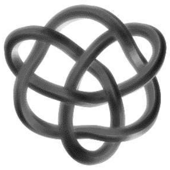


CUMC'97 - CCÉM97

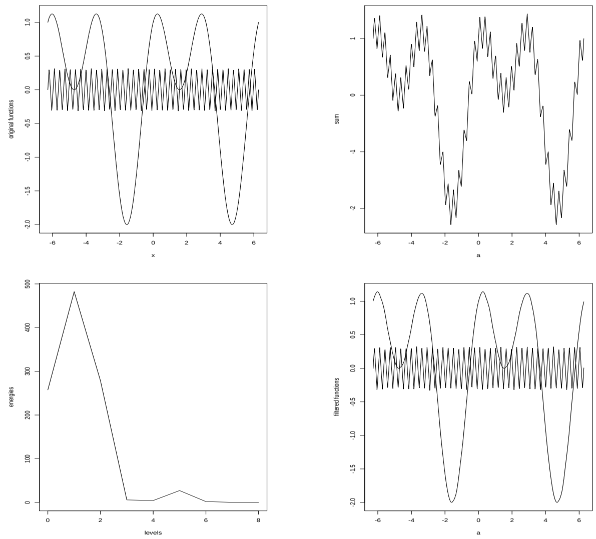
Université de Montréal

du 10 au 13 juillet 1997 • July, 10-13 1997

Wavelets and Statistics: Ten years of history
Ondellettes et statistique: 10 années d'histoire

Brani Vidakovic • *Duke University*

July 11, 1997

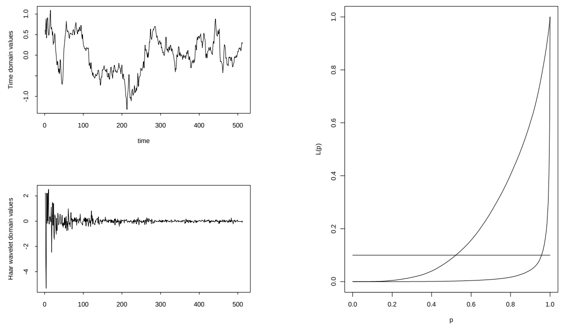


• **Wavelets filter the data.**

Two functions $y_1 = \sin x + \cos 2x$ and $y_2 = \frac{1}{5}\arcsin(\sin 20x)$, and their sum $y_1 + y_2$ are plotted on the upper panels. The lower-left panel shows the separation of "energy" to different levels in wavelet decomposition. The lower-right panel shows filtered functions.

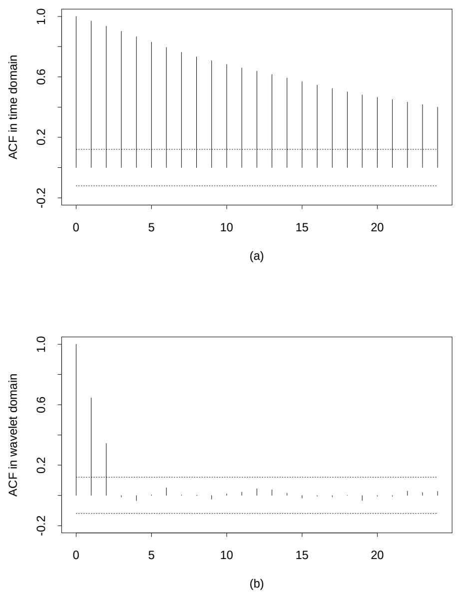
Outline

- **An Appetizer**
- **What are wavelets?**
- **Wavelets in statistics**
- **Additional reading and wavelet sources**



• **Wavelets disbalance the data.**

Left: Atmospheric turbulence measurements of u velocity component (upper panel) and their wavelet transformation (lower panel). Right: Lorentz curves [Lorentz curves measure inequality, disbalabce, concentration in data] of the original and transformed measurements. The curve corresponding to transformed measurements has higher curvature.



• Wavelets whiten the data.

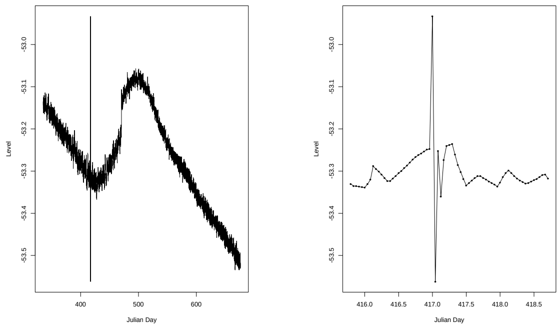
Autocovariance function for a time series in the time domain (Upper panel) and the wavelet domain (Lower panel).

California Earthquakes

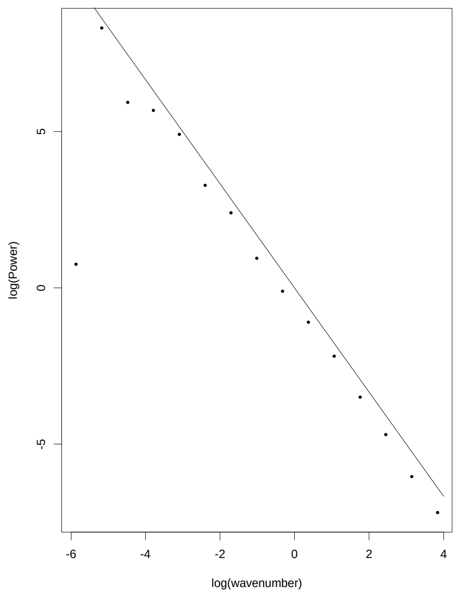
• A researcher from the Geology Department at Duke University was interested in the possibility of predicting earthquakes by monitoring water levels.

