Wavelets Associated with Periodic Basis Functions

by

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Abstract

In this paper, we investigate a class of nonstationary, orthogonal, periodic scaling functions and wavelets generated by continuously differentiable periodic functions with positive Fourier coefficients; such functions are termed *periodic basis functions.* For this class of wavelets, the decomposition and reconstruction coefficients can be computed in terms of the discrete Fourier transform, so that FFT methods apply for their evaluation. In addition, decomposition at the n^{th} level only involves 2 terms from the higher level. Similar remarks apply for reconstruction. We apply a periodic "uncertainty principle" to obtain an angle/frequency uncertainty "window" for these wavelets, and we show that for many wavelets in this class the angle/frequency localization is good.

0. Introduction

Overview. In this paper, we investigate a class of nonstationary, orthogonal, periodic scaling functions and wavelets generated by continuously differentiable periodic functions with positive Fourier coefficients; these functions, which were introduced long ago by Schoenberg [13], will be called *periodic basis functions* (PBFs). The practical motivation for our investigation came from a desire to analyze naturally periodic data arising in connection with a neural network employed by Simmers *et al.* [14] to deal with a phased-array radar antenna, while the theoretical motivation was provided by the paper [4], where a similar analysis was carried out with functions in $L^2(\mathbb{R})$ that have positive Fourier transforms. We will discuss possible applications below, but first we wish to describe our results and provide a background for them.

Various types of semiorthogonal, nonstationary periodic wavelets have already been introduced. Chui and Mhaskar [3] were the first to construct such wavelets. Somewhat later, Prestin and Quak [12] introduced a computationally efficient set of semiorthogonal, periodic wavelets. Recently, Plonka and Tasche [10] used periodic splines to construct other periodic wavelets; their decomposition and reconstruction algorithm makes heavy use of the fast Fourier transform (FFT). Quite recently, Koh, Lee, and Tan have constructed orthogonal periodic wavelets using splines [6].

Central to any wavelet analysis is the choice of the "sampling" spaces. Because PBFs provide an effective means for doing interpolation and least-squares fitting of data, even scattered data, generated by an underlying periodic function [9, 17], it is natural to work with sampling spaces $\mathcal{V}_n$ initially generated as spans of translates of a fixed PBF P; specifically, at level n the sampling space $\mathcal{V}_n$ is given by

$$\mathcal{V}_n = \mathrm{Span}\{P(\theta - 2\pi k/n),\ k = 0, \dots, n-1\}.$$

From the PBF P, we construct a scaling function φ^n whose translates $\varphi^n(\theta - 2\pi k/n)$, $k = 0, \dots, n-1$, form an orthonormal basis for $\mathcal{V}_n$. With the scaling function in hand, one can construct a wavelet ψ^n whose $n-1$ translates also comprise an orthonormal basis, this time for the orthogonal difference $\mathcal{W}_n = \mathcal{V}_{2n} \ominus \mathcal{V}_n$. When n is a power of 2, the sampling spaces are nested. For this case, we give a decomposition/reconstruction algorithm that is $\mathcal{O}(n \log_2 n)$.

The scaling functions and wavelets presented here have a number of attractive features that distinguish them from other periodic wavelets. The most important of these is that many wavelets constructed in this class have excellent localization properties. Indeed, one of the main results in this paper is that both the scaling functions and wavelets derived from a large class of PBFs, which are described in §7, have "periodic" standard deviations that are $\mathcal{O}(1/n)$ at level n, where the notion of periodic standard deviation is that described by Breitenberger in [2]. In fact, a periodic "uncertainty principle" applies in this setting, and one can show that the angle/frequency uncertainty "window" is $\mathcal{O}(\sqrt{n})$ at level n, at least for scaling functions wavelets associated with this class of PBFs. Another important distinguishing feature is that one can construct wavelets and scaling functions that not only possess the localization/uncertainty properties mentioned above, but also have any degree of smoothness desired. Finally, because PBFs can be used to stably interpolate *scattered data,* one can use the same PBF employed in constructing our periodic wavelets and scaling functions to interpolate such data, thereby providing a method of projecting such data into a sampling space.

Other attractive features, which are shared with some of the other periodic wavelets metioned earlier, is that the wavelets and scaling functions we construct have translates that are orthonormal, thus eliminating the need for dual functionals. Such wavelets also have good stability properties, because, for orthonormal bases, the Riesz constants are unity. The algorithms for analyzing with our wavelets and scaling functions are ultimately simple. The decomposition and reconstruction coefficients can be computed in terms of the discrete Fourier transform, so that FFT methods apply for their evaluation. In addition, decomposition at the n^{th} level only involves 2 terms from the higher level, as opposed to the $2n$ terms one needed with the earlier periodic wavelets. A similar result applies for reconstruction.

We want to briefly describe an application of our results to a neural network for which the input data is naturally periodic, like the one in [14]. To keep matters simple, suppose that the network is designed to fit data of the form $(x_j\,,\theta_j)$, where x_j is in the interval $[0,1]$, and θ_j is an angular variable. In the training phase, one may certainly select x_j's to be equally spaced. The difficulty is that the θ_j's will invariably be scattered. Since PBFs may interpolate (or fit via least squares) such data, one surmounts this difficulty by using a PBF basis of the form $\{P(\theta-\theta_j)\}$. In the operating phase, one would like to monitor the neural network for damage. In mathematical terms, the neural network is really a periodic function mapping $\theta \to x$. The idea is to test the operating network by using PBF wavelets to monitor its output at equally spaced θ's. To do this, one would employ wavelets constructed out of P to detect singularities, which would show up as unusually large wavelet coefficients. Such a combined approach has a cost advantage. If the neural network is designed to operate in the field, so that it has to be constructed from special purpose hardware, one needs to have a chip that will compute $P(\theta)$. Since this is the only function that has to be evaluated, only one chip is required.

Organization. The paper is organized as follows. We will finish this section with a brief discussion of PBFs, along with a few remarks about notation. In §1, we will construct the periodic scaling function φ^n. In addition to having translates that form an orthonormal basis for $\mathcal{V}_n$, we will show that it is real valued and even. In §2, we will construct the level-n wavelet, ψ^n. Like the scaling function, the wavelet is real valued, and is even about $\theta = -\pi/n$. The decomposition and reconstruction sequences are derived in §3. The density of the wavlet spaces is established in §4, while the angular localization estimates are obtained in §5. In §6, we discuss frequency localization and an angle/frequency uncertainty principle. In §7, we give an explicit family of PBFs, all of which are simple polynomials in θ on the interval $[0,2\pi]$. These PBFs belong to $C^{2\alpha-2}$ of the circle, where 2α is the degree of the polynomial. For one of these, we display graphs of both the corresponding scaling function and wavelet to illustrate angular localization. Finally, in §8 we present an overall algorithm for analysis with the PBF wavelets presented here.

Background and notation. Let $P(\theta)$ be a 2π-periodic, continuously differentiable, real-valued function having the Fourier decomposition

$$P(\theta) = \sum_{m\in\mathbb{Z}} p_m e^{im\theta},$$

where the coefficients satisfy $p_m > 0$ for all m. Of course, standard theorems from the theory of Fourier series coupled with the level of smoothness and the reality of P imply that the coefficients are summable and satisfy $p_m = p_{-m}$. Any such P will be called a *periodic basis function,* or PBF, and throughout the paper P will be a PBF, and p_m will be its m^{th} Fourier coefficient.

PBFs are special cases of a class of functions originally introduced by Schoenberg [13], the positive definite spherical functions, and are useful for doing scattered data interpolation (or least-squares) problems on the circle. Because solving such problems allow us to project data into a sampling space, we will briefly describe them. Let $\theta_1, \ldots, \theta_n$ be angles corresponding to distinct, possibly scattered, points on the unit circle, and suppose that $d_1, \ldots, d_n$ are complex numbers representing data collected at $\theta_1, \ldots, \theta_n$, repectively. We wish to find a periodic function $s(\theta) = \sum_k c_k P(\theta - \theta_k)$ for which $s(\theta_j) = d_j$. Solving this problem is equivalent to showing that the interpolation matrix with entries $P(\theta_j - \theta_k)$ is invertible. That this is the case, indeed, that the matrix is positive definite, was shown very recently by Xu and Cheney [17]. We remark that if one assumes that P is C^∞, then one may also do virtually any Hermite interpolation problem [9, §5].

In addition to PBFs, we will make heavy use of the discrete Fourier transform (DFT). We will follow the conventions employed by Folland [5, §7.6]. If a_ℓ is a periodic sequence, with period n, then the formulas for $\hat{a}_k$, the DFT of a_ℓ, and for the inverse DFT (IDFT) are

$$\text{DFT: } \mathcal{F}_d\left[\{a_\ell\}_{\ell=0}^{n-1}\right]_k := \hat{a}_k = \sum_{\ell=0}^{n-1} e^{-2\pi k\ell/n} a_\ell,$$

$$\text{IDFT: } \mathcal{F}_d^{-1}\left[\{\hat{a}_k\}_{k=0}^{n-1}\right]_\ell := a_\ell = \frac{1}{n}\sum_{k=0}^{n-1} e^{2\pi k\ell/n} \hat{a}_k.$$

Throughout the paper, the inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$ will be the standard ones for $L^2[-\pi,\pi]$. The terms "orthogonal" and "orthonormal" will always be taken in reference to the standard inner product and standard norm for $L^2[-\pi,\pi]$.

1. The scaling function.

We want to construct scaling functions associated with $P(\theta)$. To do this, suppose that we have n equally spaced angles, $2\pi k/n$, $k = 0, \ldots, n-1$. Consider the space

$$\mathcal{V}_n := \text{Span}\{P(\theta - 2\pi k/n)\}_{k=0}^{n}.$$

We want to show how to construct a function $\varphi^n \in \mathcal{V}_n$ such that

$$\{\varphi^n(\theta - 2\pi k/n)\}_{k=0}^{n}$$

is an orthonormal basis for $\mathcal{V}_n$.

We begin our construction by rearranging the (absolutely convergent) Fourier series for P as follows:

$$P(\theta) = \sum_{k=0}^{n-1} \underbrace{\sum_{m \equiv k \bmod n} p_m e^{im\theta}}_{P_k^n(\theta)} . \tag{1.1}$$

The functions P_k^n defined in (1.1) are central to the entire construction. We summarize their essential properties in the proposition below.

PROPOSITION 1.1: *The set of functions $\{P_k^n\}_{k=0}^{n-1}$ is an orthogonal basis for $\mathcal{V}_n$, with each P_k^n being as smooth as P itself, and satisfying*

$$P_k^n(\theta - 2\pi\ell/n) = e^{-2\pi ik\ell/n} P_k^n(\theta) \tag{1.2}$$

$$P_{k+n}^n(\theta) = P_k^n(\theta) \tag{1.3}$$

$$P(\theta - 2\pi\ell/n) = \sum_{k=0}^{n-1} e^{-2\pi ik\ell/n} P_k^n(\theta) \tag{1.4}$$

$$P_k^n(\theta) = \frac{1}{n}\sum_{\ell=0}^{n-1} P(\theta - 2\pi\ell/n) e^{2\pi ik\ell/n} \tag{1.5}$$

$$\overline{P_k^n(\theta)} = P_{-k}^n = P_{n-k}^n(\theta). \tag{1.6}$$

PROOF: The functions in the set $\{P_k^n\}_{k=0}^{n-1}$ are obviously mutually orthogonal in the usual inner product on $L^2[-\pi,\pi]$, because they have no common frequency components. To see that the (1.2) holds, note that if $m \equiv k \bmod n$, then $m = qn + k$ for some q. Hence, $m(-2\pi\ell/n) = -2\pi q\ell - 2\pi(k\ell/n)$. We therefore have that

$$e^{im(\theta - 2\pi\ell/n)} = e^{im\theta} e^{-2i\pi k\ell/n}.$$

Since the second factor on the right above is independent of m, it may be factored out of the series defining P_k^n, which results in (1.2). That (1.3) holds is obvious from the definition of the P_k^n's. To obtain (1.4), first make the replacement $\theta \to \theta - 2\pi\ell/n$ in (1.1), and then use (1.2). Note that for θ fixed, the right side of (1.4) is the DFT of the finite sequence $\{P_0(\theta), \ldots, P_{n-1}(\theta)\}$. Applying the inversion formula for the DFT to (1.4) yields (1.5). There are two immediate consequences of (1.5). First, $P_k^n(\theta)$ is as smooth as P, and, second, that the P_k^n's are in $\mathcal{V}_n$. Since by (1.4) $\{P_k^n\}_{k=0}^{n-1}$ spans $\mathcal{V}_n$, and since the functions in it all orthogonal and nonzero, $\{P_k^n\}_{k=0}^{n-1}$ is linearly independent as well, which establishes that it is an orthogonal basis for $\mathcal{V}_n$. Equation(1.6) can be established by first noting that the assumptions made on the PBF P imply that each Fourier coefficient

