P$ is the centroid of polygon *P*, then $A(H(y)) \le 5A(P)/9$. Based on these facts, the basic iterative algorithm proposed by Diaz and O'Rourke (1994) proceeds as follows: at the first step, choose a $x^{(1)}$ as the centroid of *P* and calculate $H(x^{(1)})$. By Lemma 2.2, $x^* \in H(x^{(1)})$; hence, denote the *Simpson_Polygon* at step one, $S^{(1)}$, by $H(x^{(1)})$. In the second step, choose a point $x^{(2)}$ as the centroid of $S^{(1)}$ and find $H(x^{(2)})$. Now we know that $x^* \in \{H((x^{(1)}) \cap H((x^{(2)})\} = \{S^{(1)} \cap H(x^{(2)})\}$. Thus, at the end of step two, update $S^{(2)}$ to $\{S^{(1)} \cap H(x^{(2)})\}$. By repeating this procedure, we get successively smaller *Simpson_Polygon*s that are guaranteed to contain x^*. Assuming that the termination criterion is to produce a *Simpson_Polygon* that has an area no more than εA, where ε is the required error bound, this idea is summarized as the following algorithm.

Algorithm *Find_Simpson* _I (from Diaz and O'Rourke (1994))

begin
Step 1: $i = 0$. Set $S^{(i)} = P$.
Step 2: while ($A(S^{(i)}) > \varepsilon A(P)$) do
{
$i = i + 1$.
Choose $x^{(i)}$ as the centroid of $S^{(i-1)}$.
Compute $H(x^{(i)})$.
Set $S^{(i)} = \{S^{(i-1)} \cap H(x^{(i)})\}$
}
end

As for the performance of Algorithm *Find_Simpson* _I of Diaz and O'Rourke, note that Lemma 2.3 guarantees that at least 4/9ths of the area of the *Simpson_Polygon* is removed at any iteration *i*. Hence, the area of $S^{(i)}$ is given as:

$$A(S^{(i+1)}) \le 5\, A(S^{(i)})/9 \text{ for } i \ge 1, \text{ with } A(S^{(1)}) \le A(P).$$

Therefore, in order to produce a *Simpson_Polygon* with area $\varepsilon A(P)$ or less, at most $O(|\log \varepsilon|)$ iterations of the algorithm are needed. However, the number of vertices of the *Simpson_Polygon* can increase by one on every iteration; hence the total time taken by Algorithm *Find_Simpson* _I to produce a *Simpson_Polygon* of area no more than $\varepsilon A(P)$ is $O(\log^2 1/\varepsilon + n \log \varepsilon)$. A second problem with Algorithm *Find_Simpson* _I is that, as noted by Diaz and O'Rourke, it can fail to exhibit pointwise convergence, i.e. converge in diameter as well as area. As mentioned previously, Diaz and O'Rourke have demonstrated that pointwise convergence can be guaranteed asymptotically by modifying the implementation of the algorithm. We will show in section 4 that the unmodified algorithm, appropriately interpreted, does converge pointwise even though the diameter may not converge to 0.

3 An Improved Version of Algorithm *Find_Simpson* _I

In this section we discuss an enhancement of Algorithm *Find_Simpson* _I that reduces its time complexity. This improvement is brought about by controlling the number of vertices of *Simpson_Polygon*. To facilitate this enhancement, we first need the concept of a *k*-Core of a convex polygon.

3.1 *k*-Core of *P*

For every given value of k, where k is any positive integer less than n-2, we define a convex polygonal subset of P called the *k-Core of P*, and designated as k-Core(P), as follows. For each $i = 1 \ldots n$ let $\{i+k+1\}$ denote $(i+k) \bmod (n) + 1$. Then define:

$$k - Core(P) = \bigcap_{i=1}^{n} Right\left[v_i, v_{\{i+k+1\}}\right]$$

See Figure 1 for an example of the 1-Core of a given convex polygon. Note that k-Core(P) may be empty. Regardless, the time required to compute it can be obtained as follows. Let $\overleftrightarrow{v_i v_{\{i+k+1\}}}^{-}$ denote the closed half-space defined by the line through v_i and $v_{\{i+k+1\}}$ that contains the polygon *Right*[v_i, $v_{\{i+k+1\}}$]. Then the *k*-Core(*P*) could also have been defined as:

$$k-Core(P)=\bigcap_{i=1}^{n}\overleftrightarrow{v_i ,v_{\{i+k+1\}}}^{-}$$

Thus finding the *k*-Core(*P*) reduces to finding the intersection of *n* linear inequalities, which can be accomplished in $O(n \log n)$ time, a standard result from computational geometry (Preparata and Shamos (1985)). Therefore

Lemma 3.1: *Given a convex polygon P with n vertices, and an integer* $k \geq 1$, *the k*-Core(*P*) *can be computed in* $O(n \log n)$ *time*.

We will want the k-Core to have nonempty interior. The next lemma gives a sufficient condition.

