ISyE 6761 Stochastic Processes I

Fall 2008

Assignment 2

Issued: August 29, 2008

Due: September 5, 2008

Problem 1

1. For $n = 1, 2, \ldots$, let a_n denote the number of ways to obtain a total of n by successive throws of a regular 6-sided die if the order in which the numbers appear on the die does not matter. For example, $a_1 = 1$ (one throw of 1), $a_2 = 2$ (one throw of 2 or 2 successive throws of 1 each), $a_3 = 3$ (one throw of 3 or one throw of 1 and one throw of 2, which is regarded as the same as one throw of 2 and one throw of 1, or 3 successive throws of 1 each). Find the generating function of $\{a_n\}$.

2. For $n = 1, 2, \ldots$, let c_n denote the number of ways to obtain a total of n by successive throws of a regular 6-sided die if the order in which the numbers appear on the die does matter. For example, $c_1 = 1$ (one throw of 1), $c_2 = 2$ (one throw of 2 or 2 successive throws of 1 each), $c_3 = 4$ (one throw of 3 or one throw of 1 followed by one throw of 2 or one throw of 2 followed by one throw of 1 or 3 successive throws of 1 each). Find the generating function of $\{c_n\}$.

Problem 2

Let $\{X_n\}$ be i.i.d. Bernoulli random variables with

$$\mathbb{P}[X_1 = 1] \quad = \quad p \quad = \quad 1 - \mathbb{P}[X_1 = 0] \quad = \quad 1 - q$$

Let $S_n := \sum_{i=1}^{n} X_i$.

1. For $n \geq 0$, $1 \leq k \leq n + 1$, show that

$$\mathbb{P}[S_{n+1} = k] \quad = \quad p\mathbb{P}[S_n = k - 1] + q\mathbb{P}[S_n = k]$$

2. Show that S_n has the binomial(n, p) distribution by using generating functions and the equation above.

Problem 3

1. Consider a sequence $\{a_n\}$. Suppose that the radius R of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$ satisfies $R > 1$. Show that for all $\beta \in \mathbb{R}$,

$$\sum_{n=1}^{\infty} |a_n| n^{\beta} \quad < \quad \infty$$

   Next suppose that the radius R of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$ satisfies $R < 1$. Show that for all $\beta \in \mathbb{R}$,

$$\sum_{n=1}^{\infty} |a_n| n^{\beta} \quad = \quad \infty$$

2. Consider a nonnegative integer valued random variable X with probability distribution $\{p_n\}$. Suppose that

$$\limsup_{n\to\infty} \sqrt[n]{p_n} \quad < \quad 1$$

   Show that all moments of X are finite.

3. Consider a positive integer valued random variable X with probability distribution $\{c/n^{\beta}\}$ where $\beta > 1$ and $c > 0$ is chosen so that $\sum_{n=1}^{\infty} c/n^{\beta} = 1$. Show that $\mathbb{E}[X^{\gamma}] < \infty$ if and only if $\gamma < \beta - 1$. Also show that

$$\limsup_{n\to\infty} \sqrt[n]{c/n^{\beta}} \quad = \quad 1$$

   and thus the results above are consistent.

Problem 4

Consider the following triangular array $\{X_{n,k}\}$ of binary valued random variables, with

$$\mathbb{P}[X_{n,k} = 1] \quad = \quad \frac{2k}{n^2} \quad = \quad 1 - \mathbb{P}[X_{n,k} = 0]$$

for $n = 2, 3, \ldots$ and $k = 1, \ldots, n$. For each $n = 2, 5, 10, 100, 1000$ do the following: Use a computer to generate $M = 10000$ replications of $X_{n,k}$ for each $k = 1, \ldots, n$; denote these replications by $X_{n,k}^{(1)}, \ldots, X_{n,k}^{(M)}$. The random variables $\{X_{n,k}^{(m)} : k = 1, \ldots, n,\ m = 1, \ldots, M\}$ should be independent (at least apparently independent — use a good random number generator). The random variables corresponding to different values of n do not have to be independent; for example, $X_{2,1}^{(1)}$ and $X_{5,1}^{(1)}$ do not have to be independent — you may want to use the same random number seed for $X_{2,1}^{(1)}$ and $X_{5,1}^{(1)}$. For each m, n and $y \in \{0, 1, \ldots, 10\}$, let

$$Y_n^{(m)}(y) \quad := \quad \begin{cases} 1 & \text{if } \sum_{k=1}^{n} X_{n,k}^{m} \leq y \\ 0 & \text{otherwise} \end{cases}$$

Next, estimate the cumulative distribution function of $\sum_{k=1}^{n} X_{n,k}$ by calculating

$$\hat{P}\left[\sum_{k=1}^{n} X_{n,k} \leq y\right] \quad = \quad \frac{1}{M} \sum_{m=1}^{M} Y_n^{(m)}(y)$$

For each $n = 2, 5, 10, 100, 1000$, plot $\hat{P}\left[\sum_{k=1}^{n} X_{n,k} \leq y\right]$ as a function of y. On the same graph, plot the cumulative distribution function of the Poisson distribution with mean 1. Describe your observations.