$p_m = \bar{p}_m = p_m > 0$, so that

$$
\begin{aligned}
\overline{P_k^n(\theta)} &= \sum_{m \equiv k \bmod n} \bar{p}_m e^{-im\theta} \\
&= \sum_{m \equiv k \bmod n} p_{-m} e^{-im\theta} \\
&= \sum_{m \equiv -k \bmod n} p_m e^{im\theta} \\
&= P_{-k}^n(\theta).
\end{aligned}
$$

Since from (1.3) we have that P_k^n is n-periodic in k, we get $P_{-k}^n = P_{n-k}^n$, which establishes the rest of (1.6). ∎

Remark 1.2: The result above establishes that the dimension of $\mathcal{V}_n$ is n, which immediately implies that $\{P(\theta - 2\pi k/n)\}_{k=0}^{n-1}$ is linearly independent, and hence is itself a basis for $\mathcal{V}_n$. Changing coordinates from this basis to $\{P_k^n\}_{k=0}^{n-1}$ and back is easy to do. Let $f \in \mathcal{V}_n$. Equation (1.4), standard linear algebra, and the definition of the DFT imply that

$$
f(\theta) = \sum_{\ell=0}^{n-1} c_\ell^n P(\theta - 2\pi\ell/n) \quad \text{if and only if} \quad f(\theta) = \sum_{k=0}^{n-1} \hat{c}_k^n P_k^n(\theta). \quad ∎
$$

Remark 1.3: The L^2-norms of the P_k^n's play an important role in all our constructions. By Parseval's Theorem, we have

$$
\nu_k^n := \|P_k^n(\theta)\|^2 = \sum_{m \equiv k \bmod n} 2\pi p_m^2.
$$

Taking norms of both sides in (1.3) and (1.6) implies this useful identity

$$
\nu_k^n = \nu_{k+n}^n = \nu_{n-k}^n. \quad ∎ \tag{1.7}
$$

We now turn to defining the analogue of the scaling function in the theory of wavelets, the function φ^n with the property that its translates form an orthonormal basis for $\mathcal{V}_n$. We will simply write down our candidate and verify that it has the desired properties.

Theorem 1.4: *If the function φ^n is defined by*

$$
\varphi^n(\theta) := \sum_{k=0}^{n-1} \frac{P_k^n(\theta)}{\sqrt{n}\|P_k^n\|} = \sum_{k=0}^{n-1} \frac{P_k^n(\theta)}{\sqrt{n\nu_k^n}}, \tag{1.8}
$$

then $\{\varphi^n(\theta - 2\pi\ell/n)\}_{\ell=0}^{n-1}$ is an orthonormal basis for $\mathcal{V}_n$. In addition, φ^n is as smooth as P, and has the absolutely convergent Fourier series decomposition $\varphi^n(\theta) = \sum_{m\in\mathbb{Z}} \varphi_m^n e^{im\theta}$, where

$$
\varphi_m^n = \frac{p_m}{\sqrt{n\nu_k^n}}, \; m \equiv k \bmod n, \; 0 \le k \le n-1. \tag{1.9}
$$

Finally, φ^n is a real-valued PBF in its own right, and is an even function of θ.

PROOF: From (1.2) and (1.8), we see that

$$\varphi^n(\theta - 2\pi\ell/n) = \sum_{k=0}^{n-1} \frac{e^{-2\pi ik\ell/n}}{\sqrt{n}\|P_k^n\|} P_k^n(\theta) = \sum_{k=0}^{n-1} \frac{e^{-2\pi ik\ell/n}}{\sqrt{n\nu_k^n}} P_k^n(\theta). \tag{1.10}$$

Using this, (1.8), and the orthogonality of the P_k^n's leads to the following chain of equations:

$$\begin{aligned}
\langle \varphi^n(\theta - 2\pi\ell/n), \varphi^n(\theta - 2\pi\ell'/n)\rangle &= \sum_{k=0}^{n-1}\sum_{k'=0}^{n-1} e^{-2\pi ik\ell/n} e^{2\pi ik'\ell'/n} \langle \frac{P_k^n(\theta)}{\sqrt{n}\|P_k^n\|}, \frac{P_{k'}^n(\theta)}{\sqrt{n}\|P_{k'}^n\|}\rangle \\
&= \sum_{k=0}^{n-1}\sum_{k'=0}^{n-1} \frac{e^{-2\pi ik\ell/n} e^{2\pi ik'\ell'/n}}{n} \delta_{k,k'} \\
&= \sum_{k=0}^{n-1} \frac{e^{2\pi ik(\ell'-\ell)/n}}{n} \\
&= \delta_{\ell,\ell'},
\end{aligned}$$

where the last step follows from the standard formula used to invert the DFT [5, §7.6]. This establishes the orthonormality of the set. That φ^n is as smooth as P follows from the fact that the P_k^n's are as smooth as P, which was established in Proposition 1.1. The Fourier series representation (1.9) follows from (1.8) and the Fourier series for the P_k^n's given in (1.1); the absolute convergence of the series follows from φ^n being at least continuously differentiable. Moreover, inspecting the series in (1.9) and using $p_m > 0$ implies that the Fourier coefficients are positive. Taken together with the smoothness of φ^n, this is sufficient for φ^n to be a PBF. To see that it is real, note that by (1.8) and (1.6)

$$\begin{aligned}
\overline{\varphi^n(\theta)} &= \sum_{k=0}^{n-1} \frac{\overline{P_k^n(\theta)}}{\sqrt{n\nu_k^n}} \\
&= \sum_{k=0}^{n-1} \frac{P_{n-k}^n(\theta)}{\sqrt{n\nu_{n-k}^n}}
\end{aligned}$$

By changing the summation index to $n-k$ and using $P_0^n = P_n^n$ and $\nu_0^n = \nu_n^n$, we see that

$$\overline{\varphi^n(\theta)} = \sum_{k=0}^{n-1} \frac{P_k^n(\theta)}{\sqrt{n\nu_k^n}} = \varphi^n(\theta),$$

and so φ^n is real. Finally, every real-valued, periodic function with real Fourier coefficients is even. ∎

REMARK 1.5; We now have three bases for $\mathcal{V}_n$: the scaling basis $\{\varphi^n(\theta - 2\pi\ell/n)\}$; the basis $\{P_k^n(\theta)\}$; and the PBF basis, $\{P(\theta - 2\pi\ell/n)\}$. Again, let $f \in \mathcal{V}_n$. From (1.10), standard linear algebra, and the definition of the DFT, we have

$$f(\theta) = \sum_{\ell=0}^{n-1} s_\ell^n \varphi^n(\theta - 2\pi\ell/n) \quad \text{if and only if} \quad f(\theta) = \sum_{k=0}^{n-1} \frac{\hat{s}_k^n}{\sqrt{n\nu_k^n}} P_k^n(\theta). \tag{1.11}$$

On the other hand, from Remark 1.2, we have

$$f(\theta)=\sum_{\ell=0}^{n-1} c_\ell^n P(\theta-2\pi\ell/n) \quad \text{if and only if} \quad f(\theta)=\sum_{k=0}^{n-1}\hat{c}_k^n P_k^n(\theta). \tag{1.12}$$

Comparing (1.11) and (1.12), we see that the coordinates relative to the scaling basis and the PBF basis are related by

$$\hat{c}_k^n=\frac{\hat{s}_k^n}{\sqrt{n\nu_k^n}}. \quad ∎ \tag{1.13}$$

Since $\mathcal{V}_n\subset\mathcal{V}_{2n}$, we may expand φ^n in terms of the scaling basis at level $2n$,

$$\varphi^n(\theta)=\sum_{\ell=0}^{2n-1} a_\ell^{2n}\varphi^{2n}(\theta-2\pi\ell/(2n)). \tag{1.14}$$

This is the *scaling relation*. We want to find the coefficients a_ℓ^{2n}, or equivalently, their DFT. This we now do.

COROLLARY 1.6: *The DFT of the a_ℓ^{2n}'s in the scaling relation satisfies*

$$\hat{a}_k^{2n}=\sqrt{\frac{2\nu_k^{2n}}{\nu_k^n}}. \tag{1.15}$$

PROOF: Note that from (1.1) we have $P_k^n=P_k^{2n}+P_{k+n}^{2n}$. Use this in (1.8) and then rearrange the sum, making use of $\nu_{k+n}^n=\nu_k^n$; the result is

$$\begin{aligned}\varphi^n(\theta)&=\sum_{k=0}^{n-1}\frac{P_k^{2n}(\theta)+P_{k+n}^{2n}(\theta)}{\sqrt{n\nu_k^n}}\\&=\sum_{k=0}^{2n-1}\frac{1}{\sqrt{n\nu_k^n}}P_k^{2n}(\theta).\end{aligned} \tag{1.16}$$

By Remark 1.5, with $n\to 2n$, and (1.16), the DFT of the a_ℓ^{2n}'s in (1.14) satisfies $\hat{a}_k^{2n}/\sqrt{2n\nu_k^{2n}}=1/\sqrt{n\nu_k^n}$. Solving for $\hat{a}_k^{2n}$ yields (1.15). ∎

We close this section by remarking that the a_ℓ^{2n} obviously depend on n. Consequently, the scaling functions and wavelets that we construct are *nonstationary.*

2. The wavelet.

The sampling space $\mathcal{V}_n$ is a proper subspace of $\mathcal{V}_{2n}$, and thus it has a nontrivial orthogonal complement in $\mathcal{V}_{2n}$; namely,

$$\mathcal{W}_n:=\mathcal{V}_{2n}\ominus\mathcal{V}_n.$$

We wish to construct a function $\psi^n \in \mathcal{W}_n$ such that the set of translates

$$\{\psi^n(\theta - 2\pi k/n)\}_{k=0}^{n}$$

is an orthonormal basis for $\mathcal{W}_n$. Such a function is a *wavelet* for $\mathcal{W}_n$.

Recall that the sets $\{P_k^{2n}\}_{k=0}^{2n-1}$ and $\{P_k^n\}_{k=0}^{n-1}$ are orthogonal bases for $\mathcal{V}_{2n}$ and $\mathcal{V}_n$, respectively. We will need the items below, which relate quantities associated with these bases:

$$P_k^n = P_k^{2n} + P_{k+n}^{2n}, \tag{2.1}$$

$$\|P_k^n\|^2 = \|P_k^{2n}\|^2 + \|P_{k+n}^{2n}\|^2, \tag{2.2}$$

$$\nu_k^n = \nu_k^{2n} + \nu_{k+n}^{2n}, \tag{2.3}$$

Equation (2.1) follows from (1.1), and (2.2) is a consequence of taking norms in (2.1). Finally, (2.3) is simply a restatement of (2.2) in terms of the ν_k's.

As we did with the scaling function, we will set forth the candidate for ψ^n and then show that it has the desired properties.