Lemma 3.2: *Given any convex polygon P with n vertices and an integer* $k \geq 1$, *if* $n \geq 3k + 3$, *then the k-Core of P has non-empty interior.*

Proof. For convenience let $R(i)$ denote *Right*[v_i, $v_{\{i+k+1\}}$]. For each *i* the polygon $R(i)$ contains *n*-*k* vertices of *P*. Therefore for any three polygons $R(a)$, $R(b)$, $R(c)$ the multiset of their vertices from *P* has cardinality $3(n\text{-}k)$. It follows from $n \geq 3k + 3$ that $3(n\text{-}k) \geq 2n+3$. Since no vertex can have multiplicity more than 3 in the multiset (as there are only 3 polygons), there must exist at least 3 vertices with multiplicity 3, i.e. that are common to all three polygons. Since no 3 vertices of *P* can be collinear, $R(a)\cap R(b)\cap R(c)$ must moreover have nonempty interior. Therefore, $In(R(a))\cap In(R(b))\cap In(R(c))$ has nonempty interior. Since this is true for every triple *a*, *b*, *c*, by Helly's Theorem for finite sets of open convex sets,

$$\bigcap_{i=1}^{n} In(R(i)) \neq \Phi.$$

This intersection is nonempty and open, hence it has nonempty interior.

We will now prove a geometric property of the *k*-Core that will be useful in containing the number of vertices in the *Simpson_Polygon* generated by Algorithm *Find_Simpson* _I. Consider any point $x \in In(k\text{-Core}(P))$ and any cut $\overline{yz}$ that passes through x. Since $x \in In(k\text{-Core}(P)) \subseteq In(P)$, y and z must occur on different edges of P. Now consider the open set obtained by omitting the two endpoints y and z from left-arc[y,z]; we claim there are at least $k+1$ vertices of P in this open set.

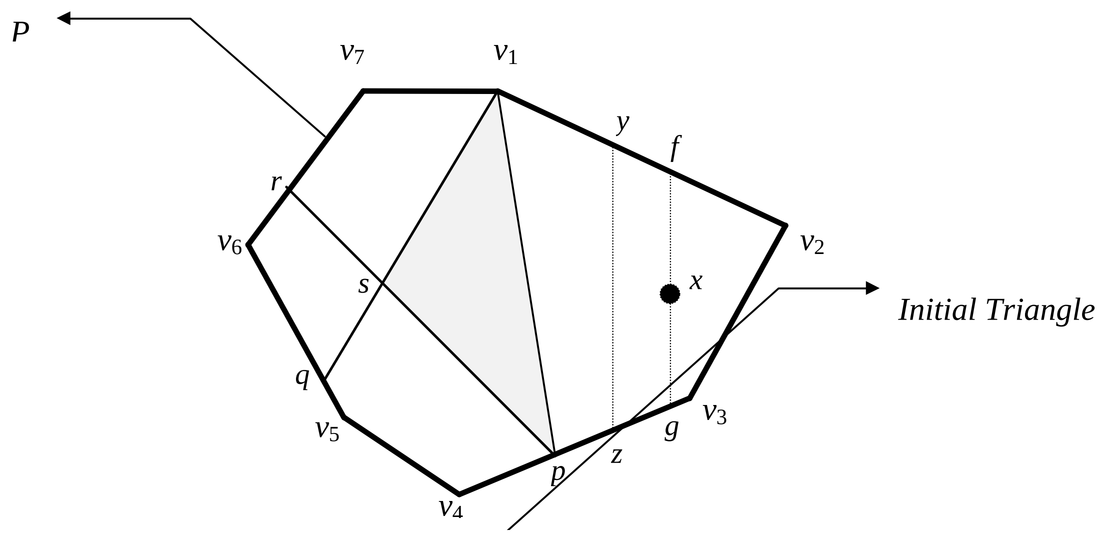

Figure 2

If to the contrary there were k or fewer vertices of P in the open halfspace $In(\overleftrightarrow{yz}^{+})$, then for convenience labeling those vertices $v_2,\ldots,v_m$ where $m \le k+1$, consider the cut $\overline{v_1, v_{k+1}}$ and the polygon $R(1)$ (using the notation from the proof of Lemma 3.2) it defines. On the one hand, y is in the half open interval $[v_1,v_2)$ and z is in the half-open interval $(v_m, v_{m+1}]$. Therefore x, which is on the segment $\overline{yz}$, is in $\overleftrightarrow{yz}^{+}$ and hence not in $In(R(1))$. (It could be on the boundary of $R(1)$ if $y = v_1$ and $z = v_{k+1}$.) On the other hand, by definition $k\text{-Core}(P) \subseteq R(1)$. Hence $x \notin In(k\text{-Core}(P))$, a contradiction.

Since there are at least $k+1$ vertices in this open segment that is obtained from left-arc[y,z], the polygon *Right*[y,z] will have at most $(n-(k+1)+2)=n-k+1$ vertices. Since *Left*[y,z] = *Right*[z,y], we have bounded the number of vertices of both polygons formed by any cut through any interior point.

Lemma 3.3: *Given any convex polygon P with n vertices, for any k³1, any cut through any point in*

In(k-Core(P)) partitions P into two polygons, each of which has no more that (n - k + 1) *vertices.*

3.2 An Enhancement of Algorithm Find_Simpson_I

Using the concept of k-Core above, we can now embellish Algorithm Find_Simpson_I so that its time complexity reduces to $O(n|\log \varepsilon|)$. In order to do so, we use Lemma 2.3 to state that:

Lemma 3.4: *Consider any cut* $\overline{yz}$ of convex polygon P *with Simpson solution* x^*. *If* $A(Left[y,z]) \leq 4A(P)/9$ (*respectively*, $A(Right[y,z]) \leq 4A(P)/9$), *then* $x^* \in Right[y,z]$ (*respectively*, $x^* \in Left[y,z]$).

