ISyE 6761 Stochastic Processes I

Fall 2008

Assignment 5

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Problem 1
Consider a Markov chain with state space $\mathcal{S}$. For any subset $\mathcal{S}' \subset \mathcal{S}$, define the closure $\mathrm{cl}(\mathcal{S}')$ to be the smallest closed set that contains $\mathcal{S}'$.

1. Prove that for any $j \in \mathcal{S}$, $\mathrm{cl}(\{j\}) = \{k \in \mathcal{S} : j \to k\}$.
   Answer: First we show that $\mathrm{cl}(\{j\}) \subset \{k \in \mathcal{S} : j \to k\}$ and then we show that $\mathrm{cl}(\{j\}) \supset \{k \in \mathcal{S} : j \to k\}$. First, show that $\{k \in \mathcal{S} : j \to k\}$ is a closed set. Consider any $i \in \{k \in \mathcal{S} : j \to k\}$ and $l \notin \{k \in \mathcal{S} : j \to k\}$. If $i \to l$, then $j \to i$ and $i \to l$ imply that $j \to l$, which contradicts $l \notin \{k \in \mathcal{S} : j \to k\}$. Thus $i \not\to l$. That is, $\{k \in \mathcal{S} : j \to k\}$ is a closed set. Next, note that $j \to j$, and thus $\{j\} \subset \{k \in \mathcal{S} : j \to k\}$. Since $\mathrm{cl}(\{j\})$ is by definition the smallest closed set that contains $\{j\}$, and $\{k \in \mathcal{S} : j \to k\}$ is a closed set that contains $\{j\}$, it follows that $\mathrm{cl}(\{j\}) \subset \{k \in \mathcal{S} : j \to k\}$. Next, consider any $i \in \{k \in \mathcal{S} : j \to k\}$ and any set $\mathcal{S}'' \subset \mathcal{S}$ that contains $\{j\}$. If $i \notin \mathcal{S}''$, then $\mathcal{S}''$ is not closed. That is, i is in every closed set that contains $\{j\}$. Therefore i is in the smallest closed set that contains $\{j\}$, that is, $i \in \mathrm{cl}(\{j\})$. Hence $\mathrm{cl}(\{j\}) \supset \{k \in \mathcal{S} : j \to k\}$. Thus we have shown that $\mathrm{cl}(\{j\}) = \{k \in \mathcal{S} : j \to k\}$.

2. For the deterministically monotone Markov chain, what is $\mathrm{cl}(\{j\})$?
   Answer: $\{j, j+1, j+2, \ldots\}$

3. If j is recurrent, show that $\mathrm{cl}(\{j\})$ is the communicating class that contains j.
   Answer: Let $\mathcal{C}(j)$ denote the communicating class that contains j. We want to show that $\mathrm{cl}(\{j\}) = \mathcal{C}(j)$. We have already shown that $\mathrm{cl}(\{j\}) = \{k \in \mathcal{S} : j \to k\}$. Thus it is sufficient to show that $\mathcal{C}(j) = \{k \in \mathcal{S} : j \to k\}$. First we show that $\mathcal{C}(j) \supset \{k \in \mathcal{S} : j \to k\}$ and then we show that $\mathcal{C}(j) \subset \{k \in \mathcal{S} : j \to k\}$.

   It was shown in Proposition 2.10.1 that if j is recurrent and $j \to k$, then $j \leftrightarrow k$, that is, j and k are in the same (recurrent) communicating class, thus, $k \in \mathcal{C}(j)$. Hence $\mathcal{C}(j) \supset \{k \in \mathcal{S} : j \to k\}$. Next, consider any $i \in \mathcal{C}(j)$, that is, $j \leftrightarrow i$. Then $j \to i$, thus $i \in \{k \in \mathcal{S} : j \to k\}$. Hence $\mathcal{C}(j) \subset \{k \in \mathcal{S} : j \to k\}$. Therefore we have shown that $\mathcal{C}(j) = \{k \in \mathcal{S} : j \to k\}$.