THEOREM 2.1: *If the function $\psi^n(\theta)$ is defined by*

$$\psi^n(\theta) = \sum_{k=0}^{2n-1} e^{ik\pi/n} \sqrt{\frac{\nu_{k+n}^{2n}}{n\nu_k^{2n}\nu_k^n}} P_k^{2n}(\theta). \tag{2.4}$$

then $\psi^n(\theta)$ is real valued and as smooth as $P(\theta)$, and the set $\{\psi(\theta - 2\pi\ell/n)\}_{\ell=0}^{n-1}$ is an orthonormal basis for $\mathcal{W}_n$. In addition, ψ^n has the absolutely convergent Fourier series decomposition $\psi^n(\theta) = \sum_{m\in\mathbb{Z}} \psi_m^n e^{im\theta}$, where

$$\psi_m^n = e^{ik\pi/n} \sqrt{\frac{\nu_{k+n}^{2n}}{n\nu_k^{2n}\nu_k^n}} p_m\,,\ \ m \equiv k \bmod(2n),\ \ 0 \le k \le 2n-1. \tag{2.5}$$

PROOF: Smoothness follows from the fact that the P_k^{2n}'s are linear combinations of translates of $P(\theta)$. That ψ^n is real is established by the following chain of equations:

$$\begin{aligned}
\overline{\psi^n(\theta)} &= \sum_{k=0}^{2n-1} e^{-ik\pi/n} \sqrt{\frac{\nu_{k+n}^{2n}}{n\nu_k^{2n}\nu_k^n}}\, \overline{P_k^{2n}(\theta)} \\
&= \sum_{k=0}^{2n-1} e^{-ik\pi/n} \sqrt{\frac{\nu_{k+n}^{2n}}{n\nu_k^{2n}\nu_k^n}} P_{2n-k}^{2n}(\theta) \quad \text{(from (1.6))} \\
&= \sum_{j=1}^{2n} e^{-i(2n-j)\pi/n} \sqrt{\frac{\nu_{2n-j+n}^{2n}}{n\nu_{2n-j}^{2n}\nu_{2n-j}^n}} P_j^{2n}(\theta) \\
&= \sum_{j=1}^{2n} e^{ij\pi/n} \sqrt{\frac{\nu_{j-n}^{2n}}{n\nu_j^{2n}\nu_j^n}} P_j^{2n}(\theta) \quad \text{(from (1.7))} \\
&= \sum_{j=0}^{2n-1} e^{ij\pi/n} \sqrt{\frac{\nu_{j+n}^{2n}}{n\nu_j^{2n}\nu_j^n}} P_j^{2n}(\theta) \quad \text{(from (1.3) \& (1.7))} \\
&= \psi^n(\theta) \quad \text{(from (2.4))}.
\end{aligned}$$

The first step in demonstrating that $\{\psi(\theta - 2\pi\ell/n)\}_{\ell=0}^{n-1}$ is an orthonormal basis for $\mathcal{W}_n$ is to show that $\psi(\theta - 2\pi\ell/n)$ is in $\mathcal{W}_n$. Since $\{P_k^n\}_{k=0}^{n-1}$ is an orthogonal basis for $\mathcal{V}_n$, showing that $\psi(\theta - 2\pi\ell/n)$ is in $\mathcal{W}_n$ amounts to showing that $\psi(\theta - 2\pi\ell/n)$ is orthogonal to each of the P_k^n's. From (1.2), with n replaced by $2n$, and (2.4), we see that

$$\psi(\theta - 2\pi\ell/n) = \sum_{k=0}^{2n-1} e^{ik\pi(1-2\ell)/n} \sqrt{\frac{\nu_{k+n}^{2n}}{n\nu_k^{2n}\nu_k^n}} P_k^{2n}(\theta) \tag{2.6}$$

From (2.1), (2.6) and the orthogonality of the P_k^{2n}'s, we have

$$\begin{aligned}&\langle \psi(\theta - 2\pi\ell/n), P_k^n(\theta)\rangle \\ &\qquad = e^{ik\pi(1-2\ell)/n} \sqrt{\frac{\nu_{k+n}^{2n}}{n\nu_k^{2n}\nu_k^n}} \|P_k^{2n}\|^2 + e^{i(k+n)\pi(1-2\ell)/n} \sqrt{\frac{\nu_{k+2n}^{2n}}{n\nu_{k+n}^{2n}\nu_{k+n}^n}} \|P_{k+n}^{2n}\|^2\end{aligned}$$

In the last equation, using $\nu_k^{2n} = \|P_k^{2n}\|^2$ and then simplifying the result yield

$$\langle \psi(\theta - 2\pi\ell/n), P_k^n(\theta)\rangle = e^{ik\pi(1-2\ell)/n} \left(\sqrt{\frac{\nu_{k+n}^{2n}\nu_k^{2n}}{n\nu_k^n}} - \sqrt{\frac{\nu_{k+2n}^{2n}\nu_{k+n}^{2n}}{n\nu_{k+n}^n}} \right).$$

Since $\nu_k^n = \nu_{k+n}^n$ and $\nu_k^{2n} = \nu_{k+2n}^{2n}$, we have that the right side of the last equation is 0; *i.e.*, $\psi(\theta - 2\pi\ell/n)$ is orthogonal to each of the P_k^n's. Consequently, $\psi(\theta - 2\pi\ell/n)$ is in $\mathcal{W}_n$ for $\ell = 0, \ldots, n-1$.

We now will show that the set $\{\psi(\theta - 2\pi\ell/n)\}_{\ell=0}^{n-1}$ is an orthonormal basis for $\mathcal{W}_n$. Since the dimension of $\mathcal{W}_n$ is $2n - n = n$, we need only demonstrate that this set is orthonormal. Again from (2.6), the orthogonality of the P_k^{2n}'s, and the fact that $\nu_k^{2n} = \|P_k^{2n}\|^2$, we obtain the orthonormality of $\{\psi(\theta - 2\pi\ell/n)\}_{\ell=0}^{n-1}$ as follows:

$$\begin{aligned}\langle \psi(\theta - 2\pi\ell/n), \psi(\theta - 2\pi\ell'/n)\rangle &= \sum_{k=0}^{2n-1} e^{2ik\pi(\ell'-\ell)/n} \left(\frac{\nu_{k+n}^{2n}}{n\nu_k^n}\right) \\ &= \sum_{k=0}^{n-1} e^{2ik\pi(\ell'-\ell)/n} \left\{ \left(\frac{\nu_{k+n}^{2n}}{n\nu_k^n}\right) + (-1)^{2\ell-2\ell'} \left(\frac{\nu_{k+2n}^{2n}}{n\nu_{k+n}^n}\right) \right\} \\ &= \sum_{k=0}^{n-1} e^{2ik\pi(\ell'-\ell)/n} \left(\frac{\nu_{k+n}^{2n} + \nu_k^{2n}}{n\nu_k^n}\right) \\ &= \frac{1}{n}\sum_{k=0}^{n-1} e^{2ik\pi(\ell'-\ell)/n} \quad \text{(from (2.3))} \\ &= \delta_{\ell,\ell'}.\end{aligned}$$

To get the Fourier series for ψ^n, replace the P_k^{2n}'s in (2.4) by their Fourier decompositions in (1.1) (with $n \to 2n$, of course). The series is absolutely convergent because ψ^n is at least continuously differentiable. ∎

REMARK 2.2: If we use (1.2)—with n replaced by $2n$ and $\ell = 1$—in (2.4), we get

$$\psi^n(\theta) = \sum_{k=0}^{2n-1} \sqrt{\frac{\nu_{k+n}^{2n}}{n\nu_k^{2n}\nu_k^n}} P_k^{2n}(\theta + \pi/n).$$

Consequently, we obtain

$$\psi^n(\theta - \pi/n) = \sum_{k=0}^{2n-1} \sqrt{\frac{\nu_{k+n}^{2n}}{n\nu_k^{2n}\nu_k^n}} P_k^{2n}(\theta). \tag{2.7}$$

If in (2.7) we again replace the P_k^{2n}'s by their Fourier series expansions, we will find that the Fourier coefficients for ψ^n are positive. Since ψ^n is real valued, this imples that $\psi^n(\theta-\pi/n)$ is a PBF that is an even function of θ. Of course, this means that the graph of $\psi^n(\theta)$ is symmetric with respect to the line $\theta = -\pi/n$. ∎

Since the wavelet ψ^n is in $\mathcal{V}_{2n}$, we may expand it in terms of the level $2n$ scaling basis,

$$\psi^n(\theta) = \sum_{\ell=0}^{2n-1} b_\ell^{2n} \varphi^{2n}(\theta - 2\pi\ell/(2n)). \tag{2.8}$$

This expansion is analogous to the scaling relation, (1.14). As in the case of the scaling relation, the DFT of the b_ℓ^{2n}'s has a simple form, which we give below.

COROLLARY 2.3: *The DFT of the b_ℓ^{2n}'s in (2.8) satisfies*

$$\hat{b}_k^{2n} = e^{ik\pi/n} \sqrt{\frac{2\nu_{k+n}^{2n}}{\nu_k^n}}. \tag{2.9}$$

PROOF: By Remark 1.5, with $n \to 2n$, (2.4), and (2.8), we have

$$\frac{\hat{b}_k^{2n}}{\sqrt{2n\nu_k^{2n}}} = e^{ik\pi/n} \sqrt{\frac{\nu_{k+n}^{2n}}{n\nu_k^{2n}\nu_k^n}}$$

Solving this equation for $\hat{b}_k^{2n}$ results in (2.9). ∎

3. Decomposition & Reconstruction.

The most efficient approach to obtaining decomposition and reconstruction formulas is to express the various functions in terms of an intermediate basis for $\mathcal{V}_{2n}$, namely, $\{P_k^{2n}\}_{k=0}^{2n-1}$.

We begin by expanding f in the basis comprising translates of φ^n and ψ^n,

$$f(\theta) = \sum_{\ell=0}^{n-1} s_\ell^n \varphi^n(\theta - 2\pi\ell/n) + \sum_{\ell=0}^{n-1} d_\ell^n \psi^n(\theta - 2\pi\ell/n). \tag{3.1}$$

In the next lemma, we will find the corresponding expansion in the P_k^{2n} basis.

LEMMA 3.1: *If $f \in \mathcal{V}_{2n}$ has the expansion (3.1), then*

$$f(\theta) = \sum_{k=0}^{2n-1} \left\{ \frac{1}{\sqrt{n\nu_k^n}} \hat{s}_k^n + \sqrt{\frac{\nu_{k+n}^{2n}}{n\nu_k^{2n}\nu_k^n}} e^{i\pi k/n} \hat{d}_k^n \right\} P_k^{2n}(\theta), \tag{3.2}$$

where $\hat{s}_k^n$ and $\hat{d}_k^n$ are the period n DFTs of the seqences $\{s_\ell^n\}_{\ell=0}^{n-1}$ and $\{d_\ell^n\}_{\ell=0}^{n-1}$, respectively.

PROOF: From (1.16) and (1.2), we have

$$\begin{aligned}
\varphi^n(\theta - 2\pi\ell/n) &= \sum_{k=0}^{2n-1} \frac{e^{-2\pi i k\ell/(2n)}}{\sqrt{n\nu_k^n}} P_k^{2n}(\theta - 2\pi(2\ell)/(2n)) \\
&= \sum_{k=0}^{2n-1} \frac{e^{-2\pi i(2k\ell)/(2n)}}{\sqrt{n\nu_k^n}} P_k^{2n}(\theta) \\
&= \sum_{k=0}^{2n-1} \frac{e^{-2\pi i k\ell/n}}{\sqrt{n\nu_k^n}} P_k^{2n}(\theta)\,.
\end{aligned} \tag{3.3}$$

A similar representation for $\psi^n(\theta - 2\pi\ell/n)$ was derived earlier, and is given in (2.6). Replacing $\varphi^n(\theta - 2\pi\ell/n)$ and $\psi^n(\theta - 2\pi\ell/n)$ in (3.1) by the right sides of (3.3) and (2.6) yields

$$f(\theta) = \sum_{\ell=0}^{n-1} s_\ell^n \left\{ \sum_{k=0}^{2n-1} \frac{e^{-2\pi i k\ell/n}}{\sqrt{n\nu_k^n}} P_k^{2n}(\theta) \right\} + \sum_{\ell=0}^{n-1} d_\ell^n \left\{ \sum_{k=0}^{2n-1} e^{ik\pi(1-2\ell)/n} \sqrt{\frac{\nu_{k+n}^{2n}}{n\nu_k^{2n}\nu_k^n}} P_k^{2n}(\theta) \right\}.$$

We may interchange the two sums above to obtain

$$f(\theta) = \sum_{k=0}^{2n-1} \left\{ \frac{1}{\sqrt{n\nu_k^n}} \left(\sum_{\ell=0}^{n-1} s_\ell^n e^{-2\pi i k\ell/n} \right) + \sqrt{\frac{\nu_{k+n}^{2n}}{n\nu_k^{2n}\nu_k^n}} e^{i\pi k/n} \left(\sum_{\ell=0}^{n-1} d_\ell^n e^{-2\pi i k\ell/n} \right) \right\} P_k^{2n}(\theta).$$

The terms in parentheses are the period-n DFTs $\hat{s}_k^n$ and $\hat{d}_k^n$; replacing them yields (3.2). ∎

In addition to the expansion in (3.1), we may also expand f in the level $2n$ scaling basis,

$$f(\theta) = \sum_{\ell=0}^{2n-1} s_\ell^{2n} \varphi^{2n}(\theta - 2\pi\ell/(2n)). \tag{3.4}$$

By Remark 1.5, we see that the expansion of f in terms of the P_k^{2n}'s is

$$f(\theta) = \sum_{k=0}^{2n-1} \frac{\hat{s}_k^{2n}}{\sqrt{2n\nu_k^{2n}}} P_k^{2n}(\theta). \tag{3.5}$$

Comparing (3.5) with (3.2) results in an expression for $\hat{s}_k^{2n}$ in terms of $\hat{s}_k^n$ and $\hat{d}_k^n$; this is the DFT version of the reconstruction formula, which takes level n scaling and wavelet basis coefficients into level $2n$ scaling coefficients, and is given as part of the following decompostion and reconstruction theorem.

