ISyE 6761 Stochastic Processes I

Fall 2008

Assignment 3

Issued: September 15, 2008

Due: September 22, 2008

Problem 1
Let X and Y be nonnegative integer valued random variables on the same probability space. For $s, t \in [-1, 1]$, define the joint generating function $P_{X,Y}$ as follows:

$$P_{X,Y}(s,t) \quad := \quad \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} s^i t^j P[X = i, Y = j]$$

The marginal generating functions are given by

$$P_X(s) \quad := \quad \sum_{i=0}^{\infty} s^i P[X = i]$$
$$P_Y(t) \quad := \quad \sum_{j=0}^{\infty} t^j P[Y = j]$$

1. Prove that X and Y are independent iff

   $$P_{X,Y}(s,t) \quad = \quad P_X(s) P_Y(t)$$

   for all $s, t \in [-1, 1]$.

2. Let

   $$P_{X+Y}(s) \quad := \quad \sum_{i=0}^{\infty} s^i P[X + Y = i]$$

   denote the generating function of $X + Y$.

   (a) Note that $P_{X+Y}(s) = P_{X,Y}(s,s)$. Thus, if X and Y are independent, then $P_{X+Y}(s) = P_{X,Y}(s,s) = P_X(s)P_Y(s)$ for all $s \in [-1, 1]$, that is, the generating function of $X + Y$ is equal to the product of the marginal generating functions of X and Y.

   (b) Give an example of jointly distributed nonnegative integer valued random variables X and Y on the same probability space, which are not independent, but for which

   $$P_{X,Y}(s,s) \quad = \quad P_X(s)P_Y(s)$$

   for all $s \in [-1, 1]$. That is, even if $P_{X,Y}(s,s) = P_X(s)P_Y(s)$ for all $s \in [-1, 1]$, it does not imply that X and Y are independent.

Problem 2

A Skip Free Negative Random Walk. Consider an iid sequence $\{X_n\}$ of random variables with values in $\{-1, 0, 1, 2, \ldots\}$. For $n \geq 1$ and $j \geq 0$, let $p_j := P[X_n = j - 1]$, where $\sum_{j=0}^{\infty} p_j = 1$. For $s \in [-1, 1]$, let $f(s) := \sum_{j=0}^{\infty} p_j s^j$. Let $S_0 = X_0 = 1$, and for $n \geq 1$ let $S_n := X_0 + X_1 + \cdots + X_n$. Let $N := \inf\{n : S_n = 0\}$. For $s \in [-1, 1]$, let $P(s) := E[s^N]$.

1. Show that $P(s) = sf(P(s))$.
2. Consider the geometric distribution with $f(s) = p/(1 - qs)$. Find the smallest solution $P(s)$ of $P(s) = sf(P(s))$.

Problem 3

Consider a binomial replacement branching model with $P(s) = q + ps$. Let $T := \inf\{n : Z_n = 0\}$.

1. Suppose that $Z_0 = 1$. Find $P[T = n]$ for $n \geq 1$.
2. Suppose that $Z_0 = i > 0$. Find $P[T = n]$ for $n \geq 1$.

Problem 4

A Point Process. Let $N(A)$ denote the number of points in a region A. Suppose that for each n, the region A can be partitioned into disjoint subsets $A_j^{(n)}, j = 1, \ldots, n$, with $A = \cup_{j=1}^{n} A_j^{(n)}$, $N(A) = \sum_{j=1}^{n} N(A_j^{(n)})$, $N(A_1^{(n)}), \ldots, N(A_n^{(n)})$ are independent, and

$$P[N(A_j^{(n)}) = 0] = \exp(-\lambda/n), \qquad P[N(A_j^{(n)}) \geq 2] \leq \frac{\lambda}{n}\delta\left(\frac{\lambda}{n}\right)$$

where $\delta : [0, \infty) \mapsto [0, \infty)$ such that $\delta(x) \downarrow 0$ as $x \downarrow 0$. Show that $N(A)$ has a Poisson distribution.