Proof. Suppose $A(Left[y,z]) \leq 4A(P)/9$ (refer to figure 2). Let $x \in Left[y,z] \setminus \overline{yz} = P \setminus Right[y,z]$ be arbitrary. Let $\overline{fg}$ denote the cut through x that is parallel to the cut $\overline{yz}$. Obviously, $Left[f,g] \subset Left[y,z]$, implying that $A(Left[f,g]) < A(Left[y,z]) \leq 4A(P)/9$ whence $A(Right[f,g] > 5A(P)/9$. By definition of $H(x)$, we have $A(H(x)) \geq A(Right[f,g]) > 5A(P)/9$. But by Lemma 2.3, the centroid w of P is a better location than x, as $H(w) \leq 5A(P)/9$, thereby proving the non-optimality of x. The complementary case is immediate because $Right[y,z] = Left[z,y]$.

Given Lemma 3.4, we now define *Initial Triangle(P)* as a triangle inside P that is constructed as follows (refer to figure 2). Starting from vertex v_1 of P, proceed clockwise on $bd(P)$ from v_1 until the point p is reached with the property that $A(Left[v_1,p]) = 4A(P)/9$. Proceeding further clockwise from p, denote by q and r two more points on the boundary of P, such that $A(Left[q,v_1] = A(Left[p,r] = 4A(P)/9$. In a clockwise traversal of the boundary of P, that begins at p and ends at v_1, r must be encountered strictly after q, because otherwise P would contain three disjoint regions, $In(Left[v_1,p])$, $In(Left[p,r])$, and $In(Left[q,v_1])$, with total area $12A(P)/9$. That implies, in turn, that the two line segments $\overline{v_1q}$ and $\overline{pr}$ will intersect in the interior of P; denote their point of intersection as s. The Initial Triangle is defined to be the triangle (v_1, p, s). Note that the Initial Triangle (i) has an area no more than $A(P)/9$, (ii) can be computed in $O(n)$ time, and (iii) is guaranteed, by Lemma 3.4, to contain x^*, since $A(Left[v_1,p]= A(Left[p,r]) = A(Left[q,v_1]) = 4A(P)/9$.

The second idea that we will use to modify Algorithm *Find_Simpson* _I is the *k*-Core(*P*) with *k*=2 (any larger k would also work). Recall that at iteration *i* of the algorithm, the *Simpson_Polygon* is given by $S^{(i)}$. Let the number of vertices in $S^{(i)}$ be denoted by $n^{(i)}$. Then by Lemmas 3.2 and 3.3 for the case of the 2-Core($S^{(i)}$), we have:

Corollary 3.5: *If* $n^{(i)} \geq 9$, *i.e., the Simpson_Polygon at iteration i has at least* 9 *vertices, the* 2-*Core*($S^{(i)}$) *has non-empty interior. Further, every point in the interior of this 2-Core has the property that any cut of P through this point divides* $S^{(i)}$ *into two smaller polygons, each of which has at most* ($n^{(i)}$-1) *vertices.*

Based on the ideas of the Initial Triangle and Corollary 3.5, the specialized version of Algorithm *Find_Simpson* _I, is presented below.

Algorithm *Find_Simpson*_II

begin
1: $i = 1$. Set $S^{(i)}$ = *Initial Triangle(P).*
2: **while** ($A(S^{(i)}) > \varepsilon\, A(P)$) **do**
{
2.1 (Vertex Reduction Loop)
If ($n^{(i)} > 8$, i.e., $S^{(i)}$ has more than 8 vertices)
{
2.1.1 Find 2-*Core*($S^{(i)}$). Choose $x^{(i+1)}$ as any point in the interior of this 2-*Core*.
2.1.2 Find $H(x^{(i+1)})$.
2.1.3 Set $S^{(i+1)} = \{ S^{(i)} \cap H(x^{(i+1)}) \}$.
}
2.2 (Area Reduction Loop)
If ($n^{(i)} \leq 8$)
{
2.2.1 Choose $x^{(i+1)}$ as the centroid of $S^{(i)}$.
2.2.2 Find $H(x^{(i+1)})$.
2.2.3 Set $S^{(i+1)} = \{ S^{(i)} \cap H(x^{(i+1)}) \}$
}
2.3 $i = i+1$
}
end.