THEOREM 3.2: *Let s_m^n and d_m^n be regarded as periodic sequences of period n, and let s_ℓ^{2n} be regarded as a periodic sequence of period $2n$. If $\hat{s}_j^n$, $\hat{d}_j^n$, $\hat{s}_j^{2n}$ denote the DFTs of these sequences, then the reconstruction formula is*

$$\hat{s}_k^{2n} = \hat{s}_k^n \sqrt{\frac{2\nu_k^{2n}}{\nu_k^n}} + e^{ik\pi/n} \hat{d}_k^n \sqrt{\frac{2\nu_{k+n}^{2n}}{\nu_k^n}}, \quad 0 \le k \le 2n-1, \tag{3.6}$$

and the decomposition formula is

$$\left.\begin{cases} \hat{s}_k^n = \dfrac{1}{2}\left(\hat{s}_k^{2n} \sqrt{\dfrac{2\nu_k^{2n}}{\nu_k^n}} + \hat{s}_{k+n}^{2n} \sqrt{\dfrac{2\nu_{k+n}^{2n}}{\nu_k^n}} \right) \\ \hat{d}_k^n = \dfrac{1}{2}\left(\hat{s}_k^{2n} \sqrt{\dfrac{2\nu_{k+n}^{2n}}{\nu_k^n}} - \hat{s}_{k+n}^{2n} \sqrt{\dfrac{2\nu_k^{2n}}{\nu_k^n}} \right) e^{-k\pi i/n} \end{cases}\right\} \quad 0 \le k \le n-1. \tag{3.7}$$

PROOF: We have already shown how to get (3.6). We want to establish (3.7). In (3.6), replace k by $k+n$, and then use the fact that ν_k^n, $\hat{s}_k^n$, and d_k^n all have period n, and ν_k^{2n} has period $2n$, to get

$$\hat{s}_{k+n}^{2n} = \hat{s}_k^n \sqrt{\frac{2\nu_{k+n}^{2n}}{\nu_k^n}} - e^{ik\pi/n} \hat{d}_k^n \sqrt{\frac{2\nu_k^{2n}}{\nu_k^n}}, \quad 0 \le k \le n-1. \tag{3.8}$$

Restrict k to $0 \le k \le n-1$ in both (3.6) and (3.8), and then solve these equations simultaneously for $\hat{d}_k^n$ and $\hat{s}_k^n$. The result is (3.7). ∎

The decompositon and reconstruction formulas are remarkably simple in DFT form. Of course, one could always take inverse DFTs and use convolutions to state them in the more usual form. Since any practical algorithm would use DFTs (or FFTs, really) to compute these convolutions, we will not bother to state them in their usual form.

4. Density of the wavelet spaces.

In this section, we will show that $\bigcup_{N\ge 0} \mathcal{V}_{2^N}$ is dense in $L^2[-\pi,\pi]$. We work with $\mathcal{V}_{2^N}$ because these spaces are nested. The argument given here was inspired by a proof of A. Brown in a different context (see the appendix to [11]).

PROPOSITION 4.1: *Let P be a continuously differentiable PBF. The union of the sampling spaces generated by P is dense in $L^2[-\pi,\pi]$.*

PROOF: In fact, we will prove that $\bigcup_{N\ge 1} \mathcal{V}_{2^N}$ is actually dense in $C_\pi[-\pi,\pi]$, the space of continuous, periodic functions on $[-\pi,\pi]$. The argument proceeds by contradiction.

If the assertion were false, there would exist a functional $\rho \in C_\pi^*[-\pi,\pi]$, $||\rho|| = 1$, for which

$$\rho\left(\bigcup_{N\geq 0} \mathcal{V}_{2^N}\right) = 0.$$

Thus, for the PBF P that generates all of the $\mathcal{V}_{2^N}$'s, we have that

$$\rho(P(\theta - 2\pi j/2^N)) = 0 \qquad \text{for all } j \in \mathbb{Z} \text{ and } N \geq 1.$$

By continuity of both P and translation, it follows that

$$\rho(P(\theta - \tilde{\theta})) = 0, \qquad \tilde{\theta} \in [-\pi,\pi]. \tag{4.1}$$

Expand P in its Fourier series,

$$P(\theta) = \sum_{m\in\mathbb{Z}} p_m e^{im\theta}. \tag{4.2}$$

Because P is continuously differentiable, its Fourier series converges uniformly and absolutely to it. Represent P in (4.1) by (4.2). Because the series is uniformly convergent, we may interchange ρ and the series. Doing so yields

$$0 = \sum_{m\in\mathbb{Z}} p_m e^{-im\tilde{\theta}} \rho(e^{im\theta}).$$

It is easy to check that the series on the right above is uniformly and absolutely convergent for all values of $\tilde{\theta}$, and therefore defines a function in $C_\pi[-\pi,\pi]$. Since this function vanishes identically, we see that all of its Fourier coefficients vanish. Since $p_m > 0$, this impies that $\rho(e^{im\theta}) = 0$ for all $m \in \mathbb{Z}$. It follows that ρ annihilates $C_\pi[-\pi,\pi]$, and consequently $\rho = 0$. Since we have assumed $||\rho|| = 1$, this is a contradiction. ∎

We remark that it should be possible to obtain rates of approximation by methods analogous to those used in [1] or in [7,8]. The proposition just proved has the obvious, but important, consequence.

COROLLARY 4.2: *The set* $\{\varphi^1(\theta)\} \bigcup_{N\geq 0} \{\psi_{2^N}(\theta - 2\pi\ell/2^N)\}_{0\leq\ell\leq 2^N-1}$ *is an orthonormal basis for* $L^2[-\pi,\pi]$.

5. Angular localization of PBF-wavelets.

We will now investigate how well the scaling functions and wavelets constructed out of PBFs localize in the angular domain. The chief measure of localization for a function f defined on the line is the variance (or its square-root: standard deviation),

$$\mathrm{var}_f(t) = \int_{\mathbb{R}} (t - t_0)^2 |f(t)|^2 dt, \quad \text{where } \; t_0 := \int_{\mathbb{R}} t|f(t)|^2 dt.$$

The notion of variance for functions defined on the circle (i.e., periodic functions) is more subtle than variance on the line, simply because angles are discontinuous functions on the circle. A notion of variance that is both invariant under rotations and gives rise to an angular version of the uncertainty principle is discussed by Breitenberger [2].

In what follows, for a continuous, periodic function f whose L^2-norm is 1, set

$$\tau(f) := \int_0^{2\pi} e^{i\theta} |f(\theta)|^2 d\theta. \tag{5.1}$$

As Breitenberger points out, if one regards $|f(\theta)|^2$ as a distribution of mass on the unit circle in the complex plane, then $\tau(f)$ is the center of mass of that distribution. The corresponding variance is easily seen to be

$$\int_0^{2\pi} |e^{i\theta} - \tau(f)|^2 |f(\theta)|^2 d\theta = 1 - |\tau(f)|^2,$$

which is defined on the circle and is invariant under rotations.

Breitenberger [2] argues that the size $1 - |\tau(f)|^2$ is a good measure of how well localized $|f(\theta)|^2$ is about τ. For example, he points out that if $|f(\theta)|^2$ approaches a point mass located at θ_0, then $\tau(f)$ approaches $e^{i\theta_0}$ and $1 - |\tau(f)|^2$ approaches 0; conversely, if $1 - |\tau(f)|^2 = 0$ then the distribution corresponds to a point mass located at $\tau(f)$. Another desirable property is that using $1 - |\tau(f)|^2$ as a measure of localization gives rise to an uncertainty principle for angular variables [2]. While there are other measures of localization (see [2] for further discussion), we feel that $1 - |\tau(f)|^2$ is a natural choice. Thus, we define the circular variance of a continuous periodic function f to be

$$\mathrm{var}_f(\theta) := 1 - |\tau(f)|^2 \tag{5.2}$$

We will give general conditions on the PBF P for there to be "good" localization for both the associated wavelets and scaling functions. When these conditions hold, it turns out that at the n^{th} level, the θ-variances of both φ^n and ψ^n are $\mathcal{O}(1/n)$; thus both the scaling function and the wavelet localize well. In what follows, we will use the following notation:

$$\tau_s^n := \tau(\varphi^n) \quad \text{and} \quad \tau_w^n := \tau(\psi^n) \tag{5.3}$$

We will begin expressing τ_s^n and τ_w^n in terms of the Fourier coefficients of the PBF P and the ν_k^n's defined in Proposition 1.1.

Proposition 5.1: *If τ_s^n and τ_w^n are defined by (5.3), then*

$$\tau_s^n = \frac{1}{n} \sum_{k=0}^{n-1} \sum_{m \equiv k (\mathrm{mod}\, n)} \frac{2\pi p_m p_{m+1}}{\sqrt{\nu_k^n \nu_{k+1}^n}} \tag{5.4s}$$

$$\tau_w^n = \frac{e^{-i\pi/n}}{n} \sum_{k=0}^{2n-1} \sum_{m \equiv k (\mathrm{mod}\, 2n)} \frac{2\pi p_m p_{m+1} \sqrt{\nu_{k+n}^{2n} \nu_{k+n+1}^{2n}}}{\sqrt{\nu_k^{2n} \nu_{k+1}^{2n} \nu_k^n \nu_{k+1}^n}} \tag{5.4w}$$

PROOF: The result follows from (1.9), (2.5), (5.1), (5.3) and Parseval's equation. ∎

In order to estimate τ_s^n and τ_w^n, we will make an assumption on the Fourier coefficients of P in addition to those already mentioned—*viz.*, $p_m > 0$ and $p_{-m} = p_m$. We will assume that the sequence

$$r_m := \frac{p_{m+1}}{p_m} - 1. \tag{5.5}$$

is square summable. As a consequence of this assumption, we have the following lemma, which will be used to facilitate the estimates to follow.

