ISyE 8803E Game Theory

Spring 2008

Assignment 2

Issued: January 28, 2008

Due: February 11, 2008

Problem 1
Consider closed sets $F, G \subset \mathbb{R}^m$, and suppose that $F \cap G = \varnothing$. Show that there exist open sets $A, B \subset \mathbb{R}^m$ such that $F \subset A$, $G \subset B$, and $A \cap B = \varnothing$.

Problem 2
Let $X \subset \mathbb{R}^m$, $Y \subset \mathbb{R}^k$, and $F : X \mapsto 2^Y$.

1. Show that if F is open, then F is lower hemi-continuous.
2. The following statement comes from Proposition 11.9(d) in Border (1985): "If F is upper hemi-continuous at x and $F(x)$ is a singleton, then F is lower hemi-continuous at x."
   (a) Give a counterexample to the statement above.
   (b) Correct the statement, and prove the correctness of your statement.
3. Show that if $F^{-}(\{y\})$ is open for all $y \in Y$, then F is lower hemi-continuous.

Problem 3
Let $X \subset \mathbb{R}^m$, $Y \subset \mathbb{R}^k$, and $F : X \mapsto 2^Y$. Show that F is lower hemi-continuous at x if and only if for all sequences $\{x_k\}_k \subset X$ such that $x_k \to x$ as $k \to \infty$, and all $y \in F(x)$, there is a sequence $\{y_k\}_k \subset Y$ such that $y_k \in F(x_k)$ for all k and $y_k \to y$ as $k \to \infty$.

Problem 4
Let $X \subset \mathbb{R}^m$ and $Y \subset \mathbb{R}^k$. Show the following:

1. Consider $F : X \mapsto 2^{\mathbb{R}^m}$. If F is upper hemi-continuous and closed valued, then the set $\{x \in X : x \in F(x)\}$ of fixed points of F is a (possibly empty) closed subset of X.
2. Consider $F, G : X \mapsto 2^Y$. If F and G are upper hemi-continuous and closed valued, then the set $\{x \in X : F(x) \cap G(x) \neq \varnothing\}$ is a (possibly empty) closed subset of X.

3. Consider $F : X \mapsto 2^Y$. If F is upper hemi-continuous, then the set $\{x \in X : F(x) \neq \varnothing\}$ is a closed subset of X.

4. Consider $F : X \mapsto 2^Y$. If F is lower hemi-continuous, then the set $\{x \in X : F(x) \neq \varnothing\}$ is an open subset of X.

Problem 5

Let $X \subset \mathbb{R}^m$, $Y \subset \mathbb{R}^k$, and $F : X \mapsto 2^Y$. Let $\bar{F} : X \mapsto 2^Y$ be defined by

$$\bar{F}(x) \quad \equiv \quad \text{closure (in } Y\text{) of } F(x)$$

Show the following:

1. F is lower hemi-continuous at x if and only if $\bar{F}$ is lower hemi-continuous at x.
2. If F is upper hemi-continuous at x, then $\bar{F}$ is upper hemi-continuous at x.
3. Show by example that it is not always true that if $\bar{F}$ is upper hemi-continuous at x, then F is upper hemi-continuous at x.

Problem 6

Let $G : X \mapsto 2^Y$ and $F : Y \mapsto 2^Z$. Let $F \circ G : X \mapsto 2^Z$ be defined by

$$(F \circ G)(x) \quad \equiv \quad \bigcup_{y \in G(x)} F(y)$$

Show the following:

1. If F and G are upper hemi-continuous, then $F \circ G$ is upper hemi-continuous.
2. If F and G are lower hemi-continuous, then $F \circ G$ is lower hemi-continuous.
3. Show by example that it may happen that F and G are closed, but $F \circ G$ is not closed.