ISyE 6761 Stochastic Processes I

Fall 2008

Assignment 6

Issued: November 17, 2008

Due: November 24, 2005

Problem 1
Consider a sequence $\{a_n\} \subset [0, \infty)$, $a \neq 0$ (not all a_n are 0), and a sequence $\{b_n\} \subset \mathbb{R}$. Suppose that

$$\lim_{n\to\infty} \frac{a_n}{\sum_{k=0}^{n} a_k} = 0$$

and

$$\lim_{n\to\infty} b_n = \bar{b}$$

Show that

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n} a_k b_{n-k}}{\sum_{k=0}^{n} a_k} = \bar{b}$$

(This result can be used for later problems.)

Problem 2
Consider a Markov chain with a state i with period

$$d(i) := \gcd\left\{n \geq 1 \; : \; p_{i,i}^{(n)} > 0\right\}$$

Show that

$$d(i) = \gcd\left\{n \geq 1 \; : \; f_{i,i}^{(n)} > 0\right\}$$

(Not surprisingly, this result can also be used for later problems.)

Problem 3
Consider a Markov chain with a communicating class $\mathcal{C}$ with period d. Show that for every $i, j \in \mathcal{C}$ there is $r_{i,j} \in \{0, 1, \ldots, d-1\}$ such that for every n such that $p_{i,j}^{(n)} > 0$, it holds that $n = qd + r_{i,j}$ for some nonnegative integer q (q depends on n, but $r_{i,j}$ does not depend on n). In addition, show that there exists $N_{i,j}$ such that $p_{i,j}^{(nd+r_{i,j})} > 0$ for all $n \geq N_{i,j}$. (By now you know that this result can also be used for later problems.)

Problem 4

Consider a Markov chain with a recurrent state i with period $d(i)$.

1. Let
$$r_{i,i}^{(n)} := \sum_{k=n+1}^{\infty} f_{i,i}^{(k)}$$
(this r is not the same as the r in the previous problem, but it seems the most natural notation). Show that
$$\sum_{k=0}^{n} r_{i,i}^{(k)} p_{i,i}^{(n-k)} = 1$$
for all $n \geq 0$.

2. Let
$$\lambda := \limsup_{n\to\infty} p_{i,i}^{(nd(i))}$$
and let $\{n_k\} \subset \mathbb{N}$ be a subsequence such that
$$\lim_{k\to\infty} p_{i,i}^{(n_k d(i))} = \lambda$$
(As you know, such a subsequence always exists.) Consider any t such that $f_{i,i}^{(t)} > 0$. Show that
$$\lambda \leq f_{i,i}^{(t)} \liminf_{k\to\infty} p_{i,i}^{(n_k d(i)-t)} + \left(1 - f_{i,i}^{(t)}\right)\lambda$$
Show that
$$\lim_{k\to\infty} p_{i,i}^{(n_k d(i)-t)} = \lambda$$

3. Conclude that
$$\lim_{k\to\infty} p_{i,i}^{(n_k d(i)-\tau)} = \lambda$$
for any τ of the form $\tau = \sum_{j=1}^{\ell} c_j t_j$ for positive integers c_j, t_j such that $f_{i,i}^{(t_j)} > 0$.

4. Show that there exists N such that for all $n \geq N$,
$$nd(i) = \sum_{j=1}^{\ell} c_j t_j$$
for positive integers c_j, t_j such that $f_{i,i}^{(t_j)} > 0$

5. Conclude that for all $n \geq N$,
$$\lim_{k\to\infty} p_{i,i}^{((n_k-n)d(i))} = \lambda$$

6. Show that for any $n_k \geq N$,

$$\sum_{j=0}^{n_k-N} r_{i,i}^{(jd(i))} p_{i,i}^{(n_k-N-j)d(i)} = 1$$

7. Show that

$$\sum_{j=0}^{\infty} r_{i,i}^{(jd(i))} = \frac{\mathbb{E}_i[\tau_i(1)]}{d(i)}$$

8. Show that, if i is positive recurrent, then

$$\lambda \sum_{j=0}^{\infty} r_{i,i}^{(jd(i))} = 1$$

9. Show that, if i is null recurrent, then $\lambda = 0$.

10. Conclude that

$$\lambda = \frac{1}{\sum_{j=0}^{\infty} r_{i,i}^{(jd(i))}} = \frac{d(i)}{\mathbb{E}_i[\tau_i(1)]}$$

for any recurrent state i.

11. Observe (you don't have to repeat the details) that the same argument used for $\lambda := \limsup_{n\to\infty} p_{i,i}^{(nd(i))}$ can be used to show that

$$\liminf_{n\to\infty} p_{i,i}^{(nd(i))} = \frac{d(i)}{\mathbb{E}_i[\tau_i(1)]}$$

12. Conclude that

$$\lim_{n\to\infty} p_{i,i}^{(nd(i))} = \frac{d(i)}{\mathbb{E}_i[\tau_i(1)]}$$

13. Conclude that, if state i is null recurrent, then

$$\lim_{n\to\infty} p_{i,i}^{(n)} = 0$$

(Note that the superscript in $p_{i,i}^{(n)}$ is now n and not $nd(i)$.)

14. Consider a Markov chain with a communicating class $\mathcal{C}$ with period d. Show that for any states $i, j \in \mathcal{C}$,

$$\lim_{n\to\infty} p_{i,j}^{(nd+r_{i,j})} = \frac{d}{\mathbb{E}_j[\tau_j(1)]}$$

Problem 5

Consider the following result. Consider a Markov chain with state i with period $d(i)$, and suppose that

$$m_{i,i}^{(2)} \quad := \quad \sum_{k=1}^{\infty} k^2 f_{i,i}^{(k)} \quad < \quad \infty$$

Then

$$\lim_{n\to\infty} \left[\sum_{k=0}^{n} p_{i,i}^{(kd(i))} - \frac{(n+1)d(i)}{\mathbb{E}_i[\tau_i(1)]} \right] \quad = \quad \frac{m_{i,i}^{(2)} - \mathbb{E}_i[\tau_i(1)]d(i)}{2(\mathbb{E}_i[\tau_i(1)])^2}$$

Show that this result implies that

$$\lim_{n\to\infty} p_{i,i}^{(nd(i))} \quad = \quad \frac{d(i)}{\mathbb{E}_i[\tau_i(1)]}$$

(Clearly, it gives even more information, because it gives a rate of convergence.)