• Water level measurements from six wells located in California, were taken every hour for approximately six years.

• The goal was to smooth the data, eliminate the noise, and inspect the signal at pre-earthquake time.

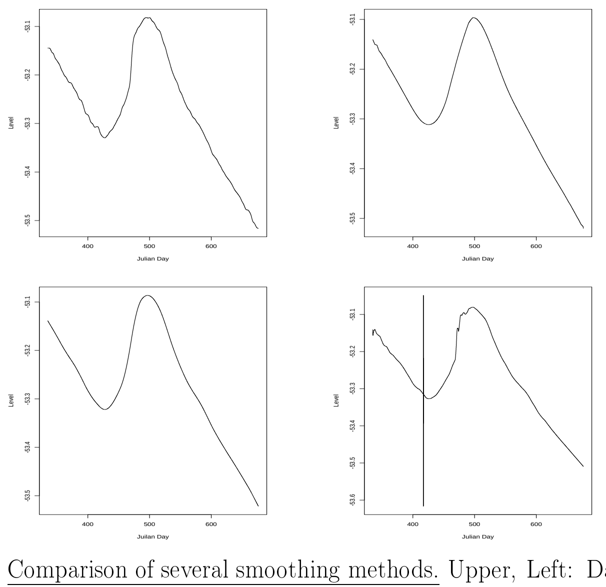


A plot of the raw data for hourly measurements collected over one year ($8192 = 2^{13}$ observations), is given in the left panel. The line-like artifact (enlarged in right panel) represents a line connecting two extreme level values at the earthquake time (Julian day of 417).



• Wavelets detect self-similarity in data.

Wavelet power spectrum of turbulence measurements and Kolmogorov's $-\frac{5}{3}$ law. Dots represent the logarithms of energy in the levels.



Comparison of several smoothing methods. Upper, Left: Data smoothed by kernel method (normal window, k=5); Upper, Right: Data smoothed by `loess` method; Lower, Left: Data smoothed by `supsmu` method; Lower, Right: Wavelet Smoothed Data

Notice that in all traditional methods the artifact of interest (earthquake jump) is lost. The application of nonlinear wavelet shrinkage to the data, results in a smooth signal with jump at earthquake time preserved.

What are wavelets?


Jean Baptiste Joseph Fourier 1768-1830 and Alfred Haar 1885-1933

The first "wavelet basis" was discovered in 1910 when Alfred Haar showed that any continuous function $f(x)$ on $[0,1]$ can be approximated by

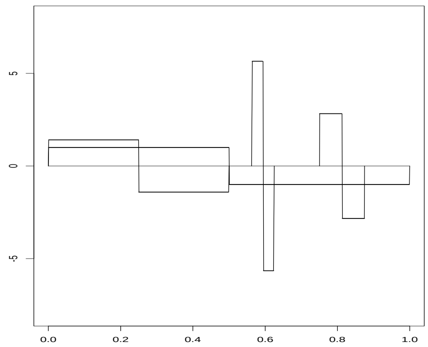
$$f_n(x) = \langle \xi_0, f\rangle \xi_0(x) + \langle \xi_1, f\rangle \xi_1(x) + \ldots + \langle \xi_n, f\rangle \xi_n(x).$$

and that, when $n \to \infty$, f_n converges to f uniformly.
The coefficients $\langle \xi_i, f\rangle$ are given by $\int \xi_i(x) f(x) dx$.

The Haar basis is very simple:

$$\begin{aligned}
\xi_0(x) &= \mathbf{1}(0 \le x \le 1),\\
\xi_1(x) &= \mathbf{1}(0 \le x \le 1/2) - \mathbf{1}(1/2 \le x \le 1),\\
\xi_2(x) &= \sqrt{2}[\mathbf{1}(0 \le x \le 1/4) - \mathbf{1}(1/4 \le x \le 1/2)],\\
&\ldots\\
\xi_n(x) &= 2^{j/2}[\mathbf{1}(k \cdot 2^{-j} \le x \le (k+1/2)\cdot 2^{-j})\\
&\quad -\mathbf{1}((k+1/2)\cdot 2^{-j} \le x \le (k+1)\cdot 2^{-j})], \ldots
\end{aligned}$$

where n is decomposed as $n = 2^j + k,\ j \ge 0,\ 0 \le k \le 2^j - 1$.



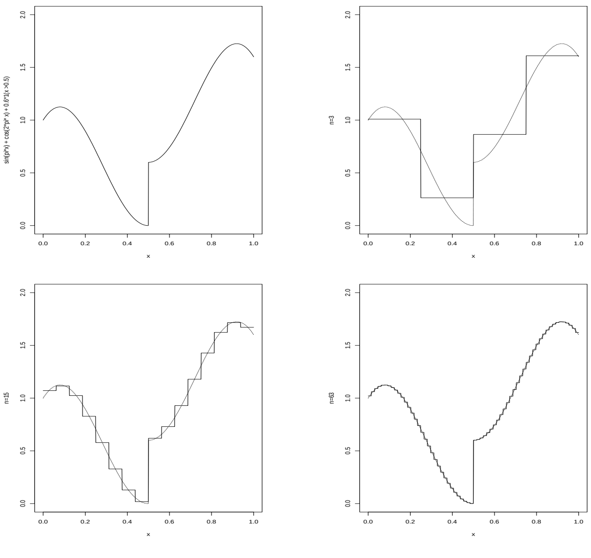
Functions ξ_1, ξ_2, ξ_{14}, and ξ_{25} from the Haar basis on $[0,1]$.