To calculate the time complexity, we have already noted that the *Initial Triangle* in step 1 can be found in $O(n)$ time. Consider now the main while-do loop. At the first iteration we have $n^{(1)} = 3$. Step 2.1 if executed decreases the number of vertices by at least one (Corollary 3.5); step 2.2 if executed can increase the number of vertices by at most one. Therefore $n^{(i)} \leq 9$ in every iteration. Moreover, Step 2.2 will be executed at least once every two consecutive iterations of the while-do loop, hence $A(S^{(i+2)}) \leq 5(S^{(i)})/9$. The O(1) bound on $n^{(i)}$ will easily guarantee that each iteration of the while-do loop can be completed in $O(n)$ time. Steps 2.1.1 and 2.2.1 take $O(n^{(i)}) = O(1)$ time. As previously stated, the polygon H(x) in steps 2.1.2 and 2.2.2 can be found in O(n) time by the algorithm of Diaz and O'Rourke(1994). Further, given $S^{(i)}$ with $n^{(i)}$ vertices, steps 2.1.3 and 2.2.3 will take $O(n + n^{(i)}) = O(n)$ time, as they require us to find the intersection of two convex polygons (Preparata and Shamos (1985)). (We can actually reduce this to $O(1)$ since all we need of $H(x)$ is the line through x, but the preceding step time dominates.)

Since each iteration takes $O(n)$ time and $A(S^{(i+2)}) \leq (5/9)^i A(P)/9$, then in order to produce a *Simpson_Polygon* with area $\varepsilon A(P)$ or less, at most $(3.4\,|\log \varepsilon| - 6.5) = O(|\log \varepsilon|)$ iterations are needed. In section 4, we show that convergence in area of order ε^3 yields convergence pointwise of order ε. Our main theorem then follows.

Theorem 3.9: *Given a convex polygon P with a uniform demand distribution, and an error bound* ε, Algorithm *Find_Simpson_*II *produces a polygon inside P of area no more than* εA, *that is guaranteed to contain x*, in* $O(n|\log \varepsilon|)$ *time. Moreover, the algorithm produces a point x that is guaranteed to be within distance* ε *of x*, in* $O(3n|\log \varepsilon|) = O(n|\log \varepsilon|)$ *time.*

3.3 Non-Uniform Demand Distribution

Here we will explore the case where the demand distribution over *P* is non-uniform. Our discussion is restricted to the high level. We ignore the issue of the form and size of the input description of the demand distribution, and we assume the existence of subroutines to perform the various low-level computations such as $H(x)$ in time $\tau(n) = \Omega(n)$ on convex polygons with n vertices. We will assume that the demand distribution is continuous and positive over the domain of P. It can be verified that Lemmas 2.1, 2.2, 3.1, 3.2, 3.3 and Corollary 3.5 hold true even in this case. However, Lemmas 2.3 and 3.4 do not work in general when area is replaced by non-uniform demand. (Likewise, the results on the pointwise convergence may not hold). Thus we can not begin with an Initial Triangle that is guaranteed to contain the Simpson point. We will begin instead with a *Simpson_Polygon* that is given by P itself, and use the results of Lemmas 3.2 and 3.3 to produce one with no more than 9 vertices. Then onwards, the algorithm will be executed similar to Algorithm *Find_Simpson*_II.

Algorithm *Find_Simpson*_III (Non-Uniform Demand)

begin
1: $i = 1$. Set $S^{(i)} = P$
2: while $(A(S^{(i)}) > \varepsilon\, A(P))$ **do**
{
2.1 (Vertex Reduction Loop)
If ($n^{(i)} > 8$, i.e., $S^{(i)}$ has more than 8 vertices)
{
2.1.1 Let $\delta^{(i)} = \lfloor (n^{(i)}\text{-}9)/3 \rfloor$.
2.2.2 Find $\delta^{(i)}$-$Core(S^{(i)})$. Choose $x^{(i+1)}$ as any point in the interior of this $\delta^{(i)}$-*Core*.
2.1.2 Find $H(x^{(i+1)})$..
2.1.3 Set $S^{(i+1)} = \{ S^{(i)} \cap H(x^{(i+1)}) \}$
}
2.2 (Area Reduction Loop)
If $(n^{(i)} \leq 8)$
{
2.2.1 Choose $x^{(i+1)}$ as the centroid of $S^{(i)}$.
2.2.2 Find $H(x^{(i+1)})$.
2.2.3 Set $S^{(i+1)} = \{ S^{(i)} \cap H(x^{(i+1)}) \}$
}
2.3 $i = i+1$
}
end.

To analyze the time complexity of Algorithm *Find_Simpson*_III, assume that $n \geq 10$. Then for the initial few, say i', iterations of the while-do loop of step 2, only the Vertex Reduction Loop of step 2.1 will be executed, until $n^{(i'+1)} \leq 9$. Hence, we will now estimate i' and the time taken by the algorithm in the first i' steps. To that end, note that by our choice of $\delta^{(i)} = \lfloor (n^{(i)}\text{-}3)/3 \rfloor$, and by Lemma 3.3, it is assured that for steps 1 through i',

$n^{(1)}$ $= n$ and for $1 \le i \le i'$
$n^{(i+1)}$ $\le n^{(i)} - \delta^{(i)} + 1 = n^{(i)} - \lfloor (n^{(i)} - 3)/3 \rfloor + 1$
$\le (2(n^{(i)} + 9)) / 3$

Given the above recurrence relation for i' such that $n^{(i'+1)} \le 9$, obviously $i' = O(\log n)$. Hence, $O(\log n)$ iterations of the while-do loop will be necessary to obtain a *Simpson* polygon with at most 9 vertices. Since each iteration will take $O(t(n)\log n^{(i)}) = O(\tau(n) \log n)$ time, the total time spent by the algorithm in the first i' steps is $O(\tau(n) \log^2 n)$.