LEMMA 5.2: *If the sequence $\{r_m\}_{m\in\mathbb{Z}}$ is square summable, and if*

$$\alpha_k^n := \sup_{m\equiv k (\operatorname{mod} n)} r_m^2 \quad \text{and} \quad \beta := \inf_{m\in\mathbb{Z}} (1+r_m)^2 \tag{5.6}$$

then these inequalities hold:

$$\sum_{k=0}^{n-1} \alpha_k^n \le \sum_{m\in\mathbb{Z}} r_m^2 \quad \text{and} \quad \beta > 0 \tag{5.7}$$

$$\sum_{m\equiv k (\operatorname{mod} n)} 2\pi (p_m - p_{m+1})^2 \le \alpha_k^n \nu_k^n \quad \text{and} \quad \nu_{k+1}^n \ge \beta \nu_k^n \tag{5.8}$$

PROOF: Since $\{r_m\}_{m\in\mathbb{Z}}$ is square summable, $r_m \to 0$ as $m \to \pm\infty$. Thus, the "sup" in the definition of α_k^n is actually a "max", and, consequently there is an integer $m_k \equiv k (\operatorname{mod} n)$ for which $\alpha_k^n = r_{m_k}^2$. Although it is possible that m_k is not unique, we can make a unique choice by picking the one for which $|m_k|$ is smallest. We now have

$$\sum_{k=0}^{n-1} \alpha_k^n = \sum_{k=0}^{n-1} r_{m_k}^2.$$

Obviously, in the sum on the right above, the m_k's are distinct. Adding in all the remaining terms of $\{r_m\}_{m\in\mathbb{Z}}$ to this sum cannot decrease it, so the first half of (5.7) holds. To establish the second half, note that, from (5.5), $p_{m+1} = (1+r_m)p_m$. Since the p_m's are all positive, $1 + r_m > 0$. In addition, as we remarked above, $r_m \to 0$ as $m \to \pm\infty$. Thus the minimum of $(1+r_m)^2$ must occur for some finite integer, and, because $1 + r_m > 0$, this mimimum, β, must be positive. We now turn to the first half of (5.8). From (5.5), we have

$$\begin{aligned} \sum_{m\equiv k (\operatorname{mod} n)} 2\pi (p_m - p_{m+1})^2 &= \sum_{m\equiv k (\operatorname{mod} n)} 2\pi r_m^2 p_m^2 \\ &\le \alpha_k^n \sum_{m\equiv k (\operatorname{mod} n)} 2\pi p_m^2 \end{aligned}$$

Recall that $\nu_k^n = \|P_k^n\|^2$. From Parseval's equation and (1.1), one has that

$$\nu_k^n = \sum_{m\equiv k (\operatorname{mod} n)} 2\pi p_m^2 \tag{5.9}$$

holds. Replacing the sum in the inequality above with the left side of the last equation yields the desired inequality. The final inequality in the lemma is the result of the following chain:

$$\begin{aligned}
\nu_{k+1}^n &= \sum_{m\equiv k (\mathrm{mod}\, n)} 2\pi p_{m+1}^2 \quad \text{(From (5.9) with } k \to k+1.\text{)} \\
&= \sum_{m\equiv k (\mathrm{mod}\, n)} 2\pi (1+r_m)^2 p_m^2 \quad \text{(From (5.5).)} \\
&\geq \beta \sum_{m\equiv k (\mathrm{mod}\, n)} 2\pi p_m^2 \quad \text{(From (5.6).)} \\
&\geq \beta \nu_k^n \quad \text{(From (5.9).)} \qquad \blacksquare
\end{aligned}$$

The next result is pertinent to localization of the scaling function, φ^n.

PROPOSITION 5.3: *If the sequence r_m defined by (5.5) is square summable, then for n sufficiently large*

$$\tau_s^n = 1 + \mathcal{O}(1/n).$$

PROOF: Since the p_m's are positive, inspection of (5.4s) shows that $\tau_s^n > 0$. From Schwarz's inequality, (5.1), and the fact that $\|\varphi^n\| = 1$, we also have that $\tau_s^n \leq 1$. Thus, we only need to show that for n large there is a positive constant C such that $1 - \tau_s^n \leq C/n$. Note that since $p_m p_{m+1} = \frac{1}{2}\left(p_m^2 + p_{m+1}^2 - (p_m - p_{m+1})^2\right)$, we have that

$$\tau_s^n = \frac{1}{n}\sum_{k=0}^{n-1} \sum_{m\equiv k (\mathrm{mod}\, n)} \frac{2\pi(p_m^2 + p_{m+1}^2 - (p_m - p_{m+1})^2)}{2\sqrt{\nu_k^n \nu_{k+1}^n}}.$$

Using (5.8) and (5.9) in the last equation yields

$$\tau_s^n \geq \frac{1}{n}\sum_{k=0}^{n-1}\left(\frac{\nu_k^n + \nu_{k+1}^n}{2\sqrt{\nu_k^n\nu_{k+1}^n}} - \frac{\alpha_k^n \nu_k}{2\sqrt{\beta(\nu_k^n)^2}}\right).$$

Simplifying the second term on the right above and using the inequality $a^2 + b^2 \geq 2ab$ on the first term on the right, we get

$$\begin{aligned}
\tau_s^n &\geq \frac{1}{n}\sum_{k=0}^{n-1}\left(1 - \frac{\alpha_k^n}{2\sqrt{\beta}}\right) \\
&\geq 1 - \frac{1}{n}\frac{\sum_{k=0}^{n-1}\alpha_k^n}{2\sqrt{\beta}} \\
&\geq 1 - \frac{1}{n}\underbrace{\frac{\sum_{m\in\mathbb{Z}} r_m^2}{2\sqrt{\beta}}}_{C}. \quad \text{(From (5.7).)}
\end{aligned}$$

Rearranging terms above implies that $1 - \tau_s^n \leq C/n$. $\blacksquare$

We now turn to the case of the wavelet.

PROPOSITION 5.4: *If the sequence r_m defined by (5.5) is square summable, then for n sufficiently large*
$$|\tau_w^n| = 1 + \mathcal{O}(1/n).$$

PROOF: From (5.4w), the equation $p_m p_{m+1} = \frac{1}{2}(p_m^2 + p_{m+1}^2 - (p_m - p_{m+1})^2)$, (5.8) and (5.9), we have

$$\begin{aligned}
|\tau_w^n| &= \frac{1}{n}\sum_{k=0}^{2n-1}\left(\sum_{m\equiv k (\mathrm{mod}\, 2n)} 2\pi(p_m^2 + p_{m+1}^2) - 2\pi(p_m - p_{m+1})^2\right)\frac{\sqrt{\nu_{k+n}^{2n}\nu_{k+n+1}^{2n}}}{2\sqrt{\nu_k^{2n}\nu_{k+1}^{2n}\nu_k^n\nu_{k+1}^n}}\\
&\geq \frac{1}{n}\sum_{k=0}^{2n-1}\left(\frac{(\nu_k^{2n}+\nu_{k+1}^{2n})\sqrt{\nu_{k+n}^{2n}\nu_{k+n+1}^{2n}}}{2\sqrt{\nu_k^{2n}\nu_{k+1}^{2n}\nu_k^n\nu_{k+1}^n}} - \frac{\alpha_k^{2n}\nu_k^{2n}\sqrt{\nu_{k+n}^{2n}\nu_{k+n+1}^{2n}}}{2\sqrt{\beta(\nu_k^{2n})^2\nu_k^n\nu_{k+1}^n}}\right)\\
&\geq \frac{1}{n}\sum_{k=0}^{2n-1}\left(\frac{(\nu_k^{2n}+\nu_{k+1}^{2n})\sqrt{\nu_{k+n}^{2n}\nu_{k+n+1}^{2n}}}{2\sqrt{\nu_k^{2n}\nu_{k+1}^{2n}\nu_k^n\nu_{k+1}^n}} - \frac{\alpha_k^{2n}}{2\sqrt{\beta}}\sqrt{\frac{\nu_{k+n}^{2n}\nu_{k+n+1}^{2n}}{\nu_k^n\nu_{k+1}^n}}\right)
\end{aligned}$$

From (2.3), we see that $\nu_{k+n}^{2n} \leq \nu_{k+n}^n$ and that $\nu_{k+n+1}^{2n} \leq \nu_{k+n+1}^n$, consequently

$$\sqrt{\frac{\nu_{k+n}^{2n}\nu_{k+n+1}^{2n}}{\nu_k^n\nu_{k+1}^n}} \leq 1.$$

Applying this and (5.7) to the previous inequality yields

$$|\tau_w^n| \geq \frac{1}{n}\underbrace{\left\{\sum_{k=0}^{2n-1}\frac{(\nu_k^{2n}+\nu_{k+1}^{2n})\sqrt{\nu_{k+n}^{2n}\nu_{k+n+1}^{2n}}}{2\sqrt{\nu_k^{2n}\nu_{k+1}^{2n}\nu_k^n\nu_{k+1}^n}}\right\}}_{S_n} - \frac{C}{n}, \tag{5.10}$$

where, as in Proposition 5.3, $C = \sum_{m\in\mathbb{Z}} r_m^2/(2\sqrt{\beta})$.

We now need to estimate S_n, the term in braces in (5.10). Break up S_n into two sums, one from $k = 0$ to $n-1$ and a second from $k = n$ to $2n-1$. In the second sum, make the change of index from $k \to k-n$, and then recombine the sums to get

$$S_n = \sum_{k=0}^{n-1}\left(\frac{(\nu_k^{2n}+\nu_{k+1}^{2n})\sqrt{\nu_{k+n}^{2n}\nu_{k+n+1}^{2n}}}{2\sqrt{\nu_k^{2n}\nu_{k+1}^{2n}\nu_k^n\nu_{k+1}^n}} + \frac{(\nu_{k+n}^{2n}+\nu_{k+n+1}^{2n})\sqrt{\nu_{k+2n}^{2n}\nu_{k+2n+1}^{2n}}}{2\sqrt{\nu_{k+n}^{2n}\nu_{k+n+1}^{2n}\nu_{k+n}^n\nu_{k+n+1}^n}}\right)$$

Recall that ν_k^n is periodic with period n, and ν_k^{2n} is periodic with period $2n$; thus the previous equation has the form

$$S_n = \sum_{k=0}^{n-1}\left(\frac{(\nu_k^{2n}+\nu_{k+1}^{2n})\sqrt{\nu_{k+n}^{2n}\nu_{k+n+1}^{2n}}}{2\sqrt{\nu_k^{2n}\nu_{k+1}^{2n}\nu_k^n\nu_{k+1}^n}} + \frac{(\nu_{k+n}^{2n}+\nu_{k+n+1}^{2n})\sqrt{\nu_k^{2n}\nu_{k+1}^{2n}}}{2\sqrt{\nu_{k+n}^{2n}\nu_{k+n+1}^{2n}\nu_k^n\nu_{k+1}^n}}\right)$$

Using (2.3) to replace ν_k^n and ν_{k+1}^n and manipulating the expression above, we get

$$S_n = \frac{1}{2}\sum_{k=0}^{n-1} \frac{\left(\frac{1}{\nu_k^{2n}} + \frac{1}{\nu_{k+1}^{2n}} + \frac{1}{\nu_{k+n}^{2n}} + \frac{1}{\nu_{k+n+1}^{2n}}\right)}{\sqrt{\left(\frac{1}{\nu_k^{2n}} + \frac{1}{\nu_{k+n}^{2n}}\right)\left(\frac{1}{\nu_{k+1}^{2n}} + \frac{1}{\nu_{k+n+1}^{2n}}\right)}}$$

Note that each term in the sum is of the form $(X+Y)/\sqrt{XY} \geq 2$, and consequently

$$S_n \geq \frac{1}{2}\sum_{k=0}^{n-1} 2 = n. \tag{5.11}$$

Combining (5.10) and (5.11) then yileds

$$|\tau_w^n| \geq 1 - C/n$$

Since $|\tau_w^n| \leq 1$, we have that $|1 - |\tau_w^n|| \leq C/n$, which completes the proof. ∎

These two propositions yield estimates on the variances for φ^n and ψ^n:

THEOREM 5.5: *If the sequence r_m defined by (5.5) is square summable, then*

$$\mathrm{var}_{\varphi^n}(\theta) = \mathcal{O}(1/n) \text{ and } \mathrm{var}_{\psi^n}(\theta) = \mathcal{O}(1/n).$$

PROOF: The result follows on combining the previous two propositions with the definition of variance given in (5.2). ∎

6. The angle/frequency uncertainty relation.

The uncertainty principle in quantum mechanics represents a trade off between two "spreads" (standard deviations), one for the position and the other for momentum. In signal analysis, there is a similar principle that again amounts to a trade off, this time between the time and frequency spreads. In [2], Breitenberger discusses a version of the uncertainty principle that amounts to a trade off between the angle and angular momentum spreads.