- The Haar wavelet decomposition of the function

$$y(x) = \sqrt{x(1-x)}\ \sin\frac{2.1\pi}{x + 0.05},\ \ 0 \le x \le 1.$$

- The function is known as the `doppler` function (Donoho and Johnstone)

- Implemented as a test function in almost all wavelet software packages.



Function $f(x) = \sin \pi x + \cos 2\pi x + 0.6 \cdot \mathbf{1}(x > 1/2),\ 0 \le x \le 1$, and three different levels of approximation in the Haar basis. Approximations f_3, f_{15}, and f_{63} are plotted.

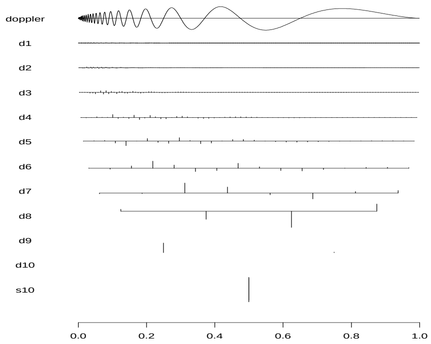
For any $n > 1$ the basis function ξ_n can be expressed as a scale-shift transformation of a single function ξ_1,

$$\xi_n(x) = \xi_{j,k}(x) = 2^{j/2}\xi_1(2^j \cdot x - k), \quad n = 2^j + k.$$

- $\xi_n,\ n \ge 1$ describe the details. • $\xi_0(x)$ is responsible for the "average."

level			j and k in the notation:	$n = 2^j + k$
	coefficients of	support	j	k
s10	ξ_0	1		
d10	ξ_1	1	$j = 0$	$k = 0$
d9	ξ_2 - ξ_3	1/2	$j = 1$	$0 \le k \le 1$
d8	ξ_4 - ξ_7	1/4	$j = 2$	$0 \le k \le 3$
d7	ξ_8 - ξ_{15}	1/8	$j = 3$	$0 \le k \le 2^3 - 1$
	...	...	...	...
d1	ξ_{512} - ξ_{1023}	$1/2^{10}$	$j = 10$	$0 \le k \le 2^{10} - 1$
	...	...	...	...

Coefficients of `doppler` function in Haar basis plotted in levels determined by the length of support of corresponding basis functions, $\xi_n,\ n \ge 0$.



`Doppler` function and its Haar basis decomposition

Multiresolution analysis (MRA) is a sequence of closed subspaces $V_n, n \in \mathbb{Z}$ in $\mathbb{L}_2(\mathbb{R})$

$$\ldots \subset V_2 \subset V_1 \subset V_0 \subset V_{-1} \subset V_{-2} \subset \ldots.$$

$$\cap_n V_n = \{\mathbf{0}\}, \quad \overline{\cup_n V_n} = \mathbb{L}_2(\mathbb{R}).$$

The hierarchy in MRA is constructed such that

- V-spaces are self similar,

$$f(x) \in V_n \text{ iff } f(2^n x) \in V_0.$$

- There exists *a scaling function* $\phi \in V_0$ whose integer translates span V_0,

$$V_0 = \{f \in \mathbb{L}_2(\mathbb{R}) | \ f(x) = \Sigma_k c_k \phi(x-k)\},$$

and for which the set $\{\phi(\bullet - k),\ k \in \mathbb{Z}\}$ is an orthonormal basis.

- (Scaling equation) Since $V_0 \subset V_{-1}$,

$$\phi(x) = \Sigma_{k \in \mathbb{Z}} h_k \sqrt{2} \phi(2x-k).$$

- **Normalization.**

$$\sum_{k \in \mathbb{Z}} h_k = \sqrt{2}.$$

- **Orthogonality.** For any $l \in \mathbb{Z}$,

$$\sum_k h_k h_{k-2l} = \delta_l.$$

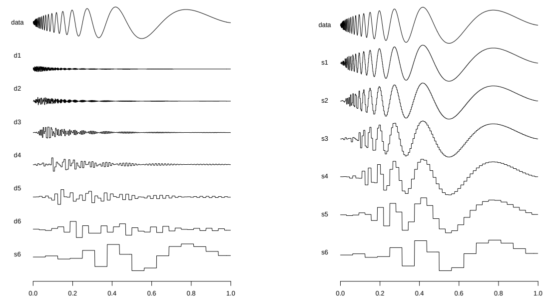
- An important special case is $l = 0$

$$\Sigma_k h_k^2 = 1.$$

$$\begin{aligned}
\phi(x) &= \phi(2x) + \phi(2x-1) = \frac{1}{\sqrt{2}}\sqrt{2}\phi(2x) + \frac{1}{\sqrt{2}}\sqrt{2}\phi(2x-1), \\
\psi(x) &= \phi(2x) - \phi(2x-1) = \frac{1}{\sqrt{2}}\sqrt{2}\phi(2x) - \frac{1}{\sqrt{2}}\sqrt{2}\phi(2x-1),
\end{aligned}$$

which gives the wavelet filter coefficients:

$$\begin{aligned}
h_0 &= h_1 = \frac{1}{\sqrt{2}}. \text{ and} \\
g_0 &= -g_1 = \frac{1}{\sqrt{2}}.
\end{aligned}$$



Left: Multiresolution Analysis of `doppler` function. The original function in V_0 is a sum of projections on V_6 and W_1–W_6 subspaces. Right: Coarsening `doppler` function by projecting it to V subspaces.

