# ISyE 8801B  Game Theory

## Fall 2001

## Assignment 2

## Issued: August 27, 2003

## Due: September 5, 2003

**Problem 1**
Consider functions $f_1, f_2 : \mathcal{X} \mapsto \mathbb{R}$. Show that

$$\inf_{x\in\mathcal{X}} f_1(x) + \inf_{x\in\mathcal{X}} f_2(x) \leq \inf_{x\in\mathcal{X}} \{f_1(x) + f_2(x)\} \leq \sup_{x\in\mathcal{X}} \{f_1(x) + f_2(x)\} \leq \sup_{x\in\mathcal{X}} f_1(x) + \sup_{x\in\mathcal{X}} f_2(x)$$

**Problem 2**
Randomization does not help for single-stage games with simultaneous decisions: Suppose we want to maximize $g(x)$ over all $x \in \mathcal{X}$. That is, the optimization problem is

$$\sup_{x\in\mathcal{X}} g(x)$$

(The following also holds if $g(x) \equiv \inf_{y\in\mathcal{Y}} G(x,y)]$ or if $g(x) \equiv G(x,y^*)]$ for some fixed $y^* \in \mathcal{Y}$.) A deterministic decision (pure strategy) chooses one particular decision $x \in \mathcal{X}$. A randomized decision (mixed strategy) chooses a probability distribution $P$ on $\mathcal{X}$ (we can choose a sufficiently rich $\sigma$-field on $\mathcal{X}$), and then a decision $x \in \mathcal{X}$ is generated according to probability distribution $P$. The objective value of such a randomized decision $P$ is $E_P[g(X)]$ (where $g(X)$ is a random variable if the $\sigma$-field on $\mathcal{X}$ is sufficiently rich).

Show that randomization does not help for such a single-stage game with simultaneous decisions, unless $\sup_{x\in\mathcal{X}} g(x) = \infty$. (There are single-stage games with sequential decisions for which randomization does help. Furthermore, there are multistage games for which randomization helps.) That is, show the following:

1. For any probability distribution $P$ on $\mathcal{X}$,

$$E_P[g(X)] \leq \sup_{x\in\mathcal{X}} g(x)$$

   That is,

$$\sup_{P\in\mathcal{P}} E_P[g(X)] = \sup_{x\in\mathcal{X}} g(x)$$

   where $\mathcal{P}$ denotes the set of probability distributions on $\mathcal{X}$.

2. If $g^* \equiv \sup_{x \in \mathcal{X}} g(x) < \infty$ and there exists no $x \in \mathcal{X}$ such that $g(x) = g^*$, that is, there exists no deterministic decision $x \in \mathcal{X}$ that attains the supremum, then there exists no $P \in \mathcal{P}$ such that $E_P[g(X)] = g^*$, that is, there exists no randomized decision $P \in \mathcal{P}$ that attains the supremum.

3. Suppose we have a static optimization problem where $g(x) < \infty$ for all $x \in \mathcal{X}$, but $\sup_{x \in \mathcal{X}} g(x) = \infty$. Show how to choose a probability distribution $P$ such that $E_P[g(X)] = \infty$. That is, there exists no deterministic decision $x \in \mathcal{X}$ that attains the supremum, but there exists a randomized decision $P \in \mathcal{P}$ that attains the supremum.

**Problem 3**

Randomization may help for games with sequential decisions: Consider a game with objective function $G(x, y)$ in which player 1 first chooses a decision $x \in \mathcal{X}$ to maximize $G(x, y)$, and thereafter player 2 observes the chosen decision $x$ of player 1 and then chooses a decision $y \in \mathcal{Y}$ to minimize $G(x, y)$. That is, the outcome of the game described above is

$$g^* \equiv \sup_{x \in \mathcal{X}} \inf_{y \in \mathcal{Y}} G(x, y)$$

We already know that if $g_1(x) \equiv \inf_{y \in \mathcal{Y}} G(x, y) > -\infty$ for the decision $x \in \mathcal{X}$ chosen by player 1 (and one would hope that this is the case in a reasonable game), then randomization cannot help player 2. However, consider the problem faced by player 1, and let us modify the game described above by allowing player 1 to make randomized decisions. A deterministic decision chooses one particular decision $x \in \mathcal{X}$. A randomized decision chooses a probability distribution $P$ on $\mathcal{X}$, and then a decision $x \in \mathcal{X}$ is generated according to probability distribution $P$. In case such a randomized decision $P$ is chosen, the objective function takes the expected value $E_P[\cdot]$ over all decisions $X$, because $X$ is now a random variable distributed according to $P$. Show that

$$\inf_{y \in Y} E_P\left[G(X, y)\right] \geq E_P\left[\inf_{y \in \mathcal{Y}} G(X, y)\right]$$

and thus

$$\sup_{P \in \mathcal{P}} \inf_{y \in \mathcal{Y}} E_P\left[G(X, y)\right] \geq \sup_{P \in \mathcal{P}} E_P\left[\inf_{y \in \mathcal{Y}} G(X, y)\right]$$