To describe this angle/angular-momentum uncertainty relation, we first recall that the variance for ℓ_z, which is the expected value of the z-component of the angular momentum operator $L_z = i\hbar\partial_\theta$, is defined by

$$\mathrm{var}_f(\ell_z) = \int_0^{2\pi} \overline{f(\theta)} L_z^2 f(\theta) d\theta - \left(\int_0^{2\pi} \overline{f(\theta)} L_z f(\theta) d\theta\right)^2,$$

where $f(\theta)$ is a smooth, periodic function having $\|f\| = 1$, If the variance in the angle θ, var_f is given by (5.2), then the variance-version of the angle/angular-momentum uncertainty relation is

$$\frac{\mathrm{var}_f(\ell_z)\mathrm{var}_f(\theta)}{1 - \mathrm{var}_f(\theta)} \geq \frac{1}{4}\hbar^2. \tag{6.1}$$

This can be stated in a way that closely resembles the standard position momentum uncertainty relation. As usual, define the spread in ℓ_z to be $\Delta\ell_z := \sqrt{\mathrm{var}_f(\ell_z)}$. For the angle variable, however, we will define the spread in θ in a somewhat different way; namely,

$$\Delta\theta := \sqrt{\frac{\mathrm{var}_f(\theta)}{1 - \mathrm{var}_f(\theta)}}$$

The angle/angular-momentum uncertainty relation then becomes a familiar

$$\Delta\ell_z\Delta\theta \geq \hbar/2.$$

We can easily derive an angle/frequency uncertainty relation from (6.1). It is a simple matter to show that if f has the Fourier series

$$f(\theta) = \sum_{m\in\mathbb{Z}} f_m e^{im\theta},$$

then one has that

$$\begin{aligned}\mathrm{var}_f(\ell_z) &= 2\pi\hbar^2 \sum_{m\in\mathbb{Z}} m^2|f_m|^2 - \hbar^2\left(2\pi\sum_{m\in\mathbb{Z}} m|f_m|^2\right)^2, \\ &=: \hbar^2\mathrm{var}_f(m),\end{aligned} \tag{6.2}$$

where $\mathrm{var}_f(m)$ is the variance in the frequency for f. Combining the last equation with (6.1) yields, after dividing by $\hbar^2$ and taking square roots,

$$\Delta m\Delta\theta \geq \frac{1}{2}, \quad \text{where } \Delta m = \sqrt{\mathrm{var}_f(m)} \text{ and } \Delta\theta = \sqrt{\frac{\mathrm{var}_f(\theta)}{1 - \mathrm{var}_f(\theta)}}. \tag{6.3}$$

This is the angle/frequency uncertainty relation. We want to estimate the size of the uncertainty "window", $\Delta m\Delta\theta$, for the case of the scaling function and the wavelet constructed in previous sections. We will first obtain a common lower bound on the frequency spread for both the scaling function ϕ^n and the wavelet ψ^n.

THEOREM 6.1: *Let $n \geq 3$, and suppose that the PBF that generates the scaling function φ^n and the wavelet ψ^n is sufficiently smooth that its Fourier series coefficients satisfy $\sum_{m\in\mathbb{Z}} m^2 p_m^2 < \infty$. If either $f = \varphi^n$ or $f = \psi^n$, then*

$$\Delta m \geq (\sqrt{3}n/6)(1 - 3/n)^{3/2} \tag{6.4}$$

PROOF: Since the scaling function φ^n is real, its Fourier series coefficients satisfy $\varphi_m^n = \overline{\varphi_{-m}^n}$. Hence, from (6.2), one sees that

$$\mathrm{var}_{\varphi^n}(m) = 2\pi \sum_{m\in\mathbb{Z}} m^2|\varphi_m^n|^2. \tag{6.5}$$

Using (1.9), rewrite this sum as

$$\operatorname{var}_{\varphi^n}(m) = 2\pi \sum_{k=0}^{n-1} \sum_{m\equiv k \bmod n} \frac{m^2 p_m^2}{n\nu_k^n}, \tag{6.6}$$

which leads to the inequality

$$\begin{aligned}
\operatorname{var}_{\varphi^n}(m) &\geq \sum_{k=0}^{n-1} \min_{m\equiv k \bmod n}\{m^2\} \frac{\sum_{m\equiv k \bmod n} 2\pi p_m^2}{n\nu_k^n} \\
&\geq (1/n) \sum_{k=0}^{n-1} \min_{m\equiv k \bmod n}\{m^2\}. \quad \text{(From (5.9).)}
\end{aligned} \tag{6.7}$$

It is easy to check that

$$\min_{m\equiv k \bmod n}\{m^2\} = \begin{cases} k^2 & \text{if } 0 \leq k \leq [n/2], \\ (n-k)^2 & \text{if } [n/2]+1 \leq k \leq n-1. \end{cases} \tag{6.8}$$

From (6.7) and (6.8), we get

$$\begin{aligned}
\operatorname{var}_{\varphi^n}(m) &\geq (1/n) \left(\sum_{k=0}^{[n/2]} k^2 + \sum_{k=[n/2]+1}^{n-1} (n-k)^2 \right) \\
&\geq (2/n) \sum_{k=1}^{[n/2]-1} k^2 \\
&\geq \frac{1}{3n}([n/2]-1)(2[n/2]-1)([n/2]) \\
&\geq \frac{1}{12n}(n-3)(n-2)(n-1) \\
&\geq \frac{1}{12} n^2 (1-3/n)^3.
\end{aligned} \tag{6.9}$$

The inequality in (6.4) for φ^n is a consequence of taking square roots in the last inequality above and using the definition of Δm in the case of $f = \varphi^n$.

We now turn to the case of the wavelet. The wavelet is real, so its Fourier series coefficients again satisfy $\psi_m^n = \overline{\psi_{-m}^n}$. Hence, from (6.2), one sees that

$$\operatorname{var}_{\psi^n}(m) = 2\pi \sum_{m\in\mathbb{Z}} m^2 |\psi_m^n|^2. \tag{6.10}$$

Using the last equation, (2.5), and (5.9) yields

$$\begin{aligned}
\operatorname{var}_{\psi^n}(m) &= \sum_{k=0}^{2n-1} \frac{\nu_{k+n}^{2n}}{n\nu_k^n} \sum_{m\equiv k \bmod 2n} \frac{2\pi m^2 p_m^2}{\nu_k^{2n}} \\
&\geq \sum_{k=0}^{2n-1} \min_{m\equiv k \bmod 2n}\{m^2\} \frac{\nu_{k+n}^{2n}}{n\nu_k^n}.
\end{aligned} \tag{6.11}$$

From this inequality and from (6.8), with n replaced by $2n$, we obtain

$$\begin{aligned}
\operatorname{var}_{\psi^n}(m) &\geq \sum_{k=0}^{n-1} k^2 \frac{\nu_{k+n}^{2n}}{n\nu_k^n} + \sum_{k=n}^{2n-1} (2n-k)^2 \frac{\nu_{k+n}^{2n}}{n\nu_k^n} \\
&\geq \sum_{k=0}^{n-1} k^2 \frac{\nu_{k+n}^{2n}}{n\nu_k^n} + \sum_{k=0}^{n-1} (n-k)^2 \frac{\nu_{k+2n}^{2n}}{n\nu_{k+n}^n} \\
&\geq \sum_{k=0}^{n-1} \min\{k^2,(n-k)^2\} \left(\frac{\nu_{k+n}^{2n}}{n\nu_k^n} + \frac{\nu_{k+2n}^{2n}}{n\nu_{k+n}^n} \right) \\
&\geq \sum_{k=0}^{n-1} \min\{k^2,(n-k)^2\} \left(\frac{\nu_{k+n}^{2n}+\nu_k^{2n}}{n\nu_k^n} \right) \quad \text{(From (1.7).)} \\
&\geq (1/n) \sum_{k=0}^{n-1} \min\{k^2,(n-k)^2\} \quad \text{(From (2.3).)} \\
&\geq (1/n) \left(\sum_{k=0}^{[n/2]} k^2 + \sum_{k=[n/2]+1}^{n-1} (n-k)^2 \right). \qquad (6.12)
\end{aligned}$$

The right side of (6.12) is identical to that of (6.9), and so the rest of the proof is the same as that for the scaling function. ∎

We now wish to obtain upper estimates on Δm for both φ^n and ψ^n. In order to do this, we will have to put additional conditions on the Fourier series coefficients for the PBF P. The results are given below.

PROPOSITION 6.2: *Let $\beta \geq \alpha > 3/2$. If there are positive constants C_1 and C_2 such that the p_m's satisfy*

$$C_1 m^{-\beta} \leq p_m \leq C_2 m^{-\alpha} \quad \text{for all } m \geq 1, \qquad (6.13)$$

then there exists a positive constant C such that for both φ^n and ψ^n

$$(6.14) \qquad \Delta m \leq C n^{1+\beta-\alpha}.$$

PROOF: We will begin with the case of the scaling function. Recalling that the reality of φ^n implies that $|\varphi_m^n| == |\varphi_{-m}^n|$, we may rewrite (6.5) as

$$\operatorname{var}_{\varphi^n}(m) = 4\pi \sum_{m=1}^{\infty} m^2 |\varphi_m^n|^2. \qquad (6.15)$$

Break the sum into pieces n in which $m \equiv k \bmod n$, parameterize each piece by $m = k + jn$, where $j = 0, 1, \ldots$, and finally replace the coefficient φ_{k+jn}^n using (1.9). The result is

$$\operatorname{var}_{\varphi^n}(m) = \frac{4\pi}{n} \sum_{k=1}^{n} \sum_{j=0}^{\infty} \frac{(k+jn)^2 p_{k+jn}^2}{\nu_k^n}. \qquad (6.16)$$

By replacing p_{k+jn} with the upper bound in (6.13), and then using the standard integral method for estimating sums, we obtain

$$\mathrm{var}_{\varphi^n}(m) \le \frac{4C_2^2\pi}{n}\sum_{k=1}^{n}\frac{k^{2-2\alpha}}{\nu_k^n}\left(1+\frac{k}{(2\alpha-3)n}\right) \le \frac{8C_2^2(\alpha-1)\pi}{(2\alpha-3)n}\sum_{k=1}^{n}\frac{k^{2-2\alpha}}{\nu_k^n}$$

From (5.9) and (6.13), $\nu_k^n \ge 2\pi p_k^2 \ge 2\pi C_1^2 k^{-2\beta}$, which may be used in the inequality above to get

$$\mathrm{var}_{\varphi^n}(m) \le \left(\frac{4(\alpha-1)C_2^2}{n(2\alpha-3)C_1^2}\right)\sum_{k=1}^{n} k^{2+2(\beta-\alpha)} \le \underbrace{\frac{\alpha-1}{2\alpha-3}\left(\frac{2C_2}{C_1}\right)^2}_{C_3^2} n^{2+2(\beta-\alpha)}. \tag{6.17}$$

To do the case of the wavelet, note we have the following chain of inequalities:

$$\begin{aligned}
\mathrm{var}_{\psi^n}(m) &= \sum_{k=0}^{2n-1}\frac{\nu_{k+n}^{2n}}{n\nu_k^n}\sum_{m\equiv k \bmod 2n}\frac{2\pi m^2 p_m^2}{\nu_k^{2n}} && \text{(From (6.11).)}\\
&\le \sum_{k=0}^{2n-1}\sum_{m\equiv k \bmod 2n}\frac{2\pi m^2 p_m^2}{n\nu_k^{2n}} && \text{(From (2.3).)}\\
&\le \mathrm{var}_{\varphi^{2n}}(m) && \text{(From (6.6).)}\\
&\le C_3^2(2n)^{2+2(\beta-\alpha)} && \text{(From (6.17).)}\\
&\le \underbrace{C_3^2 2^{2+2(\beta-\alpha)}}_{C^2} n^{2+2(\beta-\alpha)}. && (6.18)
\end{aligned}$$

Combining the definition of Δm, (6.17), the fact that $C_3 \le C$, and (6.18) yields (6.14). ∎

REMARK 6.3: If $\alpha = \beta$ in Proposition 6.2, then, for both the wavelet and scaling function, $\Delta m \le Cn$. On the other hand, if the Fourier series coefficients satisfy (6.13), then they also satisfy the conditions of Theorem 6.1. Putting the two together implies that $\Delta m = \mathcal{O}(n)$ for both φ^n and ψ^n. Thus, for this case, the Δm's are best possible, at least in terms of dependence on n.