- Whenever sequence of subspaces satisfy MRA properties there exists an orthonormal basis for $\mathbb{L}_2(\mathbb{R})$,

$$\{\psi_{jk}(x) = 2^{-j/2}\psi(2^{-j}x - k),\ j,k \in \mathbb{Z}\}$$

such that $\{\psi_{jk}(x),\ j\text{-fixed},\ k \in \mathbb{Z}\}$ is an orthonormal basis for the "difference space" $W_j = V_{j-1} \ominus V_j$.

(because of containment $W_0 \subset V_{-1}$),

$$\psi(x) = \sum_{k \in \mathbb{Z}} g_k \sqrt{2}\phi(2x-k),$$

- Quadrature mirror relation

$$g_n = (-1)^n \ h_{1-n}.$$

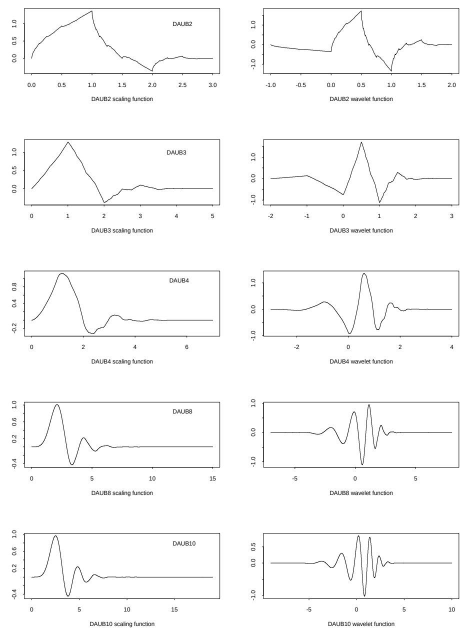
- Any $\mathbb{L}_2(\mathbb{R})$ function can be represented as

$$f(x) = \sum_{j,k} d_{jk}\psi_{jk}(x), \qquad [\mathbb{L}_2(\mathbb{R}) = \bigoplus_{j=-\infty}^{\infty} W_j.]$$

For any fixed j_0 the decomposition $\mathbb{L}_2(\mathbb{R}) = V_{j_0} \oplus \oplus_{j=-\infty}^{j=j_0} W_j$ corresponds to representation

$$f(x) = \sum_k c_{j_0,k}\phi_{j_0,k}(x) + \sum_{j \le j_0}\sum_k d_{jk}\phi_{j,k}(x).$$

Ingrid Daubechies was first to construct compactly supported orthogonal wavelets with preassigned degree of smoothness.



Graphs of scaling and wavelets functions from Daubechies family, $N = 2, 3, 4, 8$, and 10.

k	DAUB2	DAUB3	DAUB4	DAUB5
0	0.4829629131445342	0.3326705529500827	0.2303778133088966	0.160 1023979741926
1	0.8365163037378080	0.8068915093110930	0.7148465705529161	0.603 8292697971887
2	0.2241438680420134	0.4598775021184915	0.6308807679298591	0.724 3085284377723
3	-0.1294095225512604	-0.1350110200102548	-0.0279837694168604	0.138 4281459013216
4		-0.0854412738820267	-0.1870348117190935	-0.242 2948870663808
5		0.0352262918857096	0.0308413818355607	-0.032 2448695846383
6			0.0328830116668852	0.077 5714938400454
7			-0.0105974017850690	-0.006 2414902127983
8				-0.012 5807519990819
9				0.003 3357252854738

$\underset{\sim}{h}$ filters for Daubechies wavelets with $N = 2, 3, 4$ and 5 vanishing moments

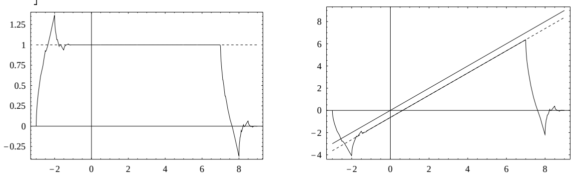
$$\begin{cases} h_0 + h_1 + h_2 + h_3 = \sqrt{2} \\ h_0^2 + h_1^2 + h_2^2 + h_3^2 = 1 \\ -h_1 + 2h_2 - 3h_3 = 0 \\ h_0\ h_2 + h_1\ h_3 = 0 \end{cases}$$

- Pollen-type Parameterization

n	h_n for $N = 2$
0	$(1 + \cos\varphi - \sin\varphi)/s$
1	$(1 + \cos\varphi + \sin\varphi)/s$
2	$(1 - \cos\varphi + \sin\varphi)/s$
3	$(1 - \cos\varphi - \sin\varphi)/s$

