ISyE 6761 Stochastic Processes I

Fall 2008

Assignment 4

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Problem 1

Consider a game where a coin is tossed independently again and again. Every time the coin turns up heads, which happens with probability $p \in (0,1)$, the player wins a dollar. Whenever the coin turns up tails, the player loses all his earnings to that point. Let $X_n(\omega)$ denote the player's accumulated earnings after the nth toss.

1. Show that $X : \Omega \mapsto \{0, 1, 2, \ldots\}^\infty$ is a Markov chain, and write down its transition probabilities. **Answer:** Consider any history $(i_0, i_1, \ldots, i_n)$. Then

$$\begin{aligned} \mathbb{P}[X_{n+1} = 0 | X_0 = i_0, X_1 = i_1, \ldots, X_n = i_n] &= \mathbb{P}[\text{coin turns up tails on throw } n+1] \\ &= 1 - p \quad = \quad \mathbb{P}[X_{n+1} = 0 | X_n = i_n] \end{aligned}$$

(in the case with $X_{n+1} = 0$, the transition probability does not even depend on the current state i_n.) Also,

$$\begin{aligned} \mathbb{P}[X_{n+1} = i_n + 1 | X_0 = i_0, X_1 = i_1, \ldots, X_n = i_n] &= \mathbb{P}[\text{coin turns up heads on throw } n+1] \\ &= p \quad = \quad \mathbb{P}[X_{n+1} = i_n + 1 | X_n = i_n] \end{aligned}$$

Also, for all $j \notin \{0, i_n + 1\}$,

$$\mathbb{P}[X_{n+1} = j | X_0 = i_0, X_1 = i_1, \ldots, X_n = i_n] \quad = \quad 0 \quad = \quad \mathbb{P}[X_{n+1} = j | X_n = i_n]$$

Transition matrix

$$P = \begin{bmatrix} 1-p & p & 0 & 0 & \cdots \\ 1-p & 0 & p & 0 & \cdots \\ 1-p & 0 & 0 & p & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

2. Show that the Markov chain is irreducible. **Answer:** Consider any two states $i, j \in \{0, 1, 2, \ldots\}$. We have to show that there is $n \in \{0, 1, 2, \ldots\}$ such that $p_{i,j}^{(n)} > 0$. If $i = j$, then $p_{i,j}^{(0)} = 1 > 0$. If $i < j$, then $p_{i,j}^{(j-i)} = p^{j-i} > 0$. If $i > j$, then $p_{i,j}^{(j+1)} = (1-p)p^j > 0$.

3. Calculate the expected hitting time $\mathbb{E}_i[\tau_i(1)]$ for each i. **Answer:** Note that to go from state i to state i in one or more steps, the process has to go from state i to state 0 and then from state 0 to state i. Thus

$$\mathbb{E}_i[\tau_i(1)] = \mathbb{E}_i[\tau_0(1)] + \mathbb{E}_0[\tau_i(1)]$$

Starting in state i, $\tau_0(1)$ is geometrically distributed with mean $1/(1-p)$, that is,

$$\mathbb{E}_i[\tau_0(1)] = \frac{1}{1-p}$$

Starting in state 0, let $L_n := \tau_0(n) - \tau_0(n-1)$, $n = 1, 2, \ldots$, denote the length of excursion n of the process until it reaches state 0 again. Note that $\{L_n\}$ is an iid sequence, and that the expected length of an excursion until it returns to state 0 is

$$\mathbb{E}[L_n] = \mathbb{E}_0[\tau_0(1)] = \frac{1}{1-p}$$

Let $N := \inf\{n : L_n > i\}$ denote the number of the first excursion from state 0 to reach state i. Note that N is a stopping time with respect to $\{L_n\}$. The probability that an excursion reaches state i before it returns to state 0 is p^i, and thus $\mathbb{E}[N] = 1/p^i$. It follows from Wald's Identity that

$$\mathbb{E}\left[\sum_{n=1}^{N} L_n\right] = \mathbb{E}[N]\mathbb{E}[L_n] = \frac{1}{(1-p)p^i}$$

Also

$$\begin{aligned} \mathbb{E}\left[\sum_{n=1}^{N} L_n\right] &= \mathbb{E}_0[\tau_i(1)] + \mathbb{E}_i[\tau_0(1)] \\ \Rightarrow \quad \mathbb{E}_i[\tau_i(1)] &= \mathbb{E}_i[\tau_0(1)] + \mathbb{E}_0[\tau_i(1)] \\ &= \mathbb{E}\left[\sum_{n=1}^{N} L_n\right] = \frac{1}{(1-p)p^i} \end{aligned}$$

