ISyE 6761 Stochastic Processes I

Fall 2005

Exam 2

December 16, 2005

1. Only 1 page of notes may be used during the exam.
2. Carefully justify any interchanges of limits or series with results given in class.
3. You may use any result from class or from any text, but you have to precisely state the result that you use.

Problem 1 [10 points]

Consider a discrete time Markov Chain X with state space $\mathcal{S}$.

1. Show that a finite communicating class cannot have a null recurrent state.
   Answer: Consider any state $i \in \mathcal{S}$. Let $\mathcal{C}(i)$ denote the communicating class that state i belongs to. Suppose that i is null recurrent and $|\mathcal{C}(i)| < \infty$. We have shown that null recurrence is a class property, and thus if i is null recurrent, then all states in $\mathcal{C}(i)$ are null recurrent. For any initial state i and any null recurrent state $j \in \mathcal{S}$, it holds that $\lim_{k\to\infty} p_{ij}^{(k)} = 0$; thus, there is K_{ij} such that $p_{ij}^{(k)} < 1/(2|\mathcal{C}(i)|) \in (0, \infty)$ for all $k \geq K_{ij}$. Let $K_i := \max\{K_{ij} : j \in \mathcal{C}(i)\} < \infty$. Then, for all $j \in \mathcal{C}(i)$, it holds that $p_{ij}^{(k)} < 1/(2|\mathcal{C}(i)|) \in (0, \infty)$ for all $k \geq K_i$. Hence, for any $k \geq K_i$, it holds that $\sum_{j\in\mathcal{C}(i)} p_{ij}^{(k)} < |\mathcal{C}(i)|/(2|\mathcal{C}(i)|) = 1/2$. However, any recurrent class is closed, and thus $\sum_{j\in\mathcal{C}(i)} p_{ij}^{(k)} = 1$ for all k, which contradicts the previous conclusion. Therefore a finite communicating class cannot have a null recurrent state.
2. Prove or disprove with a counterexample: a finite communicating class is positive recurrent.
   Answer: False. A finite communicating class can be transient. Example: $\mathcal{S} = \{0, 1\}$, $p_{01} = p_{11} = 1$. Communicating class $\mathcal{C}(0) = \{0\}$ is transient.

Problem 2 [5 points]

Consider a discrete time Markov Chain X with a countable (finite or countably infinite) state space $\mathcal{S}$, and transition matrix P. Suppose that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} P^k \quad = \quad \Pi$$

where all the rows of Π are the same, with each row sum of Π being equal to 1. Prove or disprove: Each row of Π is a stationary distribution.

Answer: True. Let π denote a row of Π. By assumption, $\pi \geq 0$ and $\sum_{i \in \mathcal{S}} \pi_i = 1$, thus π is a probability distribution.

$$\begin{aligned}
\Pi P &= \left[\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} P^k\right] P \\
&= \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} P^k P \\
&= \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} P^{k+1} \\
&= \lim_{N\to\infty} \frac{1}{N} \left(\sum_{k=1}^{N+1} P^k - P\right) \\
&= \lim_{N\to\infty} \left(\frac{N+1}{N} \frac{1}{N+1} \sum_{k=1}^{N+1} P^k - \frac{1}{N} P\right) \\
&= \lim_{N\to\infty} \frac{N+1}{N} \lim_{N\to\infty} \frac{1}{N+1} \sum_{k=1}^{N+1} P^k - \lim_{N\to\infty} \frac{1}{N} P \\
&= \Pi
\end{aligned}$$

Thus, considering each row of $\Pi P = \Pi$, it follows that $\pi^T P = \pi^T$. Therefore π is a stationary distribution.

Problem 3 [15 points]

Consider a discrete time Markov Chain X with a countable state space $\mathcal{S}$. We know that if the Markov chain is aperiodic, then a stationary distribution is also a limit distribution. Next we investigate the nature of the converse.

1. Give an example of an irreducible periodic discrete time Markov Chain X with a countable state space $\mathcal{S}$ such that $\lim_{n\to\infty} p_{ij}^{(n)}$ exists for all $i, j \in \mathcal{S}$.
   Answer: Simple random walk with i.i.d. sequence $\{Y_n\}$, $\mathbb{P}[Y_1 = 1] = p = 1 - \mathbb{P}[Y_1 = -1] = 1 - q$, $p \in (0,1)$, Markov chain X defined by $X_0 = 0$, $X_{n+1} = X_n + Y_{n+1} = \sum_{i=1}^{n+1} Y_i$. Then the Markov chain is irreducible because $p \in (0,1)$, and the Markov chain is periodic with period $d = 2$. Also, the Markov chain is null recurrent if $p = q$ and transient if $p \neq q$. In both cases, $\lim_{n\to\infty} p_{ij}^{(n)} = 0$ for all $i, j \in \mathcal{S}$.