However, at iteration $(i'+1)$, we are guaranteed that the *Simpson_Polygon* will have no more than 9 vertices. From then on, the analysis of the algorithm will be identical to that of Algorithm *Find_Simpson*_II, i.e., it will be assured that for every two consecutive iterations of the while-do loop of step 2 (which will take $O(\tau(n))$ time), at least 4/9ths of the area of the *Simpson* polygon will be removed from future consideration. Thus, even if we conservatively assume that $A(S^{(i'+1)}) = A(S^{(1)}) = A(P)$, the total amount of time taken by Algorithm *Find_Simpson*_III to produce a *Simpson* polygon of area no more than $\varepsilon A(P)$ is $O(\tau(n)(\log^2 n + |\log \varepsilon|))$. This leads to

Proposition 3.10: *Given a convex polygon P with a non-uniform demand distribution, and an error bound* ε, Algorithm *Find_Simpson*_III *produces a polygon inside P of area no more than* εA*, that is guaranteed to contain* x^**, in* $O(\tau(n)(\log^2 n + |\log \varepsilon|))$ *time.*

When the demand of the region $\mu(H(x^*)) > ½$, Lemma 4.3 in the next section easily combines with Proposition 3.10 to guarantee pointwise convergence of the algorithm, but the rate of convergence is not as explicit as one would like because it depends on the size of the gap $(\mu(H(x^*)) - ½)$, which is not known in advance.

4 Analysis of Convergence

In this section, we show that both our algorithm and Diaz and O'Rourke's original *Find_Simpson*_I algorithm have good convergence properties. We have already seen that the area of the search regions $S^{(i)}$ decreases geometrically. As Diaz and O'Rourke point out, however, the trouble is that the regions might be long and thin. They showed that for even the very simple case of a rectangle, the $S^{(i)}$ produced by their algorithm may converge to a line segment rather than a point.

It turns out that their algorithm's convergence is better than they claimed. First, *only in the case* $a^* = \frac{1}{2}$ can it occur that the $S^{(i)}$ do not converge to a point (Lemma 4.1). This simple case could be detected and dealt with separately. Second, even when the $S^{(i)}$ fail to converge, *the sequence of best candidate points from among* $x^{(1)} \ldots x^{(i)}$ *converges to the Simpson point* (Theorem 4.1). The same convergence properties hold for our enhanced algorithm.

4.1 Notation

P and S will denote convex polygons in the plane. As usual, the line segment with endpoints x, y is denoted $\overline{xy}$ and the line containing it is denoted $\overleftrightarrow{xy}$. If H is a hyperplane (line) the two halfspaces it defines are denoted H^+ and H^-. $A(S)$ as usual denotes the area of polygon S; $L(\overline{xy})$ denotes the length of a line segment $\overline{xy}$; $D(S) \equiv \max_{x \in S, y \in S} L(\overline{xy})$ denotes the diameter of polygon S..

If voters are uniformly distributed on polygon P, recall that the largest population that can be mustered against $x \in P$ is the polygon $H(x)$. Define

$$a_P(x) \equiv \frac{A(H(x))}{A(P)}$$

Our algorithm (either *Find_Simpson*_I or *Find_Simpson*_II) seeks x^*, the Simpson point of P, i.e. the point $x \in P$ that minimizes $a_P(x)$. Our key definition follows.

Definition 4.1 *Make the convention for any line G defining a side of polygon* $S \subseteq P$ *that the normal to G points away from S, that is,* $S \subseteq G^-$. *A convex polygon S is* an α-*Simpson polygon of convex polygon P* if $S \subseteq P$ *and each side of polygon S is defined by a line G such that* $A(G^- \cap P) \geq aA(P)$.

The definition applies to our algorithm in the following way. As usual let $x^{(1)} \ldots x^{(i)}$ be the sequence of points generated in the first i steps. Let

$$\hat{a}^{(i)} \equiv \min_{1 \le j \le i} a_P\left(x^{(j)}\right).$$

Let $\hat{x}(i)$ be the point at which the min is attained; thus $a\left(\hat{x}^{(i)}\right) = \hat{a}^{(i)}$. If $H\left(x^{(j)}\right)$ is defined by line G then by definition,

$$\frac{A(G^- \cap P)}{A(P)} = a_P\left(x^{(j)}\right) \ge \hat{a} .$$

Therefore the algorithm produces a sequence of $\hat{a}^{(i)}$-Simpson polygons $S^{(i)}$. The algorithm also produces a sequence $\hat{x}^{(i)} \in S^{(i)}$ of associated candidate solutions. (The candidate point is contained in the polygon – actually it is on the boundary – because it can not be cut off by another line G with smaller value $A(G^{+} \cap P)/A(P)$ £ $\hat{a}^{(i)}$. Note that the sequence is not necessarily the sequence of candidates $x^{(j)}$: $j = 1 \ldots$; rather it is the sequence of the best candidates found so far.) We summarize the preceding observations:

Claim 1 *Find_Simpson_I and Find_Simpson_II each produces a sequence of* a^i *- Simpson polygons* i = 1…., *and candidate points* $\hat{x}^i$ *contained in the respective polygons, with* $a_p\left(\hat{x}^i\right) = a^i$.