We now turn to estimating the size of the uncertainty windows for the scaling function and wavelet, given certain assumptions on the Fourier series coefficients of the PBF P.

COROLLARY 6.4: *If the p_m's satisfy (6.13) for some $\beta = \alpha > 3/2$, and if the sequence r_m defined by (5.5) is square summable, then for both the scaling function φ^n and the wavelet ψ^n*

$$\Delta m \Delta\theta = \mathcal{O}(\sqrt{n}). \tag{6.19}$$

PROOF: From Theorem 5.5, we have that $\Delta\theta^2 = \dfrac{\mathrm{var}_f(\theta)}{1-\mathrm{var}_f(\theta)} = \mathcal{O}(1/n)$. Consequently, $\Delta\theta = \mathcal{O}(1/\sqrt{n})$. On the other hand, by Remark 6.3, $\Delta m = \mathcal{O}(n)$. Putting the two together yields (6.19). ∎

We conjecture that (6.19) is the best one can do.

7. Examples.

The purpose of this section is to provide a family of PBFs that satisfy the conditions set forth in §5 and §6 above, and for one of the simpler PBFs in this family, to graphically illustrate the angular localization for both the associated PBF-scaling function and its wavelet.

We begin by recalling an elementary fact concerning Fourier series: If $f(\theta)$ and $g(\theta)$ are square-integrable, 2π-periodic functions with Fourier coefficients f_m and g_m, respectively, then the convolution of f and g, which is defined by

$$f * g(\theta) := \frac{1}{2\pi}\int_0^{2\pi} f(\theta-\phi)g(\phi)d\phi\,,$$

is a continuous, 2π-periodic function with Fourier coefficients $f_m g_m$. This fact is what we will use to construct the family described above. The following result is a consequence of it.

PROPOSITION 7.1: *Let f and g be real-valued, even, square-integrable, and 2π-periodic functions. If the product $f_m g_m$ is positive for all m, and if the seies $\sum_{m\in\mathbb{Z}}|mf_m g_m|$ is summable. then $P := f * g$ is a PBF. Moreover, if both of the sequences*

$$\left\{\frac{|f_{m+1}|}{|f_m|}-1\right\}_{m\in\mathbb{Z}} \quad \text{and} \quad \left\{\frac{|g_{m+1}|}{|g_m|}-1\right\}_{m\in\mathbb{Z}}$$

are square summable, then so is the sequence

$$\left\{\frac{f_{m+1}g_{m+1}}{f_m g_m}-1\right\}_{m\in\mathbb{Z}}.$$

Finally, if $|f_m|$ and $|g_m|$ satisfy bounds of the form

$$C_1 m^{-\beta_f} \le |f_m| \le C_2 m^{-\alpha_f} \quad \text{and} \quad C_1 m^{-\beta_g} \le |g_m| \le C_2 m^{-\alpha_g} \quad \text{for all } m\ge 1,$$

then, provided that $\alpha_f+\alpha_g > 3/2$, $f_m g_m$ satisfies (6.13), with $\alpha=\alpha_f+\alpha_g$ and $\beta=\beta_f+\beta_g$.

PROOF: The only part of the proof that requires any comment is that which concerns the square summability of the sequence $r_m = \frac{f_{m+1}g_{m+1}}{f_m g_m}-1$. If we let r_m^f and r_m^g be the corresponding quantities for $|f_m|$ and $|g_m|$, then we have

$$r_m = (r_m^f+1)(r_m^g+1)-1 = r_m^f r_m^g + r_m^f + r_m^g\,.$$

This and the square summability of r_m^f and r_m^g imply that r_m is square summable. ∎

The PBF family that we obtain will be generated recursively. The function that we begin with is the 2π-periodic, piecewise-smooth function

$$\mathcal{P}_2(\theta) := 1 + \sum_{m\neq 0}\frac{e^{im\theta}}{m^2} = 1 + \sum_{m=1}^{\infty} 2\cos(m\theta)/m^2. \tag{7.1}$$

The series above sums to a known function. From [15, eq. 13.8, p. 31], we have that the restriction of $\mathcal{P}_2$ to $0 \leq \theta \leq 2\pi$ is

$$\mathcal{P}_2(\theta) = 1 + \frac{3\theta^2 - 6\pi\theta + 2\pi^2}{6} = \frac{6-\pi^2}{6} + \frac{1}{2}(\theta-\pi)^2 \quad (0 \leq \theta \leq 2\pi). \tag{7.2}$$

The other functions in the family are then defined via

$$\mathcal{P}_{2\gamma} := \mathcal{P}_2 * \mathcal{P}_{2\gamma-2}\,, \quad \text{where } \gamma = 2, 3, \ldots. \tag{7.3}$$

The properties of the $\mathcal{P}_{2\gamma}$'s are described in our next result.

THEOREM 7.2: *For $\gamma \geq 1$, each $\mathcal{P}_{2\gamma}$ is a 2π-periodic, continuous function with Fourier series expansion*

$$\mathcal{P}_{2\gamma}(\theta) = 1 + \sum_{m \neq 0} \frac{e^{im\theta}}{m^{2\gamma}} = 1 + 2\sum_{m=1}^{\infty} \frac{\cos(m\theta)}{m^{2\gamma}}. \tag{7.4}$$

For $\gamma \geq 2$, $\mathcal{P}_{2\gamma}$ is a PBF with $2\gamma - 2$ continuous derivatives, and it has Fourier coefficients that satisfy the conditions in Theorem 5.5 and Proposition 6.2 (with $\beta = \alpha = 2\gamma$). On the interval $0 \leq \theta \leq 2\pi$ and for $\gamma \geq 1$, each $\mathcal{P}_{2\gamma}$ is a degree 2γ polynomial in θ. Finally, we have that for $0 \leq \theta \leq 2\pi$ and $|z| < 1$,

$$\Upsilon(z,\theta) := \frac{\pi\cosh(z(\theta-\pi))}{z\sinh(\pi z)} - \frac{1}{z^2} = \sum_{\gamma=1}^{\infty} (-1)^{\gamma+1} z^{2\gamma-2}\Big(\mathcal{P}_{2\gamma}(\theta) - 1\Big), \tag{7.5}$$

so that Υ is a generating function for the polynomials $\mathcal{P}_{2\gamma}|_{[0,2\pi]}$.

PROOF: The Fourier expansions in (7.4) follow on using induction in connection with the convolution theorem and the expansion for $\mathcal{P}_2$ in (7.1). Inspection of the decay of the coefficients in these expansions shows that the $\mathcal{P}_{2\gamma}$'s have the degree of smoothness claimed. Furthermore, since the Fourier coefficients for $\mathcal{P}_2$ clearly satisfy the conditions in Theorem 5.5 and Proposition 6.2 (with $\beta = \alpha = 2$), by Proposition 7.1 this is also true for the Fourier series coefficients for the other $\mathcal{P}_{2\gamma}$'s.

The statements concerning the restriction of $\mathcal{P}_{2\gamma}$ to $[0, 2\pi]$ being a degree 2γ can be established as follows. First, we have already shown this when $\gamma = 1$, the polynomial for $\mathcal{P}_2$ being given in (7.2). Next, take $\gamma > 1$. Observe that by differentiating (7.4) we have

$$\mathcal{P}_{2\gamma}''(\theta) = 1 - \mathcal{P}_{2\gamma-2}(\theta), \tag{7.6}$$

which holds for all θ and for every $\gamma \geq 2$. From (7.4), one has that $\mathcal{P}_{2\gamma}(0) = 1 + 2\sum_{m=1}^{\infty} m^{-2\gamma}$ and $\mathcal{P}_{2\gamma}'(0) = 0$. Integrating (7.6) with these initial conditions yields

$$\mathcal{P}_{2\gamma}(\theta) = 1 + 2\sum_{m=1}^{\infty} m^{-2\gamma} + \int_0^\theta \int_0^\psi \Big(1 - \mathcal{P}_{2\gamma-2}(\phi)\Big) d\phi d\psi. \tag{7.7}$$

If $\mathcal{P}_{2\gamma-2}(\theta)$ is a degree $2\gamma - 2$ polynomial on $[0, 2\pi]$, then it follows immediately from (7.7) that on $[0, 2\pi]$ the function $\mathcal{P}_{2\gamma}(\theta)$ is a degree 2γ polynomial in θ.

To establish (7.5), we start with the series found in [16, p. 190, Problem 8],

$$\sum_{m=1}^{\infty} \frac{m \sin(m\theta)}{m^2+z^2} = \frac{\pi \sinh(z(\pi-\theta))}{2\sinh(\pi z)}, \quad 0<\theta<2\pi. \tag{7.8}$$

Integrating (7.8) in $\theta \in (0, 2\pi)$ yields

$$\sum_{m=1}^{\infty} \frac{\cos(m\theta)}{m^2+z^2} = \frac{\pi \cosh(z(\pi-\theta))}{2z\sinh(\pi z)} + C(z)$$

We need to evaluate $C(z)$. Let $\theta \to 0$ above. Doing so yields

$$C(z) = \sum_{m=1}^{\infty} \frac{1}{m^2+z^2} - \frac{\pi \coth(z\pi)}{2z}$$

The series on the right in the last equation is related to the Mittag-Leffler expansion for $\coth(\pi z)$. Indeed, from [16, p. 136, Example 7], we find that $\sum_{m=1}^{\infty}(m^2+z^2)^{-1} = \pi\coth(\pi z)/(2z) - 1/(2z^2)$. Hence, $C(z) = -1/(2z^2)$. Thus, after a minor manipulation, we obtain

$$2\sum_{m=1}^{\infty} \frac{\cos(m\theta)}{m^2+z^2} = \frac{\pi \cosh(z(\pi-\theta))}{z\sinh(\pi z)} - \frac{1}{z^2} = \Upsilon(z,\theta), \tag{7.9}$$

which holds for $0<\theta<2\pi$ and $|z|<1$. From the geometric series, we see that

$$(m^2+z^2)^{-1} = \sum_{\gamma=1}^{\infty}(-1)^{\gamma+1} z^{2\gamma-2} m^{2\gamma}.$$

Inserting this into (7.9) and interchanging the two sums, which is justified for $|z|<1$ by the absolute convergence of the terms of the series in question, one finally arrives at this:

$$\begin{aligned}\Upsilon(z,\theta) &= \sum_{\gamma=1}^{\infty}(-1)^{\gamma} z^{2\gamma-2}\left(2\sum_{m=1}^{\infty}\frac{\cos(m\theta)}{m^{2\gamma}}\right)\\ &= \sum_{\gamma=1}(-1)^{\gamma+1} z^{2\gamma-2}\Big(\mathcal{P}_{2\gamma}(\theta)-1\Big). \quad \blacksquare\end{aligned}$$

REMARK 7.3: The periodicity and evenness of the function $\mathcal{P}_{2\gamma}$ translates into a symmetry for the polynomial $\mathcal{P}_{2\gamma}|_{[0,2\pi]}$. When expressed in terms $\theta-\pi$ rather than θ, it will contain only even powers. This also follows from the generating function being even in $\theta-\pi$. Here are the first few $\mathcal{P}_{2\gamma}(\theta)$'s, with $\theta\in[0,2\pi]$. (Maple V was used to compute them.)

$$\mathcal{P}_4(\theta) = \frac{360-7\pi^4}{360} + \frac{\pi^2(\theta-\pi)^2}{12} - \frac{(\theta-\pi)^4}{24}$$

$$\mathcal{P}_6(\theta) = \frac{15120-31\pi^6}{15120} + \frac{7\pi^4(\theta-\pi)^2}{720} - \frac{\pi^2(\theta-\pi)^4}{144} + \frac{(\theta-\pi)^6}{720}$$

$$\mathcal{P}_8(\theta) = \frac{604800-127\pi^8}{604800} + \frac{31\pi^6(\theta-\pi)^2}{30240} - \frac{7\pi^4(\theta-\pi)^4}{8640} + \frac{\pi^2(\theta-\pi)^6}{4320} - \frac{(\theta-\pi)^8}{40320}$$

What we say next applies to any PBF P. To obtain a plot for φ^n, we need to express φ^n in terms of P. We begin by putting $f(\theta) = \varphi^n$ in (1.11), and then comparing the two sides of the equation on the right to obtain $s^n_\ell = \delta_{\ell,0}$. The DFT of this sequence is the constant sequence $\hat{s}^n_k \equiv 1$. Equation (1.13) then implies that

$$\hat{c}^n_k = \frac{1}{\sqrt{n\nu^n_k}}.$$

Taking the inverse DFT of $\hat{c}^n_k$ then gives the coefficients to be used in the first equation in (1.12), again with $f(\theta) = \varphi^n$. That is,

$$\varphi^n(\theta) = \sum_{\ell=0}^{n-1} c^n_\ell P(\theta - 2\pi\ell/n)\,, \quad \text{where } \hat{c}^n_k = \frac{1}{\sqrt{n\nu^n_k}}. \tag{7.10}$$

The procedure for obtaining a similar equation for ψ^n is more direct. From (2.4) and (1.12), we have

$$\psi^n(\theta) = \sum_{\ell=0}^{2n-1} c^{2n}_\ell P(\theta - 2\pi\ell/(2n))\,, \quad \text{where } \hat{c}^{2n}_k = e^{ik\pi/n}\sqrt{\frac{\nu^{2n}_{k+n}}{n\nu^{2n}_k\nu^n_k}}. \tag{7.11}$$

We used (7.10) and (7.11) to make plots for both a scaling function and wavelets corresponding to the PBF P_4; see Figs. 1 through 4 below.