Problem 2

Consider a Markov chain with state space $\mathcal{S}$ that is not necessarily finite, and transition matrix P. Suppose that $P^2 = P$.

1. Prove that the Markov chain is aperiodic.
   Answer: Consider any state i. We want to show that the period $d(i)$ of state i equals 1. It follows from $P^2 = P$ that $P^n = P$ for all $n \geq 1$. If $p_{i,i} = 0$, then $p_{i,i}^{(n)} = (P^n)_{i,i} = P_{i,i} = p_{i,i} = 0$ for all $n \geq 1$, and thus $\{n \geq 1 : p_{i,i}^{(n)} > 0\} = \varnothing$, and hence by definition $d(i) = 1$. Otherwise, if $p_{i,i} > 0$, then $d(i) = 1$.

2. Prove that $p_{i,j} = p_{j,j}$ for all $i, j \in \mathcal{S}$ such that $i \leftrightarrow j$.
   Answer: First we show that $p_{i,j} \leq p_{j,j}$ for all $i, j \in \mathcal{S}$. This result does not require that $i \leftrightarrow j$. For any $n \geq 1$,

$$
\begin{aligned}
p_{i,j} &= P_{i,j} \;=\; (P^n)_{i,j} \;=\; p_{i,j}^{(n)} \\
&= \sum_{k=1}^{n} f_{i,j}^{(k)} p_{j,j}^{(n-k)} \\
&= \sum_{k=1}^{n-1} f_{i,j}^{(k)} p_{j,j} + f_{i,j}^{(n)} 1 \\
&= P_i[\tau_j(1) \leq n-1] p_{j,j} + P_i[\tau_j(1) = n] \\
&\leq p_{j,j} + P_i[\tau_j(1) = n]
\end{aligned}
$$

   Note that $\sum_{n=1}^{\infty} P_i[\tau_j(1) = n] = P_i[\tau_j(1) < \infty] \leq 1$, and thus $P_i[\tau_j(1) = n] \to 0$ as $n \to \infty$, that is, for any $\varepsilon > 0$, there is N such that $P_i[\tau_j(1) = n] < \varepsilon$ for all $n \geq N$. Thus, for all $n \geq N$,

$$p_{i,j} \;<\; p_{j,j} + \varepsilon$$

   The inequality above holds for all $\varepsilon > 0$, and thus $p_{i,j} \leq p_{j,j}$ for all $i, j \in \mathcal{S}$.

   Next we show by contradiction that if $i \leftrightarrow j$, then $p_{i,j} \not< p_{j,j}$. Because $i \leftrightarrow j$, there is n such that $p_{j,i}^{(n)} > 0$. But $p_{j,i}^{(n)} = (P^n)_{j,i} = P_{j,i} = p_{j,i}$, and thus $p_{j,i} > 0$. Suppose that $p_{i,j} < p_{j,j}$. Then

$$
\begin{aligned}
p_{j,j} &= P_{j,j} \;=\; (P^2)_{j,j} \;=\; p_{j,j}^{(2)} \\
&= \sum_{k \in \mathcal{S}} p_{j,k} p_{k,j} \\
&= p_{j,i} p_{i,j} + \sum_{k \in \mathcal{S} \setminus \{i\}} p_{j,k} p_{k,j} \\
&< p_{j,i} p_{j,j} + \sum_{k \in \mathcal{S} \setminus \{i\}} p_{j,k} p_{j,j} \\
&= \sum_{k \in \mathcal{S}} p_{j,k} p_{j,j} \\
&= p_{j,j}
\end{aligned}
$$

   which is a contradiction. Thus $p_{i,j} \not< p_{j,j}$, and therefore $p_{i,j} = p_{j,j}$ for all $i, j \in \mathcal{S}$ such that $i \leftrightarrow j$.