The rest of the analysis depends only on these properties, so we can drop the superscript i.

4.2 Claims and Lemmas

Claim 4.2 *For any points x, y in convex polygon* S *s.t.* $L\left(\overline{xy}\right) = D(S)$ *and any distinct points p, q in S such that* $\overline{pq}$ *is perpendicular to* $\overleftrightarrow{xy}$, *we have* $L\left(\overline{pq}\right) \le \frac{2A(S)}{D(S)}$. *In particular, for any point* $r \in S$ *the distance from r to* $\overline{xy}$ *is at most* $\frac{2A(S)}{D(S)}$.

Proof: The quadrilateral $xpyq \subseteq S$ by convexity. It has area $\frac{1}{2} L\left(\overline{pq}\right) L\left(\overline{xy}\right)$.

Claim 4.3 *Let* x, y *in convex polygon* S *be s.t.* $L(\overline{xy}) = D(S)$. *Then there exists a side* $\overline{x_i x_{i+1}}$ *of* S *contained in* $\overleftrightarrow{xy}^{+}$ *such that* $\overleftrightarrow{x_i x_{i+1}}$ *makes acute angle less than* $q = \arctan\left\{\frac{4A(S)}{D(S)^2}\right\}$ *with line* $\overleftrightarrow{xy}$. *Symmetrically* S *has a different side* $\overline{y_j y_{j+1}}$ *in* $\overleftrightarrow{xy}^{-}$ *whose defining line makes acute angle less than* q *with line* $\overleftrightarrow{xy}$.

Proof: Denote the extreme points of S as $\{x = x_0, x_1, \ldots x_m = y = y_0, y_1, \ldots, y_n = x\}$ traveling clockwise from *x*. It is possible that *m* = 1 (or later symmetrically *n* = 1) in which case the side $\overline{xy}$ makes angle 0 with itself. Otherwise $m > 1$. Let x_t be an x_i at maximum distance to $\overleftrightarrow{xy}$. By Claim 4.2 this distance is at most $\frac{2A(S)}{D(S)}$. Without loss of generality assume x_t is closer to y than to x. Then the angle formed by points x_t, x, y has tangent at most

$$\frac{\frac{2A(S)}{D(S)}}{\frac{1}{2}L(\overline{xy})}.$$

Finally, by convexity of S, $\overline{xx_t} \subset \overleftrightarrow{x_{t-1}x_t}^{-}$ (informally, the point x_{t-1} lies "above" the segment $\overline{xx_t}$), hence the line defining side $\overline{x_{t-1}x_t}$ makes at least as small an angle with $\overleftrightarrow{xy}$. Repeat the argument to find a side of S on the other side of $\overleftrightarrow{xy}$.

Claim 4.4 *Let F and G be lines defining edges of* S *as proved to exist in claim 4.3. Let* $Q = P \cap F^{-} \cap G^{-}$. *Then*

$$A(Q) \leq 4A(S)\left(\frac{D(P)}{D(S)}\right)^2$$

Proof: Let F (resp. G) intersect $\overleftrightarrow{xy}$ at f (resp. G). Define triangle F_Δ as follows: it has vertex f, and incident to f it has two sides, one lying in F and containing x_t, the other lying in $\overleftrightarrow{xy}$ and containing y, each of length $D(P)$. Similarly define triangle G_Δ. Visually there are two cases, depending on whether f and g are both on the same side of $\overline{xy}$ or whether $\overline{xy} \subseteq \overline{fg}$, but in either case $Q \subseteq F_\Delta \cup G_\Delta$. Hence $A(Q) \le A(F_\Delta) + A(G_\Delta)$. Now $\sin q \le \tan q$, so

$$A(F_\Delta) = \frac{1}{2} D(P)^2 \sin q \le \frac{1}{2} D(P)^2 \frac{4A(S)}{D(S)^2} = 2A(S)\frac{D(P)^2}{D(S)^2}.$$

The area of G_Δ is bounded by the same quantity and the claimed bound follows.

Lemma 4.1 *Let* $\frac{1}{2} \le \hat{a} < 1$ *and let S be an* $\hat{a}$ *- Simpson polygon of P. Then*

$$D(S)\sqrt{\hat{a} - \frac{1}{2}} \le D(P)\sqrt{2\frac{A(S)}{A(P)}}.$$

Proof: Let F and G be lines according to claim 4.3. As in claim 4.4, let $Q = P \cap F^- \cap G^-$. By inclusion-exclusion,

$$A(Q) \ge A(P) - A(P \cap F^+) - A(P \cap G^+) \ge A(P)(1 - (1 - \hat{a}) - (1 - \hat{a})) = 2\left(\hat{a} - \frac{1}{2}\right)A(P).$$

(The second inequality holds because F and G are lines defining sides of S, and S is a $\hat{a}$-Simpson polygon of P.) Combining this inequality with the upper bound on $A(Q)$ from claim 4.4 gives the result.

Lemma 4.1 tells us that the diameter of the inner (Simpson) polygon shrinks as its area shrinks, at rate square root of the reduction in area. However, Lemma 4.1 only guarantees this if the candidate value $\hat{a}$ is bounded away from ½, and we do not know α* in advance. It is very easy to check for the case $\hat{a} = \frac{1}{2}$: determine the horizontal and vertical lines which bisect the area of P (any two nonparallel normal vectors will do). Let y be the intersection of these two lines. Then y is the only possible solution. That is, either $a_p(y) = \frac{1}{2} = a^*$ or $a^* > \frac{1}{2}$.