This means that the information that player 2 obtains before making a decision $y \in \mathcal{Y}$ has important consequences, as described next. First, player 1 chooses $P$. If thereafter the decision $x$ is generated according to $P$, and then player 2 gets to observe $x$ and then chooses $y$ (such a player is sometimes called an omniscient adversary), then the outcome of the game is given by

$$\sup_{P \in \mathcal{P}} E_P\left[\inf_{y \in \mathcal{Y}} G(X, y)\right] = \sup_{P \in \mathcal{P}} E_P\left[g_1(X)\right]$$

We know that

$$\sup_{P \in \mathcal{P}} E_P\left[g_1(X)\right] = \sup_{x \in \mathcal{X}} g_1(x) = \sup_{x \in \mathcal{X}} \inf_{y \in \mathcal{Y}} G(x, y)$$

and if $\sup_{x \in \mathcal{X}} g_1(x) < \infty$, then randomization does not help player 1, and player 1 is just as well off making a deterministic decision $x$, and hence we are back at the problem

$$\sup_{x \in \mathcal{X}} \inf_{y \in \mathcal{Y}} G(x, y)$$

However, if player 1 chooses $P$, and thereafter player 2 has to choose $y \in \mathcal{Y}$, knowing player 1's choice of $P$ but not the resulting decision $x$ (such a player is sometimes called an oblivious adversary), then the outcome is given by

$$\sup_{P \in \mathcal{P}} \inf_{y \in \mathcal{Y}} E_P \left[ G(X, y) \right]$$

Assume that $g^* < \infty$. (If $g^* = \infty$, then player 1 can choose a probability distribution $P$ on $\mathcal{X}$ such that $E_P \left[ \inf_{y \in \mathcal{Y}} G(X, y) \right] = \infty$, and thus the expected value of the outcome is $\infty$ for both the case with the omniscient adversary and the case with the oblivious adversary.) In this case player 1 can exploit the property that

$$\sup_{P \in \mathcal{P}} \inf_{y \in \mathcal{Y}} E_P \left[ G(X, y) \right] \quad \geq \quad \sup_{P \in \mathcal{P}} E_P \left[ \inf_{y \in \mathcal{Y}} G(X, y) \right]$$

to make randomized decisions that are better, in expectation, than deterministic decisions. For ease of comparison, suppose that $x^*$ is an optimal deterministic decision for player 1, that is, $G(x^*, y) \geq g^*$ for all $y \in \mathcal{Y}$. For many games, a decision $x^*$ is good for some values of $y$ and not good for other values of $y$. Suppose that player 1 has a set $\{x_1, \ldots, x_n\} \subset \mathcal{X}$ of $n$ optimal deterministic decisions, each of which is good on some subset of $\mathcal{Y}$ and reasonable on the rest of $\mathcal{Y}$. (Obviously, there are many examples of games for which this holds—think about this a little.) Specifically, suppose $\mathcal{Y}$ can be partitioned into $n$ subsets, $\mathcal{Y} = \cup_{i=1}^{n} \mathcal{Y}_i$, with $\mathcal{Y}_i \cap \mathcal{Y}_j = \varnothing$ for all $i \neq j$, such that $G(x_i, y) \geq g^*$ for all $y \in \mathcal{Y}$ and

$$g_i^* \quad \equiv \quad \inf_{y \in \mathcal{Y}_i} G(x_i, y) \quad > \quad g^*$$

(that is, $x_i$ is good on subset $\mathcal{Y}_i$) for each $i \in \{1, \ldots, n\}$. (Note that, in general, for any $x$,

$$\inf_{y \in \mathcal{Y}_i} G(x, y) \quad \geq \quad \inf_{y \in \mathcal{Y}} G(x, y)$$

because $\mathcal{Y}_i \subset \mathcal{Y}$. Thus note that it is very reasonable to assume that one can do the construction described above for a given game.) Now show how to construct a randomized decision $\tilde{P}$ such that

$$\inf_{y \in \mathcal{Y}} E_{\tilde{P}} \left[ G(X, y) \right] \quad > \quad \sup_{x \in \mathcal{X}} \inf_{y \in \mathcal{Y}} G(x, y) \quad = \quad g^*$$

that is, player 1 can choose a randomized decision $\tilde{P}$ that gives a strictly better outcome than can be obtained with deterministic decisions.

Next, relax the assumptions above. Suppose that player 1 has a set $\{x_1, \ldots, x_n\} \subset \mathcal{X}$ of $n$ deterministic decisions that are not necessarily optimal deterministic decisions anymore, but each of which is still good on some subset of $\mathcal{Y}$ and reasonable on the rest of $\mathcal{Y}$. Make these ideas precise, and give sufficient conditions for the construction of a randomized decision $\tilde{P}$ that uses $\{x_1, \ldots, x_n\}$ and that gives a strictly better outcome than can be obtained with deterministic decisions.