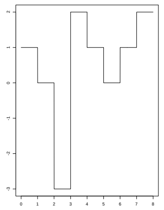
Pollen parameterization for four-tap filters. $[s = 2\sqrt{2}]$

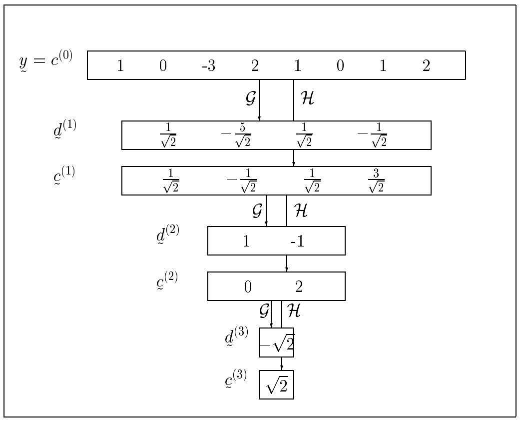
- Strang - Fix condition $[\sum_k \phi(x-k) = 1,\ \sum_k (x - \text{Const})\phi(x-k) = x]$



Let $\underset{\sim}{y} = (1, 0, -3, 2, 1, 0, 1, 2)$. The values $f(n) = y_n,\ n = 0, 1, \ldots, 7$ are interpolated by piecewise constant function. Assume that f belongs to Haar's multiresolution space V_0.

$$\underset{\sim}{h} = \{h_0, h_1\} = \{1/\sqrt{2}, 1/\sqrt{2}\};\ \underset{\sim}{g} = \{g_0, g_1\} = \{1/\sqrt{2}, -1/\sqrt{2}\}.$$



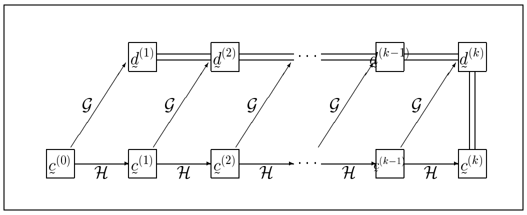


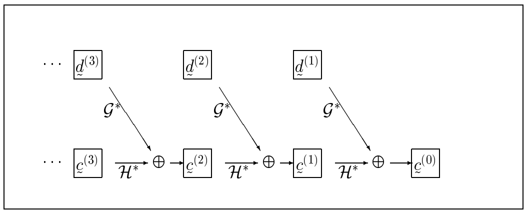
Discrete Wavelet Transformations

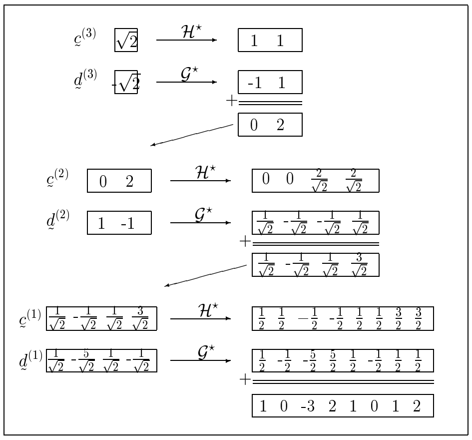
Fourier Methods	Fourier Integrals	Fourier Series	Discrete Fourier Transformations
Wavelet Methods	Continuous Wavelet Transformations	Wavelet Series	Discrete Wavelet Transformations

- **The Cascade Algorithm [Mallat, 1989]**

$$\mathcal{H}: \quad c_{j+1,l} = \sum_k h_{k-2l} c_{j,k}$$
$$\mathcal{G}: \quad d_{j+1,l} = \Sigma_k g_{k-2l} c_{j,k}$$
$$\mathcal{H}^*, \mathcal{G}^*: \quad c_{j,k} = \Sigma_l c_{j+1,l} h_{k-2l} + \Sigma_l d_{j+1,l} g_{k-2l}.$$







$$(\mathcal{H}a)_k = \Sigma_n h_{n-2k} a_n$$
$$(\mathcal{G}a)_k = \Sigma_n g_{n-2k} a_n.$$

$$\underset{\sim}{y} \longrightarrow (\mathcal{G}\underset{\sim}{y}, \mathcal{G}\mathcal{H}\underset{\sim}{y}, \mathcal{G}\mathcal{H}^2\underset{\sim}{y}, \ldots, \mathcal{G}\mathcal{H}^{k-1}\underset{\sim}{y}, \mathcal{H}^k\underset{\sim}{y}).$$

$$(\mathcal{H}^*a)_n = \Sigma_k h_{n-2k} a_k$$
$$(\mathcal{G}^*a)_n = \Sigma_k g_{n-2k} a_k.$$

Some General References and Sources

http://www.isds.duke.edu/~brani/wavelet.html

• Meyer, Y. (1992). *Wavelets and Operators.* Cambridge Studies in Advanced Mathematics **37.** Cambridge University Press.

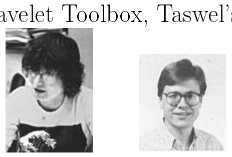
• Daubechies, I. (1992). *Ten Lectures on Wavelets.* Society for Industrial and Applied Mathematics.

• Strang G. and Nguyen T. (1996). Wavelets and Filter Banks, Wellesley-Cambridge Press, email: gs@math.mit.edu

• Vidakovic, B. and Müller, P. (1994). Wavelets for Kids - Tutorial Introduction. Discussion Paper **94-13**, ISDS, Duke University.