Even when $a^* > \frac{1}{2}$, Lemma 4.1 does not give a satisfactory convergence rate because a^* may be very close to $\frac{1}{2}$. We will need another lemma to get rid of the $(\hat{a} - \frac{1}{2})$ term.

Lemma 4.2 *Let S be any convex polygon in the plane. Let line H divide S into regions each with area at least t times the area of S, that is, let H be such that $A(H^+ \cap S) \geq tA(S)$ and $A(H^- \cap S) \geq tA(S)$ where $0 < t \leq \frac{1}{2}$. Then*

$$L(H \cap S) \geq \frac{tA(S)}{D(S)}. \tag{1}$$

Proof: Consider the set of lines parallel to H, which have nonempty intersection with S. Parameterize these lines by $H_x : 0 \leq x \leq d_H$, where d_H is the length of the projection of S onto the normal (line perpendicular to) H. Note $d_H \leq D(S)$. Define $f(x) = L(H_x \cap S)$, the length of the line segment formed by line H_x and S. Thus

$$A(S) = \int_0^{d_H} f(x)dx. \tag{2}$$

Let $0 < \hat{x} < d_H$ be the value for which $H = H_{\hat{x}}$. Thus $f(\hat{x}) = L(H \cap S)$ and

$$tA(S) \leq \int_0^{\hat{x}} f(x)dx \leq (1-t)A(S) \Rightarrow \int_{\hat{x}}^{d_H} f(x)dx \geq tA(S). \tag{3}$$

Observe that the function $f(x)$ is concave. This follows from the convexity of *S*. We consider two cases:

Case 1: $\hat{x} \geq \arg\max_x f(x)$. In this case, by concavity of f, $f(\hat{x}) \geq f(x) \forall x > \hat{x}$. Hence,

$$tA(S) \leq \int_{\hat{x}}^{d_H} f(x)dx \leq (d_H - \hat{x}) f(\hat{x}) \leq d_H f(\hat{x})$$

$$\Rightarrow f(\hat{x}) \geq \frac{tA(S)}{d_H}$$

Case 2: $\hat{x} \leq \arg\max_x f(x)$. By concavity of f, $f(\hat{x}) \geq f(x) \forall x < (\hat{x})$. Hence,

$$tA(S) \le \int_0^{\hat{x}} f(x)dx \le (\hat{x} - 0) f(\hat{x}) \le d_H f(\hat{x})$$

$$\Rightarrow f(\hat{x}) \ge \frac{tA(S)}{d_H}$$

4.3 Convergence

Theorem 4.1 *Let* $a_P(\hat{x}) = \hat{a} \le b < 1$ *and let* S *containing* $\hat{x}$ *be an* $\hat{a}$ *-Simpson polygon of P. Then the distance between* $\hat{x}$ *and x*, the Simpson point of* P*, is at most*

$$\sqrt[3]{\frac{4}{1-b}}\, D(P) \sqrt[3]{\frac{A(S)}{A(P)}}\,.$$

Proof: As given in the statement of the theorem, $a_P(\hat{x}) = \hat{a}$, and $x*$ is the optimum point with $a*_P = a_P(x*)$.

Consider the line segment $T = \overline{\hat{x}x*}$. Let H_1 (resp. H_2) be the line perpendicular to T through $\hat{x}$ (resp. $x*$). Denote by H_i^+ the halfspace not containing T, for i = 1,2. Then $A(H_1^+ \cap P) \ge (1-\hat{a})A(P)$ and $A(H_2^+ \cap P) \ge (1-a*)A(P)$. Therefore every line H perpendicularly intersecting T satisfies the condition of lemma 4.2 with $t = 1-\hat{a}$. By that lemma, for each such H, $L(H \cap P) \ge (1-\hat{a})\frac{A(P)}{D(P)}$.

Therefore, the area in P "between" H_1 and H_2 satisfies

$$A(P \cap H_1^- \cap H_2^-) \ge L(T)(1-\hat{a})\frac{A(P)}{D(P)}.$$

Hence,

$$\hat{a}A(P) \ge A(H_1^- \cap P) \ge A(H_2^+ \cap P) + L(T)(1-\hat{a})\frac{A(P)}{D(P)} \ge (1-a*)A(P) + L(T)(1-\hat{a})\frac{A(P)}{D(P)}.$$

From this and $a* \le \hat{a} \le b$ we obtain

$$L(T) \le \frac{2}{1-b}\left(\hat{a} - \frac{1}{2}\right)D(P). \tag{4}$$

Because in our successive relaxation algorithm S always contains both $\hat{x}$ and x^*, we have $L(T) \leq D(S)$. Combining the square of this inequality with the inequality (4), and applying lemma 4.1, we have

$$(L(T))^3 \leq \frac{2}{1-b}(D(S))^2 D(P)\left(\hat{a} - \frac{1}{2}\right) \leq \frac{2}{1-b}(D(S))^2 D(P) \frac{2D(P)^2 A(S)}{A(P)D(S)^2} = \frac{4}{1-b}(D(P))^3 \frac{A(S)}{A(P)}.$$