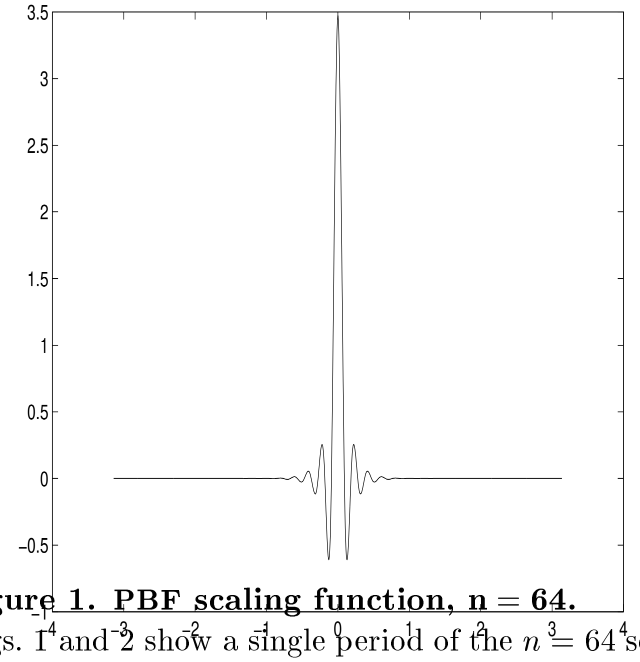

Figure 1. PBF scaling function, n = 64.

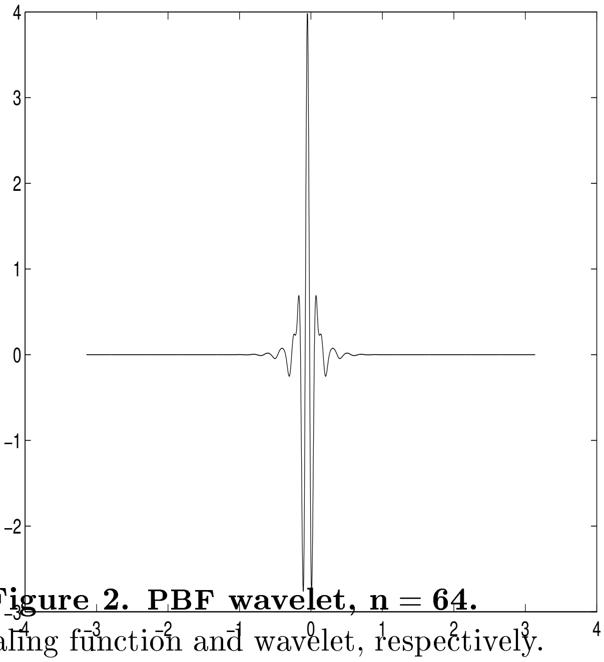

Figure 2. PBF wavelet, n = 64.

Figs. 1 and 2 show a single period of the $n = 64$ scaling function and wavelet, respectively. It is clear from Figs. 1 and 2 that both the scaling function and the wavelet are well localized. Fig. 1 also clearly shows that the scaling function is even about $\theta = 0$. In Fig. 2, one can see that the graph of the wavelet has its center slightly to the left of $\theta = 0$.

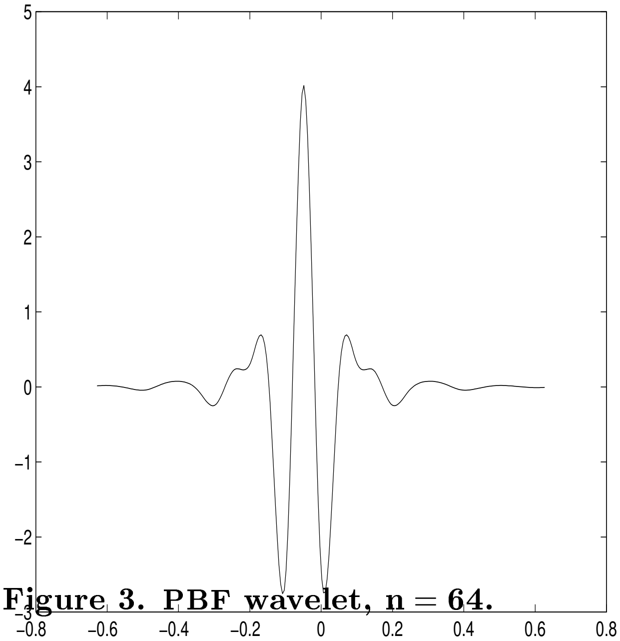

Figure 3. PBF wavelet, n = 64.

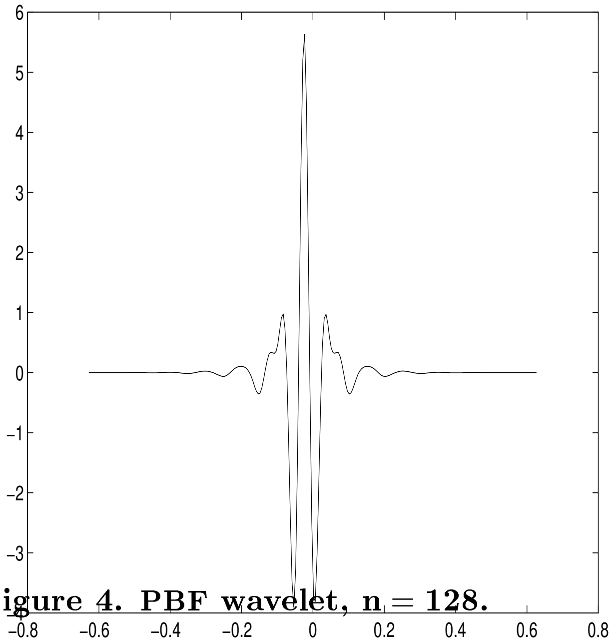

Figure 4. PBF wavelet, n = 128.

Figs. 3 and 4 show a blow-up of the supports for the $n = 64$ and $n = 128$ wavelets constructed out of P_4. Fig. 4 serves to illustrate what we predicted earlier: the support of the wavelet approximately halves when n is doubled.

8. Algorithms.

For any type of wavelet analysis to be useful, one must have an algorithm for its implementation. Here, we will present an algorithm for doing a PBF-wavelet analysis. Our initial sampling space will be $\mathcal{V}_{2^N}$.

Step 1. Fit the data to a function f in the highest level sampling space $\mathcal{V}_{2^N}$. (This can be done in several ways; we will describe a few below.) In general, f will have the form

$$f(\theta) = \sum_{\ell=0}^{2^N-1} c_\ell^{2^N} P(\theta - 2\pi\ell/2^N),$$

where P is the PBF used to generate the analysis. Take the FFT of the $c_\ell^{2^N}$'s, and compute the top-level FFT of the scaling coefficients via (1.13).

Step 2. Use the decomposition formula given in (3.7) to compute the level 2^{N-1} FFT of the scaling and wavelet coefficients. Repeat this step $N+1$ times to get the FFT of the wavelet coefficents at all levels.

Step 3. For a given level, take the inverse FFT to find the wavelet coefficients at that level. One may then use the coefficients to detect noise, singularities, and so on. One can also filter the coefficients to remove noise.

Step 4. Take the FFT of the processed coefficients. Use the reconstruction formula given in (3.6) to get back the FFT version of the scaling coefficients at each level. Use the $\hat{s}_k^{2^N}$'s to obtain $\hat{c}_k^{2^N}$'s via (1.13), and take the inverse FFT of the $\hat{c}_k^{2^N}$'s and reconstruct f via the formula in **Step 1**.

The only part of the algorithm that requires discussion is fitting a function in $\mathcal{V}_{2^N}$ to the data. There are several ways to do this, depending on the form of the data and what type of fit is desired.

The simplest case is that in which a continuous (but unknown) function $F(\theta)$ is sampled at 2^N equally spaced angles; that is, one has available

$$y_j := F(2\pi j/2^N)\,, \quad 0 \le j \le 2^N - 1.$$

One can project F into $\mathcal{V}_{2^N}$ in two ways.

The first is to interpolate F using the PBF P. This requires solving 2^N equations,

$$y_j = \sum_{\ell=0}^{2^N-1} c_\ell^{2^N} P(2\pi(j-\ell/2^N)\,, \quad 0 \le j \le 2^N - 1. \tag{8.1}$$

Because P is a PBF, the interpolation matrix $A_{j,\ell} = P(2\pi(j-\ell/2^N)$ is strictly positive [17,9], and therefore invertible. The fastest way to obtain the solution, however, is not to directly invert the matrix, but rather to use FFT methods similar to those mentioned in [10, §6]. The right side of (8.1) is a discrete convolution. Taking the FFT of both sides yields

$$\hat{y}_k = \hat{c}_k^{2^N} \mathcal{F}_d\left[\{P(2\pi\ell/2^N)\}_{\ell=0}^{2^N-1}\right]_k . \tag{8.2}$$

One may solve this for the $\hat{c}_k^{2^N}$. If all that is desired is an interpolant, one may take the inverse FFT of the $\hat{c}_k^{2^N}$'s to get the coefficients, and use f in **Step 1** as an interpolant for F. To do a wavelet decomposition, one would start with **Step 2** using the $\hat{c}_k^{2^N}$'s found from (8.2), and then proceed with the rest of the algorithm.

There is a second method, one that amounts to calculating the L^2 projection of F onto $\mathcal{V}_{2^N}$. To use this method, one must satisfy two conditions: (1) the scaling function and wavelet must be well localized at level 2^N; and, (2) the function F should not oscillate too fast over intervals having lengths of order $2^{-N+1}\pi$. When these conditions hold, one has that

$$\begin{aligned}
s_\ell^{2^N} &= \int_{-\pi}^{\pi} F(\theta)\varphi^{2^N}(\theta - 2\pi\ell/2^N)d\theta \\
&= \int_{-\pi}^{\pi} F(\theta + 2\pi\ell/2^N)\varphi^{2^N}(\theta)d\theta \\
&\approx F(2\pi\ell/2^N)\underbrace{\int_{-\pi}^{\pi}\varphi^{2^N}(\theta)d\theta}_{2\pi\varphi_0^{2^N}} \\
&\approx \frac{2\pi y_\ell p_0}{\sqrt{2^N\nu_0^{2^N}}} \quad \text{(From (1.9).)}
\end{aligned} \tag{8.3}$$

We remark that it is possible to turn the calculation above into a proof of (8.3). Doing this requires using the localization results from §6 and §7; we omit the details.

One can also handle scattered data. Again, this can be done in two ways. One way is to use the PBF to construct an interpolant F_{int} for the scattered data. That this can always be done follows from the invertibility of the interpolation matrix, which was established in [17]. The second way is to use *discrete* least sqaures to get the coefficients in f in **Step 1**.

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