Alternative approach to determine $\mathbb{E}\left[\sum_{n=1}^{N} L_n\right]$ (implicitly it still uses the fact that N is a stopping time with respect to $\{L_n\}$): Starting in state 0, call an excursion a successful excursion if it reaches state i, which happens with probability p^i, and an unsuccessful excursion otherwise, which happens with probability $1 - p^i$. Note that

$$\begin{aligned} \mathbb{E}[L_n] &= \frac{1}{1-p} \\ \Rightarrow \quad \mathbb{P}[\text{successful excursion}]\mathbb{E}[L_n|\text{successful excursion}] & \\ +\mathbb{P}[\text{unsuccessful excursion}]\mathbb{E}[L_n|\text{unsuccessful excursion}] &= \frac{1}{1-p} \\ \Rightarrow \quad p^i\left(i + \frac{1}{1-p}\right) + \left(1 - p^i\right)\mathbb{E}[L_n|\text{unsuccessful excursion}] &= \frac{1}{1-p} \end{aligned}$$

$$\begin{aligned}
\Rightarrow \quad \mathbb{E}[L_n|\text{unsuccessful excursion}] &= \frac{\frac{1}{1-p} - p^i\left(i + \frac{1}{1-p}\right)}{1-p^i} \\
&= \frac{1 - p^i\left(i(1-p)+1\right)}{(1-p)(1-p^i)}
\end{aligned}$$

Thus, by conditioning on whether the first excursion is a success or not,

$$\begin{aligned}
\mathbb{E}\left[\sum_{n=1}^{N} L_n\right] &= \mathbb{P}[\text{first excursion successful}]\mathbb{E}\left[\sum_{n=1}^{N} L_n|\text{first excursion successful}\right] \\
&\quad + \mathbb{P}[\text{first excursion unsuccessful}]\mathbb{E}\left[\sum_{n=1}^{N} L_n|\text{first excursion unsuccessful}\right] \\
&= p^i\mathbb{E}[L_n|\text{successful excursion}] \\
&\quad + \left(1-p^i\right)\left(\mathbb{E}[L_n|\text{unsuccessful excursion}] + \mathbb{E}\left[\sum_{n=1}^{N} L_n\right]\right) \\
&= p^i\left(i + \frac{1}{1-p}\right) + \left(1-p^i\right)\left(\frac{1-p^i\left(i(1-p)+1\right)}{(1-p)(1-p^i)} + \mathbb{E}\left[\sum_{n=1}^{N} L_n\right]\right) \\
\Rightarrow \quad p^i\mathbb{E}\left[\sum_{n=1}^{N} L_n\right] &= p^i\left(i + \frac{1}{1-p}\right) + \frac{1-p^i\left(i(1-p)+1\right)}{1-p} \quad = \quad \frac{1}{1-p} \\
\Rightarrow \quad \mathbb{E}\left[\sum_{n=1}^{N} L_n\right] &= \frac{1}{(1-p)p^i}
\end{aligned}$$

4. Classify the Markov chain. **Answer:** The Markov chain is irreducible (one class) and positive recurrent.

Problem 2

Show that in an irreducible discrete time Markov chain with N states, it is possible to go from any state to any other state in N steps or less. **Answer:** Consider any states $i, j \in \mathcal{S}$. The Markov chain being irreducible implies that there is n such that $p_{ij}^{(n)} > 0$. Note that

$$\begin{aligned}
p_{ij}^{(n)} &:= \mathbb{P}[X_n = j|X_0 = i] \\
&= \frac{\mathbb{P}[X_0 = i, X_n = j]}{\mathbb{P}[X_0 = i]} \\
&= \sum_{k_1,\ldots,k_{n-1}\in\mathcal{S}} \frac{\mathbb{P}[X_0 = i, X_1 = k_1, \ldots, X_{n-1} = k_{n-1}, X_n = j]}{\mathbb{P}[X_0 = i]} \quad > \quad 0
\end{aligned}$$

Thus there is a sequence of states $i = k_0, k_1, \ldots, k_{n-1}, k_n = j \in \mathcal{S}$ such that

$$\mathbb{P}[X_0 = i, X_1 = k_1, \ldots, X_{n-1} = k_{n-1}, X_n = j] = \mathbb{P}[X_0 = i]p_{k_0,k_1}p_{k_1,k_2}\cdots p_{k_{n-2},k_{n-1}}p_{k_{n-1},k_n} \quad > \quad 0$$