Theorem 4.1 tells us that any algorithm which generates a sequence of Simpson polygons, with area converging to 0, must also converge pointwise. To apply Theorem 4.1 to our algorithm, we can take $b = \frac{5}{9}$ because the initial triangle provides a value no larger. Given that after $2m$ iterations, $A(S) \leq \left(\frac{5}{9}\right)^m A(P)$, we conclude:

Corollary 4.1 *Algorithm Find_Simpson_II pointwise converges at rate*:

$$L(\overline{\hat{a}a^*}) \leq \sqrt[3]{9} D(P) \sqrt[3]{\frac{A(S)}{A(P)}} \leq \sqrt[3]{9} D(P) \left(\frac{5}{9}\right)^{\frac{m}{6}}$$

where m *is the number of iterations*.

Similarly, Theorem 1 applies to the original Diaz-O'Rourke algorithm. We can take $b = \frac{5}{9}$ again because of the centroid property stated earlier.

Corollary 4.2 *Algorithm Find_Simpson_I pointwise converges at rate*:

$$L(\overline{\hat{a}a^*}) \leq \sqrt[3]{9} D(P) \sqrt[3]{\frac{A(S)}{A(P)}} \leq \sqrt[3]{9} D(P) \left(\frac{5}{9}\right)^{\frac{m}{3}}$$

where m is the number of iterations.

4.4 Extensions to the non-uniform case

Some of the convergence properties extend readily to the case of non-uniform convergence. The main idea is to use some of the results about area unchanged, and to alter some other results using bounds on voter population density.

Assume that a continuous probability measure $m(x)$ is defined on *P*. By compactness of P, *m* attains a maximum value $m_{\max} / A(P)$. Retaining the definitions of length *L*() and area *A*(), define the demand of a region as $V(S) \equiv \int_S m(x)dx$. The definition of $a_P()$ changes to

$$a_P(x) \equiv \frac{V(H(x))}{V(P)} = V(H(x))$$

since $V(P) = 1$. The definition of Simpson polygon changes to:

Definition 4.2 *A convex polygon* S *is an* a *-Simpson polygon of convex polygon P if* $S \subseteq P$ *and each side of polygon* S *is defined by a line* G *such that* $V(G^- \cap P) \geq a$.

Claims 4.2, 4.3, and 4.4 are all in terms of $A()$ and remain unchanged. Lemma 4.1 and its proof change as follows:

Lemma 4.3 *Let* $\frac{1}{2} < \hat{a} < 1$ *and let* S *be an* $\hat{a}$ *- Simpson polygon of* P *. Then*

$$D(S)\sqrt{\hat{a} - \frac{1}{2}} \leq D(P)\sqrt{2m_{\max}\frac{A(S)}{A(P)}}.$$

Proof: Let *F* and *G* be lines according to claim 3. As in claim 4, let $Q = P \cap F^- \cap G^-$. By inclusion-exclusion,

$$V(Q) \geq V(P) - V(P \cap F^+) - V(P \cap G^+) \geq (1 - (1 - \hat{a}) - (1 - \hat{a})) = 2\left(\hat{a} - \frac{1}{2}\right).$$

(The second inequality holds because *F* and *G* are lines defining sides of *S*, and *S* is a $\hat{a}$ -Simpson polygon of *P*.) On the other hand, $V(Q) \leq m_{\max} A(Q)/A(P)$.

Combining these two inequalities with the upper bound on $A(Q)$ from claim 4.4 gives the result.

If $m(x) > 0$, there are finite nonzero bounds on minimum density and ratio of densities as well. However, this does not seem to be sufficient to generalize lemma 4.2 and Theorem 4.1.

5 Conclusions and Future Research

We have investigated the problem of finding the Simpson Point, also called the (1|1) centroid, of a convex polygon where the demand is continuously distributed over it. We have presented an enhancement of the Diaz and O'Rourke's iterative search algorithm for finding the Center of Area that solves the version of the problem when the demand distribution is uniform, reducing the time requirements from $O(|\log \varepsilon|^2 + n|\log \varepsilon|)$ to $O(n|\log \varepsilon|)$. We have also proved pointwise convergence (at unchanged time order cost) for both our version and Diaz and O'Rourke's original algorithm. Last, we outlined a high level modification of our enhanced algorithm for the problem when the demand distribution is continuous but non-uniform, which runs in time $O(\tau(n)(\log^2 n + |\log \varepsilon|))$ and pointwise converges if the value is bounded away from 1/2. One immediate avenue for future research is to address the problem of finding the Simpson Point of a uniformly distributed population on a non-convex polygon. This problem is asymptotically close to a special case of non-uniform continuous distributions on convex polygons. Although Diaz and O'Rourke have developed an algorithm to find the Center of Area of non-convex polygons, it is not known yet if that also resolves the Simpson Point problem in the uniform and/or the non-uniform demand distribution case. Another avenue is to develop representations of non-uniform distributions and associated low-level subroutines to calculate centroids, $H(x)$, etc. Yet another open question is whether pointwise convergence, with a rate independent of the gap $(\mu(H(x^*)) - ½)$, holds in these more general settings.

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