ISyE 8872 Topics in Nonlinear Optimization

Fall 2001

Assignment 1 (Revision)

Issued: August 21, 2001

Due: August 28, 2001

Problem 1
Assume that $\mathcal{L} \subset \mathbb{R}^n$ is a linear subspace, and that $\mathcal{B}^1$ and $\mathcal{B}^2$ are two bases for $\mathcal{L}$.

(1) Show that $|\mathcal{B}^1| = |\mathcal{B}^2|$, where $|\mathcal{B}|$ denotes the number of elements (cardinality) of set $\mathcal{B}$.

(2) Show that the expression of any element of $\mathcal{L}$ as a linear combination of the elements of $\mathcal{B}^1$ is unique.

(3) Represent a linear combination by its vector of coefficients. Show that the function that maps the expression of an element of $\mathcal{L}$ as a linear combination of the elements of $\mathcal{B}^1$ to the expression of the same element as a linear combination of the elements of $\mathcal{B}^2$ is an invertible linear function. Show how to construct the matrix representation of the abovementioned function and its inverse.

Problem 2
Consider $\mathcal{S}^1, \mathcal{S}^2 \subset \mathbb{R} \cup \{+\infty\}$. If you need additional assumptions for a result to hold, then state those assumptions, and motivate why the assumptions are needed. Show that:

(1)

$$\inf\{\mathcal{S}^1 + \mathcal{S}^2\} \;=\; \inf \mathcal{S}^1 + \inf \mathcal{S}^2$$

(2)

$$\inf\{\mathcal{S}^1 \cup \mathcal{S}^2\} \;=\; \min\{\inf \mathcal{S}^1, \inf \mathcal{S}^2\}$$

(3)

$$\inf\{\mathcal{S}^1 \cap \mathcal{S}^2\} \;\geq\; \max\{\inf \mathcal{S}^1, \inf \mathcal{S}^2\}$$

Give an example where strict inequality holds.

Problem 3
Consider $\mathcal{A}^1, \mathcal{A}^2, \mathcal{B}^1, \mathcal{B}^2 \subset \mathcal{V}$, for some vector space $\mathcal{V}$. Suppose that $\mathcal{A}^1 \subseteq \mathcal{B}^1$ and $\mathcal{A}^2 \subseteq \mathcal{B}^2$. Show that:

(1)

$$\mathcal{A}^1 \cup \mathcal{A}^2 \;\subseteq\; \mathcal{B}^1 \cup \mathcal{B}^2$$

(2)

$$\mathcal{A}^1 \cap \mathcal{A}^2 \subseteq \mathcal{B}^1 \cap \mathcal{B}^2$$

(3)

$$\mathcal{A}^1 + \mathcal{A}^2 \subseteq \mathcal{B}^1 + \mathcal{B}^2$$

That is, set union, set intersection, and set addition together with the $\subseteq$ relation behaves like real number addition together with the $\leq$ relation. However, give examples where:

(a) $\mathcal{A}^1 \cup \mathcal{A}^2 = \mathcal{A}^1$, but $\mathcal{A}^2 \neq \varnothing$.

(b) $\mathcal{A}^1 \cap \mathcal{A}^2 = \mathcal{A}^1$, but $\mathcal{A}^2 \neq \mathcal{V}$.

(c) $\mathcal{A}^1 + \mathcal{A}^2 = \mathcal{A}^1$, but $\mathcal{A}^2 \neq \{0\} \subset \mathcal{V}$.

That is, although real number addition has a unique zero element ($a + b = a \Leftrightarrow b = 0$, with $a, b \in \mathbb{R}$), the same does not hold for set union, set intersection, and set addition.