The Probability of Majority Rule Instability in the 2D Euclidean Model with an Even Number of Voters

Craig A. Tovey

School of ISyE, Georgia Tech, Atlanta Ga 30332 U.S.A.

Abstract

The classic instability theorems of Euclidean voting theory definitively treat all cases except that of an even number of voters in 2 dimensions. For that case, all that has been known is that the set of stable configurations is neither measure zero nor measure one. We prove that instability occurs with probability converging rapidly to 1 as the population increases.

Key words: spatial voting, equilibrium, stability, Euclidean preferences, majority rule

1. Instability in Two Dimensions

1.1. Introduction.

The Downsian or spatial model of voting under Euclidean or quadratic concave preferences is a widely used model of group choice and has found extensive empirical application as well, particularly in 2 dimensions (11; 6, e.g.). The work of Plott (10), McKelvey (9; 7; 8), Rubenstein (12), Schofield (13), Banks (2), Saari (3), and Banks, Duggan, and LeBreton (1) has shown that the probability of equilibrium is 0 in three or more dimensions, and in two dimensions when the number of voters is odd. To be more precise, the set of configurations for which equilibrium exists is measure 0 for these cases. In contrast, one dimension always admits of an equilibrium. The case of 2 dimensions with an even number of voters has never been resolved. All that is known is that the set of configurations for which equilibrium exists, and the set for which equilibrium does not exist, both have positive measure. (The first fact can be established by considering $2n-1$ points at the vertices of a regular polygon on $2n-1$ vertices, and one point at the polygon's center. The center point is an equilibrium, and small perturbations of the $2n$ points do not disturb the equilibrium. The second fact is established by considering $2n$ points at the vertices of a regular $2n$-gon. No equilibrium exists, and nonexistence is unaffected by small perturbations of the $2n$ points.) In this paper we show that the probability of equilibrium converges to 0 exponentially rapidly in this case, if voters are sampled i.i.d. from any nonsingular distribution.

1.2. Definitions.

In the 2D Euclidean spatial model of voting, n voter ideal points are located in $\Re^2$ and voters prefer policies (points) closer to their ideal points under the Euclidean norm. This model is equivalent to the more general case of convex quadratic preferences, (see (4)). An equilibrium point is one that can not be dislodged by majority vote. That is, y is an equilibrium point iff there does not exist x such that strictly more than $\frac{n}{2}$ of the ideal points are closer to x than to y, under the Euclidean norm.

The case of n even differs from other cases because the answer is not simply zero or one. Therefore, the probability measure from which voter ideal points are sampled could affect the answer. However, it will turn out that, aside from a nonsingularity condition, the particular measure matters little.

We say that probability measure μ on $\Re^2$ is *nonsingular* if for every line L, $\mu(L) = 0$. Nonsingularity is a weaker condition than absolute continuity with respect to Lebesgue measure; for example, the uniform distribution on the edge of a circle satisfies the former but not the latter condition.

1.3. Main Result.

Theorem 1.1. *Let μ be any nonsingular probability distribution on $\Re^2$. Let $2m$ voter ideal points be sampled independently at random according to μ. The probability the resulting configuration possesses an equilibrium is less than*

$$\frac{\sqrt{8\pi m}}{e^{m-\frac{1}{12m}}}. \tag{1}$$

Corollary 1.2. *Under the assumptions of Theorem 1, as $m \to \infty$, equilibrium does not occur a.e.*

Proof. Since μ is nonsingular, with probability 1 no three points are collinear. Similarly, with probability 1 no three lines formed by pairs of points intersect at a point. Therefore with probability 1, if there is an equilibrium point it occurs at a voter ideal point. Let $v_1, \ldots, v_{2m}$ denote the ideal points.

Let E denote the event that v_1 is an equilibrium point. We bound $Pr(E)$. Consider the vertical line in the plane passing through v_1. By nonsingularity, the probability is 0 that the line passes through another ideal point. If there are more than m ideal points to the left (respectively right) of the line, v_1 is not an equilibrium point, because it would be defeated by a proposal immediately to its left (respectively right). Define E_1 to be the event that there are m points on one side of the line and $m-1$ on the other. Then $Pr(E) = Pr(E_1)Pr(E|E_1)$. Now $Pr(E_1) = \frac{2}{2m} = \frac{1}{m}$ because the horizontal component of v_1 would have to be either the mth or $m+1$th largest out of $2m$ values.

We now bound $Pr(E|E_1)$. By symmetry, we may assume, when we condition on the event E_1, that there are $m-1$ points to the left of the line and m to the right. See Figure 1. Pin the line at v_1 and color the two rays emanating from v_1 black and white. Keeping the line pinned at v_1 rotate the line clockwise 180 degrees. The white ray traverses the area to the right and the black ray the area to the left of the original vertical line. As the line rotates, it crosses over the other v_i in the plane. The only way for v_1 to be an equilibrium is if the first crossing is white, the next crossing is black, and so on until the last crossing is white. For if there were two consecutive white (black) crossings, more than $n/2$ ideal points would lie in the open half-plane defined by a line through v_1, whence v_1 would not be a core point (4). Let E_2 denote the event that the crossings are interleaved, i.e., alternate in color. Then $Pr(E|E_1) = Pr(E_2|E_1)Pr(E|E_1 \bigcap E_2) = Pr(E_2|E_1)$.