• Software FREE: wavethresh (Guy Nason, U. Bristol); wavelab (Donoho and coauthors, Stanford)

• Software Commercial: S-plus Wavelets, Mathematica Wavelet Explorer, Matlab Wavelet Toolbox, Taswel's Wavebox.



Ingrid Daubechies (Princeton U.) and David Donoho (Stanford U.)

Nonparametric Regression

• $t_i, i = 1, \ldots, N$ a sequence of equally-spaced points.
• $f_i = f(t_i), i = 1, \ldots, N$
• $y_i = f(t_i) + \epsilon_i$ – observations.
• ϵ_i i.i.d. $\mathcal{N}(0, \sigma^2)$, σ^2 unknown.
The problem of interest is estimating $f_i = f(t_i)$.

• Wavelet shrinkage in statistics was formally developed and explored in the work of Donoho and Johnstone (a series of their papers is available at `http://playfair.Stanford.EDU/reports/donoho/`)

OBSERVATIONS → D.W.T → SHRINK → Inv. D. W. T → ESTIMATOR

• Magnitudes of wavelet coefficients characterize regularity.
(Meyer, 1992, 179p, Th 5.) A function $f(x) = \Sigma_{k\in\mathbb{Z}} c_{ok}\phi(x-k) + \Sigma_{j\geq 0, k\in\mathbb{Z}} d_{jk}\psi_{jk}(x)$ belongs to the class C^s if and only if in a MRA of regularity $r > s$, the wavelet coefficients satisfy:
(i) $|c_{ok}| \leq C_0$,
(ii) $|d_{jk}| \leq C_1 2^{-j/2} 2^{-js}$.

PART 2. Wavelets and Statistics

• Nonparametric Regression (Denoising, Function Estimation, Wavelet Shrinkage

• Density Estimation

• Classification, Factor Analysis, Wavelet Based Principal Components

• Time Series

• Change Point Estimation and Detection of Outliers

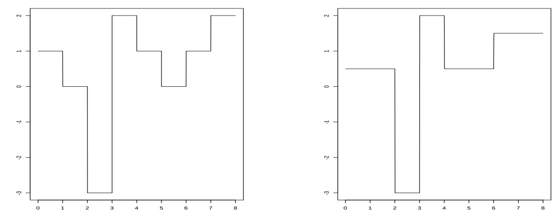
• Bayesian Methods

• Statistical Calculation

The data vector $(1, 0, -3, 2, 1, 0, 1, 2)$ is transformed into the vector

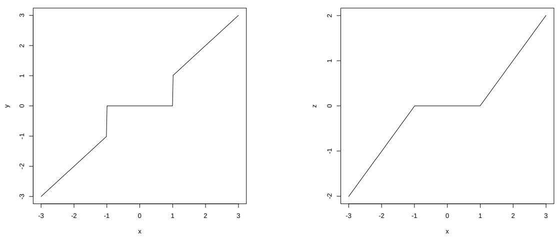
$$(\frac{1}{\sqrt{2}}, -\frac{5}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 1, -1, -\sqrt{2}, \sqrt{2}).$$

If all coefficients less than 0.9 are replaced by zeroes, then the resulting ("thresholded") vector is $(0, -\frac{5}{\sqrt{2}}, 0, 0, 1, -1, -\sqrt{2}, \sqrt{2})$.



Two standard choices are: **hard** and **soft** thresholding with corresponding transformations given by:

$$T^{hard}(d, \lambda) = d\ \mathbf{1}(|d| > \lambda),$$
$$T^{soft}(d, \lambda) = (d - sgn(d)\lambda)\ \mathbf{1}(|d| > \lambda).$$



Thresholds

Universal

$$\lambda^U = \sqrt{2\log n}\ \hat\sigma,$$

which Donoho and Johnstone call *universal*. This threshold is one of the first proposed and provides an easy, fast, and automatic thresholding.

- Model induced thresholding and Shrinkage [Mallat, Bayesian Methods, Testing hypotheses methods]
- Crossvalidation Methods [Nason, Wang, Weirich, Warhola]
- Risk (MSE) Considerations and Mimicking Oracle [Donoho-Johnstone, Bruce, Hong-Ye]
- Another class of useful functions are shrinkage (tapering) functions. A function S from that class exhibits the following properties:

$$S(d) \sim 0, d \text{ small}; \quad S(d) \sim d, d \text{ large}.$$

The last step in the model specification is the selection of a prior on θ.

$$\theta \sim t_n(0,\tau).$$

In general, Bayes rules are "shrinkers," and their shape in many cases has a desirable property for wavelet shrinkage: it shrinks small-magnitude arguments heavily and large-magnitude arguments only slightly.

Let $f(d|\theta) \sim \mathcal{DE}(\theta, \frac{1}{\sqrt{2\mu}})$, and let $\pi(\theta)$ be a prior on θ satisfying the symmetry condition:

$$\pi(\theta) = \pi(-\theta),\ \theta \in R.$$

Then the Bayes rule with respect to the squared error loss is:

$$\delta(d) = d - \frac{\Pi_1'(c) - \Pi_2'(c)}{\Pi_1(c) + \Pi_2(c)},$$

where Π_1 and Π_2 are the one-sided Laplace transforms of functions $\pi(\theta+d)$ and $\pi(\theta-d)$, $\theta \in (0,\infty)$, and $c = \sqrt{2\mu}$. [1]

[1] $\Pi(c) = \int_0^\infty e^{-ct}\pi(t)dt$.