Suppose any state repeats in this sequence, say $k_l = k_m$ with $l < m$. Then all the states between k_l and k_m can be eliminated from the sequence:

$$p_{k_0,k_1} \cdots p_{k_{l-1},k_l} p_{k_m,k_{m+1}} \cdots p_{k_{n-1},k_n} \quad \geq \quad p_{k_0,k_1} p_{k_1,k_2} \cdots p_{k_{n-2},k_{n-1}} p_{k_{n-1},k_n} \quad > \quad 0$$

By induction, all duplicated states can be eliminated from the sequence. Because the Markov chain has N states, the resulting sequence has N or fewer states. The Markov chain can go from state i to state j along the resulting sequence of states in N steps or less.

Problem 3

For a discrete time Markov chain $X : \Omega \mapsto \mathcal{S}^\infty$, use only basic identities in probability and the Markov property

$$\mathbb{P}[X_{n+1} = j | X_0 = i_0, X_1 = i_1, \ldots, X_n = i_n] \quad = \quad \mathbb{P}[X_{n+1} = j | X_n = i_n] \quad = \quad p_{i_n,j}$$

for all histories $(i_0, i_1, \ldots, i_n)$ and all j, to prove that

$$\mathbb{P}[X_n = j | X_{n_1} = i_1, \ldots, X_{n_k} = i_k] \quad = \quad \mathbb{P}[X_n = j | X_{n_k} = i_k]$$

for all states $i_1, \ldots, i_k$ and all j, whenever $n_1 < n_2 < \cdots < n_k < n$. **Answer:** First show that

$$\begin{aligned} & \mathbb{P}[X_{n+1} = k_1, \ldots, X_{n+m} = k_m | X_0 = i_0, \ldots, X_n = i_{n-1}, X_n = i] \\ = \quad & \mathbb{P}[X_{n+1} = k_1, \ldots, X_{n+m} = k_m | X_n = i] \end{aligned}$$

for all states $i_0, \ldots, i_{n-1}, i, k_1, \ldots, k_m$.

$$\begin{aligned} & \mathbb{P}[X_{n+1} = k_1, \ldots, X_{n+m} = k_m | X_0 = i_0, \ldots, X_{n-1} = i_{n-1}, X_n = i] \\ = \quad & \mathbb{P}[X_{n+1} = k_1 | X_0 = i_0, \ldots, X_{n-1} = i_{n-1}, X_n = i] \\ & \times \mathbb{P}[X_{n+2} = k_2 | X_0 = i_0, \ldots, X_{n-1} = i_{n-1}, X_n = i, X_{n+1} = k_1] \\ & \times \cdots \times \mathbb{P}[X_{n+m} = k_m | X_0 = i_0, \ldots, X_{n-1} = i_{n-1}, X_n = i, X_{n+1} = k_1, \ldots, X_{n+m-1} = k_{m-1}] \\ = \quad & p_{i,k_1} p_{k_1,k_2} \cdots p_{k_{m-1},k_m} \end{aligned}$$

$$\begin{aligned} & \mathbb{P}[X_{n+1} = k_1, \ldots, X_{n+m} = k_m | X_n = i] \\ = \quad & \frac{\mathbb{P}[X_n = i, X_{n+1} = k_1, \ldots, X_{n+m} = k_m]}{\mathbb{P}[X_n = i]} \\ = \quad & \sum_{i_0,\ldots,i_{n-1} \in \mathcal{S}} \frac{\mathbb{P}[X_0 = i_0, \ldots, X_{n-1} = i_{n-1}, X_n = i, X_{n+1} = k_1, \ldots, X_{n+m} = k_m]}{\mathbb{P}[X_n = i]} \\ = \quad & \sum_{i_0,\ldots,i_{n-1} \in \mathcal{S}} \mathbb{P}[X_0 = i_0, \ldots, X_{n-1} = i_{n-1}, X_n = i] \mathbb{P}[X_{n+1} = k_1 | X_0 = i_0, \ldots, X_{n-1} = i_{n-1}, X_n = i] \\ & \quad \frac{\times \cdots \times \mathbb{P}[X_{n+m} = k_m | X_0 = i_0, \ldots, X_{n-1} = i_{n-1}, X_n = i, X_{n+1} = k_1, \ldots, X_{n+m-1} = k_{m-1}]}{\mathbb{P}[X_n = i]} \\ = \quad & \sum_{i_0,\ldots,i_{n-1} \in \mathcal{S}} \frac{\mathbb{P}[X_0 = i_0, \ldots, X_{n-1} = i_{n-1}, X_n = i]}{\mathbb{P}[X_n = i]} p_{i,k_1} p_{k_1,k_2} \cdots p_{k_{m-1},k_m} \\ = \quad & \frac{\mathbb{P}[X_n = i]}{\mathbb{P}[X_n = i]} p_{i,k_1} p_{k_1,k_2} \cdots p_{k_{m-1},k_m} \\ = \quad & p_{i,k_1} p_{k_1,k_2} \cdots p_{k_{m-1},k_m} \end{aligned}$$

