ISyE 8813C Game Theory

Fall 2013

Assignment 1 (Revision)

Issued: August 20, 2013

Due: August 27, 2013

Problem 1
Consider a function $g(x_1, x_2)$, with decisions $x_1 \in \mathcal{X}_1$ and $x_2 \in \mathcal{X}_2$.

1. Show that

$$\sup_{x_1 \in \mathcal{X}_1} \sup_{x_2 \in \mathcal{X}_2} g(x_1, x_2) \quad = \quad \sup_{x_2 \in \mathcal{X}_2} \sup_{x_1 \in \mathcal{X}_1} g(x_1, x_2) \quad = \quad \sup_{(x_1, x_2) \in \mathcal{X}_1 \times \mathcal{X}_2} g(x_1, x_2)$$

2. Show that

$$\sup_{x_1 \in \mathcal{X}_1} \inf_{x_2 \in \mathcal{X}_2} g(x_1, x_2) \quad \leq \quad \inf_{x_2 \in \mathcal{X}_2} \sup_{x_1 \in \mathcal{X}_1} g(x_1, x_2)$$

3. Give an example of a function $g(x_1, x_2)$ such that

$$\sup_{x_1 \in \mathcal{X}_1} \inf_{x_2 \in \mathcal{X}_2} g(x_1, x_2) \quad < \quad \inf_{x_2 \in \mathcal{X}_2} \sup_{x_1 \in \mathcal{X}_1} g(x_1, x_2)$$

4. Suppose two decision makers compete in the following way. Decision maker 1 chooses $x_1 \in \mathcal{X}_1$ with the objective to maximize $g(x_1, x_2)$, and decision maker 2 chooses $x_2 \in \mathcal{X}_2$ with the objective to minimize $g(x_1, x_2)$. Decision maker 1 chooses first, and decision maker 2 chooses second. Is the outcome given by

$$\sup_{x_1 \in \mathcal{X}_1} \inf_{x_2 \in \mathcal{X}_2} g(x_1, x_2)$$

   or by

$$\inf_{x_2 \in \mathcal{X}_2} \sup_{x_1 \in \mathcal{X}_1} g(x_1, x_2)$$

   Is it better to choose first or to choose second?

Problem 2
Let $\{C_i \subseteq V_i \,:\, i \in \{1, \ldots, n\}\}$ be a collection of convex sets in vector spaces V_i. Show that the Cartesian product $C_1 \times \cdots \times C_n$ is a convex set in $V_1 \times \cdots \times V_n$.

Problem 3

Let $\{C_i \subseteq V_i : i \in \{1, \ldots, n\}\}$ be a collection of convex sets in vector spaces V_i. A function $f : C_1 \times \cdots \times C_n \mapsto \mathbb{R}$ is called componentwise convex if, for each $i \in \{1, \ldots, n\}$, and each $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in C_1 \times \cdots \times C_{i-1} \times C_{i+1} \times \cdots \times C_n$, the function $f(x_1, \ldots, x_{i-1}, \cdot, x_{i+1}, \ldots, x_n) : C_i \mapsto \mathbb{R}$ is convex.

1. Show that if f is convex then f is componentwise convex.
2. Give an example of a componentwise convex function f that is not convex.

Problem 4

A function $f : C \mapsto \mathbb{R}$ is called quasiconvex if all its sublevel sets are convex, that is, if the sets $\{x \in C : f(x) \leq a\}$ are convex for all $a \in \mathbb{R}$.

1. Show that if f is convex then f is quasiconvex.
2. Give an example of a quasiconvex function f that is not convex.

Problem 5

Let $f : C \mapsto \mathbb{R}$ be a strictly convex function on a convex subset C of a vector space.

1. Show that f has at most one minimum on C.
2. Give an example of a strictly convex function f on a convex set C such that f does not have a minimum on C.