An Example of Bayesian Shrinkage.

- $d|\theta \sim f(d;\theta)$ model, likelihood;
  $\theta \sim \pi(\theta)$ – prior
  $\theta|d \sim \pi^*(\theta; d)$ – posterior
- Bayes Theorem: $\pi^*(\theta; d) \propto f(d;\theta)\pi(\theta)$.
  $E\theta|d = \int \theta\pi^*(\theta; d)d\theta$ – Bayes Rule

$$y_i = f_i + \epsilon_i, \quad i = 1,\dots,N.$$

Wavelet Transformation ↓

$$d_i = \theta_i + \epsilon_i', \quad i = 1,\dots,N.$$

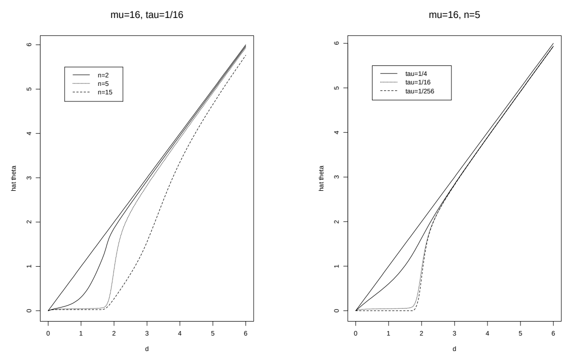
$$d|\theta,\sigma^2 \sim N(\theta,\sigma^2), \quad \sigma^2 \text{ unknown.}$$

To complete our model we specify priors for σ^2 and θ.

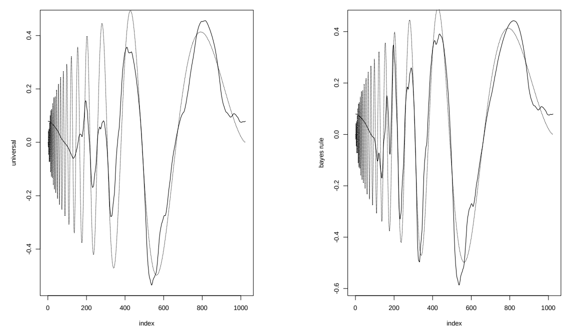
$$\sigma^2 \sim \mathcal{E}(\mu). \ \ (f(\sigma^2|\mu) = \mu e^{-\mu\sigma^2}).$$

$$d|\theta \sim \mathcal{DE}(\theta, \frac{1}{\sqrt{2\mu}}),$$

with the density given by $f(d|\theta) = \frac{1}{2}\sqrt{2\mu}e^{-\sqrt{2\mu}|d-\theta|}$.



Bayes rules for selected values of n, τ, and μ.



Wavelets and Density Estimation

- Čencov (1962).

$$f(x) = \Sigma_{j \in J} a_j \psi_j(x),$$

where $\{\psi_j,\ j \in J\}$ is a complete orthonormal system of functions in $\mathbb{L}_2$, and J is an appropriate set of indices.

- a_j can be expressed as

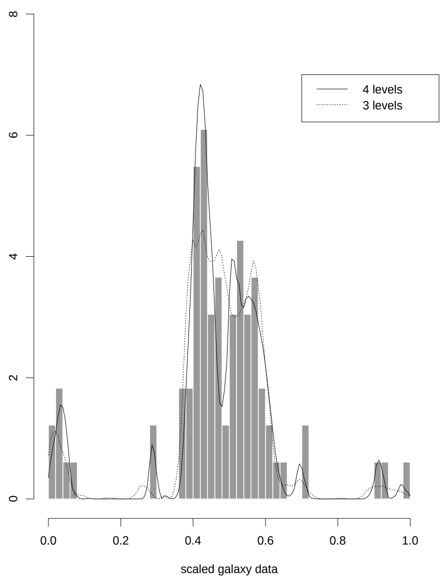
$$\begin{aligned} a_j &= \langle f, \psi_j \rangle \\ &= \int \psi_j(x) f(x) dx \\ &= E\psi_j(X). \end{aligned}$$

- $\underset{\sim}{X} = (X_1, X_2, \ldots, X_n)$ be a sample from the unknown distribution f.

$$\hat{a}_j = \Sigma_{i=1}^n \psi_j(X_i),$$

- Estimator of $f(x)$:

$$\hat{f}(x) = \Sigma_{j \in J} \hat{a}_j \psi_j(x).$$



$$1 = \int f = \langle \sqrt{f}, \sqrt{f} \rangle,$$

if

$$\sqrt{f} = \Sigma_{j \in J} a_j \psi_j(x),$$

then

$$||a_j||_{\ell_2}^2 = \Sigma_j a_j^2 = 1,$$

by the Parseval identity.

This result motivates re-normalizing empirical coefficients $\hat{a}_j$ $||\hat{a}_j||_{\ell_2}^2 = \Sigma_j \hat{a}_j^2 = 1$, will ensure $\int \hat{f} = 1$.