2. Show that if the Markov chain is irreducible with period $d > 1$, then for any initial state $i \in \mathcal{S}$, it cannot hold that $p_{ij}^{(n)} \to \pi_{ij}$ for all $j \in \mathcal{S}$ as $n \to \infty$, where $(\pi_{ij} : j \in \mathcal{S})$ is a probability distribution. (Thus even if we allow the limit to depend on i, the limit result cannot hold.)
   Answer: If $(\pi_{ij} : j \in \mathcal{S})$ is a probability distribution, then $\pi_{ij} > 0$ for some

$j \in \mathcal{S}$. Suppose that $\pi_{ij} > 0$. We know that there is $r_{ij} \in \{0, 1, \ldots, d-1\}$ such that $p_{ij}^{(n)} = 0$ if $n \notin \{r_{ij}, d + r_{ij}, 2d + r_{ij}, \ldots\}$. Specifically, $p_{ij}^{(qd+r_{ij}+1)} = 0$ for all integers q. Thus $\lim_{q\to\infty} p_{ij}^{(qd+r_{ij}+1)} = 0 < \pi_{ij}$, which contradicts the hypothesis that $\lim_{n\to\infty} p_{ij}^{(n)} = \pi_{ij} > 0$.

3. Give an example of a discrete time Markov Chain X with a countable state space $\mathcal{S}$, such that the period $d(j) = d > 1$ for all $j \in \mathcal{S}$, and for some initial state $i \in \mathcal{S}$, it holds that $p_{ij}^{(n)} \to \pi_{ij}$ for all $j \in \mathcal{S}$ as $n \to \infty$, where $(\pi_{ij} : j \in \mathcal{S})$ is a probability distribution.
   Answer: Let $\mathcal{S} = \{1, 2, 3, 4\}$, $p_{1,2} = 1$, $p_{2,1} = 1 - p$, $p_{2,3} = p/2$, $p_{2,4} = p/2$ for some $p \in (0, 1)$, $p_{3,4} = p_{4,3} = 1$. Then $d(j) = d = 2$ for all $j \in \mathcal{S}$. States $1, 2$ are transient, states $3, 4$ are positive recurrent. Suppose $X_0 = 1$ (the same result holds if $X_0 = 2$). Let $\tau_{\{3,4\}} := \inf\{n \geq 1 : X_n \in \{3, 4\}\}$. It is easy to see that, because $p > 0$, $\mathbb{P}_1[\tau_{\{3,4\}} < \infty] = 1$, that is $\mathbb{P}_1[\tau_{\{3,4\}} \leq n] \to 1$ as $n \to \infty$. By symmetry, $\mathbb{P}_1[X_{\tau_{\{3,4\}}} = 3] = \mathbb{P}_1[X_{\tau_{\{3,4\}}} = 4] = 1/2$, hence for all n, $\mathbb{P}_1[X_n = 3|\tau_{\{3,4\}} \leq n] = \mathbb{P}_1[X_n = 4|\tau_{\{3,4\}} \leq n] = 1/2$. Thus

$$\begin{aligned} p_{13}^{(n)} &= \mathbb{P}_1[X_n = 3] \\ &= \mathbb{P}_1[X_n = 3, \tau_{\{3,4\}} \leq n] \\ &= \mathbb{P}_1[\tau_{\{3,4\}} \leq n]\mathbb{P}_1[X_n = 3|\tau_{\{3,4\}} \leq n] \\ &= \mathbb{P}_1[\tau_{\{3,4\}} \leq n]1/2 \quad \to \quad 1/2 \end{aligned}$$