Next consider the result to be shown.

$$
\begin{aligned}
& \mathbb{P}[X_n = j | X_{n_1} = i_1, X_{n_2} = i_2, \ldots, X_{n_k} = i_k] \\
= \; & \frac{\mathbb{P}[X_n = j, X_{n_1} = i_1, X_{n_2} = i_2, \ldots, X_{n_k} = i_k]}{\mathbb{P}[X_{n_1} = i_1, X_{n_2} = i_2, \ldots, X_{n_k} = i_k]} \\
= \; & \sum_{j_0^{(1)},\ldots,j_0^{(n_1)},j_1^{(1)},\ldots,j_1^{(n_2-n_1-1)},j_2^{(1)},\ldots,j_2^{(n_3-n_2-1)},\ldots,j_k^{(1)},\ldots,j_k^{(n-n_k-1)} \in \mathcal{S}} \\
& \frac{\begin{array}{c}\mathbb{P}[X_0 = j_0^{(1)}, \ldots, X_{n_1-1} = j_0^{(n_1)}, X_{n_1} = i_1, X_{n_1+1} = j_1^{(1)}, \ldots, X_{n_2-1} = j_1^{(n_2-n_1-1)}, X_{n_2} = i_2, \\ X_{n_2+1} = j_2^{(1)}, \ldots, X_{n_3-1} = j_2^{(n_3-n_2-1)}, \ldots, X_{n_k} = i_k, X_{n_k+1} = j_k^{(1)}, \ldots, X_{n-1} = j_k^{(n-n_k-1)}, X_n = j]\end{array}}{\mathbb{P}[X_{n_1} = i_1, X_{n_2} = i_2, \ldots, X_{n_k} = i_k]} \\
= \; & \sum_{j_0^{(1)},\ldots,j_k^{(n-n_k-1)} \in \mathcal{S}} \mathbb{P}[X_0 = j_0^{(1)}, \ldots, X_{n_k} = i_k] \\
& \qquad \frac{\times \mathbb{P}[X_{n_k+1} = j_k^{(1)}, \ldots, X_{n-1} = j_k^{(n-n_k-1)}, X_n = j | X_0 = j_0^{(1)}, \ldots, X_{n_k} = i_k]}{\mathbb{P}[X_{n_1} = i_1, X_{n_2} = i_2, \ldots, X_{n_k} = i_k]} \\
= \; & \sum_{j_0^{(1)},\ldots,j_k^{(n-n_k-1)} \in \mathcal{S}} \mathbb{P}[X_0 = j_0^{(1)}, \ldots, X_{n_k} = i_k] \\
& \qquad \frac{\times \mathbb{P}[X_{n_k+1} = j_k^{(1)}, \ldots, X_{n-1} = j_k^{(n-n_k-1)}, X_n = j | X_{n_k} = i_k]}{\mathbb{P}[X_{n_1} = i_1, X_{n_2} = i_2, \ldots, X_{n_k} = i_k]} \quad \text{(from first part of proof)} \\
= \; & \sum_{j_0^{(1)},\ldots,j_{k-1}^{(n_k-n_{k-1}-1)} \in \mathcal{S}} \left(\frac{\mathbb{P}[X_0 = j_0^{(1)}, \ldots, X_{n_k} = i_k]}{\mathbb{P}[X_{n_1} = i_1, X_{n_2} = i_2, \ldots, X_{n_k} = i_k]} \right. \\
& \qquad \left. \times \sum_{j_k^{(1)},\ldots,j_k^{(n-n_k-1)} \in \mathcal{S}} \frac{\mathbb{P}[X_{n_k} = i_k, X_{n_k+1} = j_k^{(1)}, \ldots, X_{n-1} = j_k^{(n-n_k-1)}, X_n = j]}{\mathbb{P}[X_{n_k} = i_k]} \right) \\
= \; & \left(\sum_{j_0^{(1)},\ldots,j_{k-1}^{(n_k-n_{k-1}-1)} \in \mathcal{S}} \frac{\mathbb{P}[X_0 = j_0^{(1)}, \ldots, X_{n_k} = i_k]}{\mathbb{P}[X_{n_1} = i_1, X_{n_2} = i_2, \ldots, X_{n_k} = i_k]} \right) \\
& \qquad \times \left(\sum_{j_k^{(1)},\ldots,j_k^{(n-n_k-1)} \in \mathcal{S}} \frac{\mathbb{P}[X_{n_k} = i_k, X_{n_k+1} = j_k^{(1)}, \ldots, X_{n-1} = j_k^{(n-n_k-1)}, X_n = j]}{\mathbb{P}[X_{n_k} = i_k]} \right) \\
& \qquad \text{(note that the second sum does not depend on the variables in the first sum)} \\
= \; & \frac{\mathbb{P}[X_{n_1} = i_1, X_{n_2} = i_2, \ldots, X_{n_k} = i_k]}{\mathbb{P}[X_{n_1} = i_1, X_{n_2} = i_2, \ldots, X_{n_k} = i_k]} \times \frac{\mathbb{P}[X_{n_k} = i_k, X_n = j]}{\mathbb{P}[X_{n_k} = i_k]} \\
= \; & \mathbb{P}[X_n = j | X_{n_k} = i_k]
\end{aligned}
$$