Galaxy Example

According to the Big Bang theory, matter in the universe expanded at a tremendous rate. Gravitational forces caused the formation of galaxies. Astronomers speculate that gravitational pull led to clustering of galaxies and research indicates the presence of super-clusters of galaxies surrounded by large voids (string-and-filament pattern). Measurements have recently become available for the distances between our galaxy and others. Distance is estimated by the red shift in the light spectrum in a fashion similar to how the Doppler effect measures the changes in speed via changes in sound. Under the expansion-universe paradigm, the furthest (from our galaxy) galaxies must be moving at greater velocities, because the distances and velocities are proportional. If, in reality, the galaxies are clumped, the velocities should have a multimodal distribution, each mode corresponding to a cluster. In the region of *Corona Borealis* the velocities of 82 galaxies were measured. The relative measurement error is believed to be smaller than 0.5%.

9172	9350	9483	9558	9775	10227	10406	16084	16170	18419	18552	18600
18927	19052	19070	19330	19343	19349	19440	19473	19529	19541	19547	19663
19846	19856	19863	19914	19918	19973	19989	20166	20175	20179	20196	20215
20221	20415	20629	20795	20821	20846	20875	20986	21137	21492	21701	21814
21921	21960	22185	22209	22242	22249	22314	22374	22495	22746	22747	22888
22914	23206	23241	23263	23484	23538	23542	23666	23706	23711	24129	24285
24289	24366	24717	24990	25633	26960	26995	32065	32789	34279		

$\phi(x) = ?$

- Let ϕ be the scaling function of DAUB#N wavelet. The support of ϕ is $[0, K]$, $K = 2N - 1$. Let $x \in (0, 1)$, and let $dyad(x) = \{d_1, d_2, \ldots d_n, \ldots\}$ be the set of 0-1 digits in dyadic representation of x $(x = \Sigma_{j=1}^{\infty} d_j 2^{-j})$. By $dyad(x, n)$ we denote the subset of the first n digits from $dyad(x)$, i.e., $dyad(x, n) = \{d_1, d_2, \ldots d_n\}$.
- Let $\underset{\sim}{h} = (h_0, h_1, \ldots, h_{2N})$ be the filter vector of DAUB #N Define two $K \times K$ matrices as
- $T_0 = (h_{2i-j-1})_{1 \le i,j \le K}$, and $T_1 = (h_{2i-j})_{1 \le i,j \le K}$.
- (Daubechies and Lagarias, 1992.)

$$\lim_{n \to \infty} T_{d_1} \cdot T_{d_2} \cdot \ldots \cdot T_{d_n} = \begin{bmatrix} \phi(x) & \phi(x) & \ldots & \phi(x) \\ \phi(x+1) & \phi(x+1) & \ldots & \phi(x+1) \\ \vdots & & & \\ \phi(x+K-1) & \phi(x+K-1) & \ldots & \phi(x+K-1) \end{bmatrix}.$$

- DAUB 2 case. • The filter is $(\frac{1+\sqrt{3}}{4}, \frac{3-\sqrt{3}}{4}, \frac{3+\sqrt{3}}{4}, \frac{1-\sqrt{3}}{4})$.
- T_0 and T_1 are:

$$T_0 = \begin{bmatrix} \frac{1+\sqrt{3}}{4} & 0 & 0 \\ \frac{3-\sqrt{3}}{4} & \frac{3+\sqrt{3}}{4} & \frac{1+\sqrt{3}}{4} \\ 0 & \frac{1-\sqrt{3}}{4} & \frac{3-\sqrt{3}}{4} \end{bmatrix}, \quad \text{and} \quad T_1 = \begin{bmatrix} \frac{3+\sqrt{3}}{4} & \frac{1+\sqrt{3}}{4} & 0 \\ \frac{1-\sqrt{3}}{4} & \frac{3-\sqrt{3}}{4} & \frac{3+\sqrt{3}}{4} \\ 0 & 0 & \frac{1-\sqrt{3}}{4} \end{bmatrix}.$$

- If $x = 0.45$, then $dyad(0.45, 20) = \{$ 0, 1, 1, 1, 0, 0, 1, 1, 0, 0, 1, 1, 0, 0, 1, 1, 0, 0, 1, 1 $\}$. The approximations of $\phi(0.45), \phi(1.45)$, and $\phi(2.45)$ are in the rows of

$$\prod_{i \in dyad(0.45,20)} T_i = \begin{bmatrix} 0.86480582 & 0.86480459 & 0.86480336 \\ 0.08641418 & 0.08641568 & 0.08641719 \\ 0.04878000 & 0.04877973 & 0.04877945 \end{bmatrix}.$$

The Daubechies and Lagarias algorithm gives only the values of the scaling function. However, most of the evaluation needed involves the mother wavelet. Visit: http://www.isds.duke.edu/ brani/datapro.html

Also visit: http://www.isds.duke.edu/ brani/local.html and fetch “Estimation of Square Root of Density by Wavelets.”

Homework

• Plot a graph of the scaling function from the Pollen library for $\varphi = \pi/4$. Use Daubechies-Lagarias Algorithm.

Additional Reading

• S+Plus Wavelets (Manual by Andrew Bruce and Hong-Ye Gao)

• Gilbert Walter: Wavelets and Other Orthogonal Systems with Applications, CRC Press Inc.

• Todd Ogden: Essential Wavelets for Statistical Applications and Data Analysis, Birkhauser, Boston

• Hernandez and Weiss: A First Course on Wavelets, CRC Press.

• 1000's of references with linked papers in ps format at:

http://liinwww.ira.uka.de/bibliography/Theory/Wavelets/