   as $n \to \infty$. Similarly, $p_{13}^{(n)} \to 1/2$ as $n \to \infty$. Thus, with probability $1/2$, $X_n = 3$ for all odd n sufficiently large and $X_n = 4$ for all even n sufficiently large, and with probability $1/2$, $X_n = 3$ for all even n sufficiently large and $X_n = 4$ for all odd n sufficiently large. Therefore, $\lim_{n\to\infty} p_{13}^{(n)} = \lim_{n\to\infty} p_{14}^{(n)} = 1/2$ and $\lim_{n\to\infty} p_{11}^{(n)} = \lim_{n\to\infty} p_{12}^{(n)} = 0$. Thus $\pi_{11} = \pi_{12} = 0$, and $\pi_{13} = \pi_{14} = 1/2$, and therefore $(\pi_{1j} : j \in \mathcal{S})$ is a probability distribution. Similarly, $\pi_{21} = \pi_{22} = 0$, and $\pi_{23} = \pi_{24} = 1/2$.

Problem 4 [40 points]

Consider a discrete time Markov Chain X with a finite state space $\mathcal{S}$, and transition matrix $P \in \mathbb{R}^{n\times n}$, with left eigenvalues $\lambda_1, \ldots, \lambda_n$ and corresponding left eigenvectors $v_1, \ldots, v_n$, and right eigenvalues $\mu_1, \ldots, \mu_n$ and corresponding right eigenvectors $w_1, \ldots, w_n$. Suppose that $v_1, \ldots, v_n$ are linearly independent, and that $w_1, \ldots, w_n$ are linearly independent. Let $V = [v_1 \ \cdots \ v_n] \in \mathbb{R}^{n\times n}$ be the matrix with columns $v_1, \ldots, v_n$, and let $W = [w_1 \ \cdots \ w_n] \in \mathbb{R}^{n\times n}$ be the matrix with columns $w_1, \ldots, w_n$.

1. Write P^k in terms of $\mu_1, \ldots, \mu_n$, $v_1, \ldots, v_n$, and $w_1, \ldots, w_n$.
   Answer:

$$\begin{aligned} P &= W\text{diag}(\mu_1, \cdots, \mu_n)W^{-1} \\ \Rightarrow \quad P^k &= W\text{diag}(\mu_1, \ldots, \mu_n)^k W^{-1} \end{aligned}$$

$$
\begin{aligned}
&= W\,\mathrm{diag}(\mu_1^k, \ldots, \mu_n^k) V^T \\
&= [\mu_1^k w_1 \ \cdots \ \mu_n^k w_n] V^T \\
&= \sum_{i=1}^n \mu_i^k w_i v_i^T
\end{aligned}
$$

2. Show that all the eigenvalues of P satisfy $|\mu_i| \leq 1$.
Answer: Index the eigenvalues so that $|\mu_1| \geq |\mu_2| \geq \cdots \geq |\mu_n|$. Suppose that $|\mu_1| = \cdots = |\mu_m| > |\mu_{m+1}|$. Note that, by the definition of eigenvectors, $w_i \neq 0$ and $v_i \neq 0$. By linear independence, $\sum_{i=1}^m w_i v_i^T \neq 0$. Choose any $j, l \in \{1, \ldots, n\}$ so that $\sum_{i=1}^m (w_i)_j (v_i)_l \neq 0$. Suppose that $\sum_{i=1}^m (w_i)_j (v_i)_l > 0$; the argument is the same if $\sum_{i=1}^m (w_i)_j (v_i)_l < 0$. Next we show by contradiction that $|\mu_1| \leq 1$. Suppose that $|\mu_1| > 1$. Then

$$
\sum_{i=1}^m |\mu_i|^k (w_i)_j (v_i)_l \quad = \quad |\mu_1|^k \sum_{i=1}^m (w_i)_j (v_i)_l \quad \to \quad \infty
$$

as $k \to \infty$. Also,

$$
\begin{aligned}
\frac{\sum_{i=1}^m |\mu_i|^k (w_i)_j (v_i)_l}{\left|\sum_{i=m+1}^n \mu_i^k (w_i)_j (v_i)_l\right|} \quad &\geq \quad \frac{\sum_{i=1}^m |\mu_i|^k (w_i)_j (v_i)_l}{\sum_{i=m+1}^n |\mu_i|^k \left|(w_i)_j (v_i)_l\right|} \\
&\geq \quad \frac{|\mu_1|^k}{|\mu_{m+1}|^k} \, \frac{\sum_{i=1}^m (w_i)_j (v_i)_l}{\sum_{i=m+1}^n \left|(w_i)_j (v_i)_l\right|} \quad \to \quad \infty
\end{aligned}
$$