The key step now is to condition on the locations of all the black crossings. We will show that regardless of their locations, $Pr(E_2|E_1)$ is small. The half-plane where the white crossings will occur is divided into sectors. (See Figure 1). These sectors are defined by the black crossings, which we have conditioned on. For perfect interleaving, exactly one of the m white crossings must fall into each sector $1, 2, \cdots, m$. Let p_j denote the probability that a voter point drawn from μ falls into sector $1 \leq j \leq m$, conditioned on the event that it falls into one of these sectors. Then $\sum_j^{m-1} p_j = 1$, and the probability of perfect interleaving is $(m)!\prod_{j=1}^{m} p_j$. By Jensen's inequality, this quantity is maximized over the possible values of $p_1 \ldots p_m$ when $p_j = \frac{1}{m}\forall j$, in which case it equals $(m)!/(m)^m$.

Therefore $Pr(E) \leq \frac{2}{2m}m!/m^m$. By symmetry, the same bound holds for the probability that V_i is an equilibrium, for each $2 \leq i \leq 2m$. The probability that there exists an an equilibrium point is therefore bounded by $2m\frac{2}{2m}m!/m^m \leq \sqrt{8\pi m}e^{\frac{1}{12m}-m}$ by the extended Stirling approximation (5).

For the corollary, let I_m be the indicator function that equals 1 if there is an equilibrium and equals 0 if there is not. Consider the infinite sum over m of (1): it obviously converges because of the e^{-m} term. Therefore, with probability 1, $limsup_{m\to\infty}\ I_m = 0$, and the corollary follows by the Borel-Cantelli Theorem . □

Theorem 1 is a rather powerful result, as it allows the sampling to occur from *any* nonsingular distribution μ. For very small m the probability of an equilibrium point may be fairly large. For instance the probability is 1 when there are two or four voters ($m = 1, 2$). So Theorem 1 may be counterintuitive to those familiar with the behavior of small cases. But convergence is rapid. For 12 voters the probability bound given by (1) is approximately 2%.

References

[1] J.S. Banks, J. Duggan and M. Le Breton, *Social choice and electoral competition in the general spatial model*, Journal of Economic Theory, **126** (2006), 194–234.

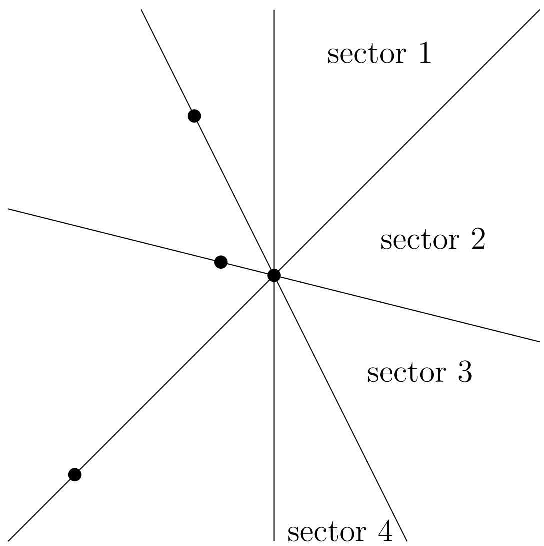

Figure 1: The three points to the left of the vertical line define the m=4 sectors

[2] J.S.Banks,*Singularity Theory and Core Existence in the Spatial Model*, Journal of Mathematical Economics **24** (1995), 523-536.

[3] D. Saari, *The Generic Existence of a Core for q-Rules*, Economic Theory **9**, 219-260.

[4] O.A.Davis, M.H. DeGroot and M. J. Hinich, *Social Preference Orderings and Majority Rule*, Econometrica **40** (1972), 147–157.

[5] W. Feller, *An Introduction to Probability Theory and Its Applications, Vol. 1, 3rd ed.*, Wiley, New York (1968), 50–53.

[6] G.H. Jeong, *Testing the Predictions of the Multidimensional Spatial Voting Model with Roll Call Data*, Political Analysis **16** (2008), 179–196.

[7] R.D.McKelvey, *General conditions for global intransitivies in formal voting models*, Econometrica **47** (1979),1085–1112.

[8] R.D. McKelvey and N. J. Schofield, *Generalized Symmetry Conditions at a Core Point* , Econometrica **55** (1987), 923–933.

[9] R.D. McKelvey and R.E. Wendell, *Voting equilibria in multidimensional choice spaces*, Mathematics of Operations Research **1** (1976), 144–158.

[10] C. Plott, *A notion of equilibrium and its possibility under majority rule*, American Economic Review **57** (1967),787–806.

[11] K. Poole and H. Rosenthal, *D-NOMINATE After 10 Years: An Update to Congress: A Political-Economic History of Roll Call Voting*, Legislative Studies Quarterly **26** (2001),5–29.

[12] A. Rubinsein, *A note about the 'nowhere denseness' of societies having an equilibrium under majority rule*, Econometrica **47** (1979), 511–514.

[13] N. Schofield, *Generic instability of Majority Rule*, Review of Economic Studies **50** (1983), 696–705.