ISyE 8872 Topics in Nonlinear Optimization

Fall 2001

Assignment 2

Issued: August 28, 2001

Due: September 4, 2001

Problem 1
If you need additional assumptions for a result to hold, then state those assumptions, and motivate why the assumptions are needed.

(1) Consider $f, g : \mathcal{X} \mapsto \mathbb{R}$. Show that

$$\inf_{x \in \mathcal{X}} f(x) + \inf_{x \in \mathcal{X}} g(x) \;\leq\; \inf_{x \in \mathcal{X}} \{f(x) + g(x)\}$$

Give an example where strict inequality holds.

(2) Consider $f : \mathcal{X} \mapsto \mathbb{R}$ and $g : \mathcal{Y} \mapsto \mathbb{R}$. Show that

$$\inf_{x \in \mathcal{X}} f(x) + \inf_{y \in \mathcal{Y}} g(y) \;=\; \inf_{x \in \mathcal{X}, y \in \mathcal{Y}} \{f(x) + g(y)\}$$

(3) Consider $f : \mathcal{X} \times \mathcal{Y} \mapsto \mathbb{R}$. Show that

$$\inf_{x \in \mathcal{X}} \{\inf_{y \in \mathcal{Y}} f(x, y)\} \;=\; \inf_{y \in \mathcal{Y}} \{\inf_{x \in \mathcal{X}} f(x, y)\} \;=\; \inf_{x \in \mathcal{X}, y \in \mathcal{Y}} f(x, y)$$

(4) Consider $f : \mathcal{X} \times \mathcal{Y} \mapsto \mathbb{R}$. Show that

$$\sup_{x \in \mathcal{X}} \{\inf_{y \in \mathcal{Y}} f(x, y)\} \;\leq\; \inf_{y \in \mathcal{Y}} \{\sup_{x \in \mathcal{X}} f(x, y)\}$$

Give an example where strict inequality holds.

Problem 2
For each of the following statements, either prove the statement or give a counterexample:

(1) Let L_1, L_2 be two linear subspaces in V. Then $L_1 \cup L_2$ is a linear subspace in V.

(2) Let A_1, A_2 be two affine manifolds in V. Then $A_1 \cup A_2$ is an affine manifold in V.

(3) Let C_1, C_2 be two convex sets in V. Then $C_1 \cup C_2$ is a convex set in V.

(4) Let K_1, K_2 be two cones in V. Then $K_1 \cup K_2$ is a cone in V.

(5) Let $\{L_\alpha \subseteq V : \alpha \in S\}$ be an arbitrary collection of linear subspaces L_α in V. Then $\bigcap\{L_\alpha \subseteq V : \alpha \in S\}$ is a linear subspace in V.

(6) Let $\{A_\alpha \subseteq V : \alpha \in S\}$ be an arbitrary collection of affine manifolds A_α in V. Then $\bigcap\{A_\alpha \subseteq V : \alpha \in S\}$ is an affine manifold in V.

(7) Let $\{C_\alpha \subseteq V : \alpha \in S\}$ be an arbitrary collection of convex sets C_α in V. Then $\bigcap\{C_\alpha \subseteq V : \alpha \in S\}$ is a convex set in V.

(8) Let $\{K_\alpha \subseteq V : \alpha \in S\}$ be an arbitrary collection of cones K_α in V. Then $\bigcap\{K_\alpha \subseteq V : \alpha \in S\}$ is a cone in V.

(9) Let $\{K_\alpha \subseteq V : \alpha \in S\}$ be an arbitrary collection of convex cones K_α in V. Then $\bigcap\{K_\alpha \subseteq V : \alpha \in S\}$ is a convex cone in V.

(10) A cone $K \subseteq V$ is convex if and only if $K + K \subseteq K$.

(11) A set $K \subseteq V$ is convex if and only if $K + K \subseteq K$.

(12) A set $K \subseteq V$ is a linear subspace if and only if K is a convex cone and $-K \subseteq K$.

(13) Let $\{L_i \subseteq V_i : i \in \{1, \dots, n\}\}$ be a collection of linear subspaces. Then the Cartesian product $L_1 \times \cdots \times L_n$ is a linear subspace in $V_1 \times \cdots \times V_n$.

(14) Let $\{A_i \subseteq V_i : i \in \{1, \dots, n\}\}$ be a collection of affine manifolds. Then the Cartesian product $A_1 \times \cdots \times A_n$ is an affine manifold in $V_1 \times \cdots \times V_n$.

(15) Let $\{C_i \subseteq V_i : i \in \{1, \dots, n\}\}$ be a collection of convex sets. Then the Cartesian product $C_1 \times \cdots \times C_n$ is a convex set in $V_1 \times \cdots \times V_n$.

(16) Let $\{K_i \subseteq V_i : i \in \{1, \dots, n\}\}$ be a collection of cones. Then the Cartesian product $K_1 \times \cdots \times K_n$ is a cone in $V_1 \times \cdots \times V_n$.

(17) Let $\{L_i \subseteq V : i \in \{1, \dots, n\}\}$ be a collection of linear subspaces in V, and $c_1, \dots, c_n \in F$. Then the direct sum $c_1 L_1 + \cdots + c_n L_n$ is a linear subspace in V.

(18) Let $\{A_i \subseteq V : i \in \{1, \dots, n\}\}$ be a collection of affine manifolds in V, and $c_1, \dots, c_n \in F$. Then the direct sum $c_1 A_1 + \cdots + c_n A_n$ is an affine manifold in V.

(19) Let $\{C_i \subseteq V : i \in \{1, \dots, n\}\}$ be a collection of convex sets in V, and $c_1, \dots, c_n \in \mathbb{R}$. Then the direct sum $c_1 C_1 + \cdots + c_n C_n$ is a convex set in V.

(20) Let $\{K_i \subseteq V : i \in \{1, \dots, n\}\}$ be a collection of cones in V, and $c_1, \dots, c_n \in \mathbb{R}$. Then the direct sum $c_1 K_1 + \cdots + c_n K_n$ is a cone in V.

Problem 3
Assume that $L \subseteq V$ is a linear subspace and $A \subseteq V$ is an affine manifold. Show that:

(1) for any $x^0 \in V$, the set $L + x^0$ is an affine manifold;

(2) for any $x^0 \in A$, the set $A - x^0$ is a linear subspace which does not depend on x^0.

(3) A is a linear subspace if and only if $0 \in A$.