as $k \to \infty$. Thus there is K such that

$$
\sum_{i=1}^m |\mu_i|^k (w_i)_j (v_i)_l \quad > \quad 4
$$

and

$$
\frac{\sum_{i=1}^m |\mu_i|^k (w_i)_j (v_i)_l}{\left|\sum_{i=m+1}^n \mu_i^k (w_i)_j (v_i)_l\right|} \quad > \quad 2
$$

for all $k \geq K$. Choose $k \geq K$ such that k is even, and thus $\mu_i^k = |\mu_i|^k$ for all i. Then

$$
\begin{aligned}
P_{jl}^k \quad &= \quad \left(\sum_{i=1}^n \mu_i^k w_i v_i^T\right)_{jl} \\
&= \quad \sum_{i=1}^n \mu_i^k (w_i)_j (v_i)_l \\
&= \quad \sum_{i=1}^m \mu_i^k (w_i)_j (v_i)_l + \sum_{i=m+1}^n \mu_i^k (w_i)_j (v_i)_l \\
&\geq \quad \sum_{i=1}^m |\mu_i|^k (w_i)_j (v_i)_l - \left|\sum_{i=m+1}^n \mu_i^k (w_i)_j (v_i)_l\right|
\end{aligned}
$$

$$\begin{aligned} &> \sum_{i=1}^{m} |\mu_i|^k (w_i)_j (v_i)_l - \frac{1}{2} \sum_{i=1}^{m} |\mu_i|^k (w_i)_j (v_i)_l \\ &= \frac{1}{2} \sum_{i=1}^{m} |\mu_i|^k (w_i)_j (v_i)_l \quad > \quad 2 \end{aligned}$$

However, this contradicts the fact that $P_{jl}^k \in [0,1]$. Therefore $|\mu_i| \leq 1$ for all i.

3. Identify one eigenvalue, call it μ_1, and corresponding right eigenvector w_1, of P.
Answer: Let $\mathbf{1} := (1, 1, \ldots, 1) \in \mathbb{R}^n$. Observe that

$$P\mathbf{1} \quad = \quad \mathbf{1}$$

Thus $\mu_1 = 1$ is a right eigenvalue and $w_1 = \mathbf{1}$ is a corresponding right eigenvector.

4. Suppose that $\mu_2, \ldots, \mu_n$ satisfy $|\mu_2| < 1, \ldots, |\mu_n| < 1$. Write $\lim_{k\to\infty} P^k$ in terms of $\mu_1, \ldots, \mu_n$, $v_1, \ldots, v_n$, and $w_1, \ldots, w_n$, and simplify as much as possible.
Answer:

$$\begin{aligned} P^k &= \sum_{i=1}^{n} \mu_i^k w_i v_i^T \\ \Rightarrow \quad \lim_{k\to\infty} P^k &= \lim_{k\to\infty} \sum_{i=1}^{n} \mu_i^k w_i v_i^T \\ &= \sum_{i=1}^{n} \lim_{k\to\infty} \mu_i^k w_i v_i^T \\ &= w_1 v_1^T \quad = \quad \mathbf{1} v_1^T \end{aligned}$$

5. Again suppose that $\mu_2, \ldots, \mu_n$ satisfy $|\mu_2| < 1, \ldots, |\mu_n| < 1$. Write a stationary distribution for the Markov chain in terms of the eigenvalues and eigenvectors of P. Use the given information to prove that your claimed stationary probabilities add up to 1.
Answer: It follows from above that

$$\lim_{k\to\infty} P_{ij}^k \quad = \quad (v_1)_j$$

for all $i \in \{1, \ldots, n\}$. Thus, if $\sum_{j=1}^n (v_1)_j = 1$, then v_1 is a limit distribution, and thus v_1 is a stationary distribution for the Markov chain. For all $i \in \{1, \ldots, n\}$,

$$\begin{aligned} \sum_{j=1}^{n} P_{ij} &= 1 \\ \Rightarrow \quad \sum_{j=1}^{n} P_{ij}^k &= 1 \quad \forall\, k \\ \Rightarrow \quad \lim_{k\to\infty} \sum_{j=1}^{n} P_{ij}^k &= 1 \end{aligned}$$

$$\begin{aligned} \Rightarrow \quad \sum_{j=1}^{n} \lim_{k\to\infty} P_{ij}^{k} &= 1 \\ \Rightarrow \quad \sum_{j=1}^{n} (\mathbf{1} v_1^T)_{ij} &= 1 \\ \Rightarrow \quad \sum_{j=1}^{n} (v_1)_j &= 1 \end{aligned}$$

Therefore v_1 is a limit distribution, and thus v_1 is a stationary distribution for the Markov chain.