Problem 4

Suppose that $X : \Omega \mapsto \mathcal{S}^\infty$ is a discrete time Markov chain with a countable state space $\mathcal{S}$.

Let $f : \mathcal{S} \mapsto \mathcal{S}$ be an arbitrary function. Let $Y : \Omega \mapsto \mathcal{S}^\infty$ be given by $Y_n(\omega) := f(X_n(\omega))$. Is Y a Markov chain? Prove or give a counterexample. If you give a counterexample, also give a sufficient condition on f for Y to be a Markov chain. **Answer:** No, not necessarily. Counterexample: Let X be a simple random walk on the integers with $p = q = 1/2$ and $X_0 = 0$. Let $f(x) = [x]^+ := \max\{x, 0\}$. Then

$$\begin{aligned}
\mathbb{P}[f(X_2) = 0, f(X_3) = 0] &= \mathbb{P}[X_0 = 0, X_1 = -1, X_2 = -2, X_3 = -3] \\
&\quad + \mathbb{P}[X_0 = 0, X_1 = -1, X_2 = -2, X_3 = -1] \\
&\quad + \mathbb{P}[X_0 = 0, X_1 = -1, X_2 = 0, X_3 = -1] \\
&\quad + \mathbb{P}[X_0 = 0, X_1 = 1, X_2 = 0, X_3 = -1] \\
&= 4\left(\frac{1}{2}\right)^3 = \frac{1}{2}
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{P}[f(X_2) = 0] &= \mathbb{P}[X_0 = 0, X_1 = -1, X_2 = -2] \\
&\quad + \mathbb{P}[X_0 = 0, X_1 = -1, X_2 = 0] \\
&\quad + \mathbb{P}[X_0 = 0, X_1 = 1, X_2 = 0] \\
&= 3\left(\frac{1}{2}\right)^2 = \frac{3}{4}
\end{aligned}$$

thus

$$\mathbb{P}[f(X_3) = 0 | f(X_2) = 0] = \frac{\mathbb{P}[f(X_2) = 0, f(X_3) = 0]}{\mathbb{P}[f(X_2) = 0]} = \frac{1/2}{3/4} = \frac{2}{3}$$

However,

$$\begin{aligned}
\mathbb{P}[f(X_3) = 0 | f(X_0) = 0, f(X_1) = 1, f(X_2) = 0] &= \mathbb{P}[f(X_3) = 0 | X_0 = 0, X_1 = 1, X_2 = 0] \\
&= \mathbb{P}[X_3 = -1 | X_0 = 0, X_1 = 1, X_2 = 0] \\
&= \mathbb{P}[X_3 = -1 | X_2 = 0] = \frac{1}{2}
\end{aligned}$$

Thus

$$\mathbb{P}[f(X_3) = 0 | f(X_0) = 0, f(X_1) = 1, f(X_2) = 0] \neq \mathbb{P}[f(X_3) = 0 | f(X_2) = 0]$$

If f is a one-to-one function, then Y is a Markov chain.