3. Write a stationary distribution π in terms of P.

   Answer: First we show that for any state i and any transient state j, $p_{i,j} = 0$. It follows from Proposition 2.6.3 that $\sum_{n=1}^{\infty} p_{ij}^{(n)} < \infty$. Thus $\sum_{n=1}^{\infty} p_{ij} < \infty$. This implies that $p_{i,j} = 0$. Next, it follows from $\sum_{k\in\mathcal{S}} p_{i,k} = 1$ that $p_{i,k} > 0$ for some $k \in \mathcal{S}$, and thus there are some recurrent states $k \in \mathcal{S}$. Consider any recurrent state $k \in \mathcal{S}$. Let $\mathcal{C}(k)$ denote the (recurrent) communicating class that contains k. Note that $\sum_{i\in\mathcal{C}(k)} p_{k,i} = 1$ ($\mathcal{C}(k)$ is closed), and it follows from the previous part that for all $i, l \in \mathcal{C}(k)$, $p_{i,l} = p_{l,l} = p_{k,l}$. Consider the vector π given by the row of P corresponding to state k. Clearly $\sum_{i\in\mathcal{S}} \pi_i = \sum_{i\in\mathcal{S}} p_{k,i} = 1$. Also,

$$\begin{aligned}
(\pi P)_l &= \sum_{i\in\mathcal{S}} \pi_i P_{i,l} \\
&= \sum_{i\in\mathcal{S}} p_{k,i} p_{i,l} \\
&= \sum_{i\in\mathcal{C}(k)} p_{k,i} p_{i,l} \\
&= \sum_{i\in\mathcal{C}(k)} p_{k,i} p_{k,l} \\
&= p_{k,l} \;=\; \pi_l
\end{aligned}$$

   Therefore π is a stationary distribution of the Markov chain.

Problem 3

Show that a simple random walk with state space $\mathcal{S} = \{0, 1, 2, \ldots\}$ and transition probabilities

$$P_{0,1} = 1, \quad P_{n,n+1} = p, \quad P_{n,n-1} = q = 1 - p \quad \forall\, n \geq 1$$

is positive recurrent if $0 < p < q$.

Answer: We show that the Markov chain is irreducible and has a (unique) stationary distribution. Then it follows from Theorem 2.13.2 that the Markov chain is positive recurrent. First note that $0 < p < q$ implies that the Markov chain is irreducible. Next, we write down the balance equations:

$$\begin{aligned}
\pi_0 &= q\pi_1 \\
\pi_1 &= \pi_0 + q\pi_2 \\
\pi_2 &= p\pi_1 + q\pi_3 \\
\pi_n &= p\pi_{n-1} + q\pi_{n+1} \quad \forall\, n \geq 2 \\
\Rightarrow \quad \pi_1 &= \frac{1}{q}\,\pi_0 \\
\pi_2 &= \frac{p}{q^2}\,\pi_0 \\
\pi_3 &= \frac{p^2}{q^3}\,\pi_0 \\
\pi_n &= \frac{p^{n-1}}{q^n}\,\pi_0 \quad \forall\, n \geq 1
\end{aligned}$$

Also

$$\begin{aligned}
\sum_{n=0}^{\infty} \pi_n &= 1 \\
\Rightarrow \quad \pi_0 + \sum_{n=1}^{\infty} \frac{p^{n-1}}{q^n}\, \pi_0 &= 1 \\
\Rightarrow \quad \pi_0 + \frac{1/q}{1-p/q}\, \pi_0 &= 1 \quad \text{if } p/q < 1 \\
\Rightarrow \quad \pi_0 &= \frac{1}{1+\frac{1/q}{1-p/q}} \quad = \quad \frac{q-p}{2q} \\
\Rightarrow \quad \pi_n &= \frac{p^{n-1}}{q^n}\, \frac{q-p}{2q} \quad \forall\ n \geq 1
\end{aligned}$$