6. Again suppose that $\mu_2, \ldots, \mu_n$ satisfy $|\mu_2| < 1, \ldots, |\mu_n| < 1$. Completely classify the Markov chain, and justify your classification.
Answer: The Markov chain is not necessarily irreducible, for example

$$P = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

has eigenvalues $0, 1$, with corresponding left eigenvectors $(-1, 1)$ and $(0, 1)$ and corresponding right eigenvectors $(1, 0)$ and $(1, 1)$. Thus P satisfies the assumptions above, but the Markov chain is not irreducible.
Consider any stationary distribution π of the Markov chain. Then

$$\begin{aligned} \pi^T P &= \pi^T \\ \Rightarrow \quad \pi^T P^k &= \pi^T \quad \forall\, k \\ \Rightarrow \quad \lim_{k\to\infty} \pi^T P^k &= \pi^T \\ \Rightarrow \quad \pi^T \mathbf{1} v_1^T &= \pi^T \\ \Rightarrow \quad v_1^T &= \pi^T \end{aligned}$$

because $\pi^T \mathbf{1} = \sum_{i\in\mathcal{S}} \pi_i = 1$. Thus the Markov chain has a unique stationary distribution $\pi = v_1$. Hence the Markov chain has exactly one positive recurrent class, and possibly some transient states (it was shown above that a Markov chain with a finite state space cannot have a null recurrent state). It was shown above that v_1 is also a limit distribution. It was shown above that a periodic class of states cannot have a limit distribution. Thus the positive recurrent class of the Markov chain is aperiodic.

7. Suppose the Markov chain has exactly one positive recurrent class with period $d > 1$. What can you conclude about the eigenvalues of P? Justify your conclusions.
Answer: As observed before, P has an eigenvalue $\mu_1 = 1$ with corresponding right eigenvector $w_1 = \mathbf{1}$. It has been shown that $|\mu_i| \leq 1$ for all i. It has also been shown that if $|\mu_2| < 1, \ldots, |\mu_n| < 1$, then the unique stationary distribution is also a limit distribution. However, it has been shown that if the positive recurrent class is periodic, then it cannot have a limit distribution. Thus it cannot hold that $|\mu_2| < 1, \ldots, |\mu_n| < 1$. Hence there has to be another eigenvalue, besides μ_1, say μ_2, such that $|\mu_2| = 1$. Next we show that, more specifically, there has to be an eigenvalue,

say μ_2, such that $\mu_2 = -1$. By contradiction. Suppose that $\mu_1 = \cdots = \mu_m = 1$ and $|\mu_{m+1}| < 1, \ldots, |\mu_n| < 1$. Then

$$\begin{aligned} P^k &= \sum_{i=1}^{n} \mu_i^k w_i v_i^T \\ \Rightarrow \quad \lim_{k\to\infty} P^k &= \lim_{k\to\infty} \sum_{i=1}^{n} \mu_i^k w_i v_i^T \\ &= \sum_{i=1}^{n} \lim_{k\to\infty} \mu_i^k w_i v_i^T \\ &= \sum_{i=1}^{m} w_i v_i^T \end{aligned}$$

However, that contradicts the result that if state i has period $d > 1$, then $p_{ii}^{(k)} = P_{ii}^k$ does not converge as $k \to \infty$. Therefore, there has to be an eigenvalue, say μ_2, such that $\mu_2 = -1$.

8. Suppose that the Markov chain has two recurrent classes. For concreteness, you may suppose that

$$P = \begin{bmatrix} a & b & 0 & 0 & 0 \\ c & d & 0 & 0 & 0 \\ 0 & 0 & e & f & 0 \\ 0 & 0 & g & h & 0 \\ i & j & k & l & m \end{bmatrix}$$

where $m < 1$. What can you conclude regarding the eigenvalues and eigenvectors of P?

Answer: P has at least 2 eigenvalues equal to 1, say $\mu_1 = \mu_2 = 1$, with corresponding right eigenvectors $w_1 = (1, 1, 0, 0, (i+j)/(1-m))$ and $w_2 = (0, 0, 1, 1, (k+l)/(1-m))$.

Problem 5 [30 points]

Consider a pure renewal process with i.i.d. inter-renewal times $X_1, X_2, \ldots$, with probability distribution function F, with $F(0) < 1$ and $\mathbb{E}[X_1] < \infty$. Let $T_0 := 0$ and $T_n := \sum_{i=1}^n X_i$. Let $N(t) := \max\{n \geq 0 : T_n \leq t\}$. Let $A(t) := t - T_{N(t)}$ denote the age at time t, and let $B(t) := T_{N(t)+1} - t$ denote the residual life at time t. Consider $x, y \geq 0$.

1. Derive a renewal equation for

$$Z(t) := \mathbb{P}[A(t) > x, B(t) > y]$$

Answer: First note that $\mathbb{P}[A(t) > x, B(t) > y] = 0$ if $t \leq x$, because $A(t) \leq t$.

$$\mathbb{P}[A(t) > x, B(t) > y] = \mathbb{E}(\mathbb{P}[A(t) > x, B(t) > y | X_1])$$

$$
\begin{aligned}
&= \int_{[0,\infty)} \mathbb{P}[A(t) > x, B(t) > y | X_1 = u]\, F(du) \\
&= \int_{[0,t]} \mathbb{P}[A(t) > x, B(t) > y | X_1 = u]\, F(du) \\
&\quad + \int_{(t,\infty)} \mathbb{P}[A(t) > x, B(t) > y | X_1 = u]\, F(du)
\end{aligned}
$$

Note that

$$
\begin{aligned}
\int_{(t,\infty)} \mathbb{P}[A(t) > x, B(t) > y | X_1 = u]\, F(du) &= \int_{(t,\infty)} \mathbb{I}_{\{t>x\}} \mathbb{I}_{\{u-t>y\}}\, F(du) \\
&= \mathbb{I}_{(x,\infty)}(t) \int_{(t+y,\infty)} F(du) \\
&= \mathbb{I}_{(x,\infty)}(t)[1 - F(t+y)]
\end{aligned}
$$

Also,

$$
\begin{aligned}
\int_{[0,t]} \mathbb{P}[A(t) > x, B(t) > y | X_1 = u]\, F(du) &= \int_{[0,t]} \mathbb{P}[A(t-u) > x, B(t-u) > y]\, F(du) \\
&= \int_{[0,t]} Z(t-u)\, F(du) \quad = \quad F * Z(t)
\end{aligned}
$$

Thus

$$
Z(t) \quad = \quad z(t) + F * Z(t)
$$

where

$$
z(t) \quad := \quad \mathbb{I}_{(x,\infty)}(t)[1 - F(t+y)]
$$

2. Solve the renewal equation for Z.
Answer: Let $U(t) := \sum_{n=0}^{\infty} F^{n*}(t)$. Then

$$
\begin{aligned}
Z(t) &= U * z(t) \quad = \quad \int_{[0,t]} z(t-u)\, U(du) \\
&= \int_{[0,t]} \mathbb{I}_{(x,\infty)}(t-u)[1 - F(t-u+y)]\, U(du) \\
&= \int_{[0,t-x)} [1 - F(t-u+y)]\, U(du)
\end{aligned}
$$

3. Suppose that F is non-arithmetic (non-lattice). Determine

$$
\lim_{t\to\infty} Z(t)
$$

Specify and verify any additional conditions needed.
Answer: We need z to be directly Riemann integrable. First check whether z is

dRi. Note that $z(t) := \mathbb{I}_{(x,\infty)}(t)[1 - F(t+y)]$, thus $z(t) = 0$ on $[0, x]$ and $z(t) \geq 0$ and nonincreasing on (x, ∞). In addition,

$$\begin{aligned}
\int_0^\infty z(t)\,dt &= \int_0^\infty \mathbb{I}_{(x,\infty)}(t)[1 - F(t+y)]\,dt \\
&= \int_x^\infty [1 - F(t+y)]\,dt \\
&= \int_{x+y}^\infty [1 - F(u)]\,du \\
&\leq \int_0^\infty [1 - F(u)]\,du \quad = \quad \mathbb{E}[X_1] \quad < \quad \infty
\end{aligned}$$

Thus z is dRi on (x, ∞), and hence z is dRi on $[0, \infty)$. Therefore it follows from the key renewal theorem that

$$\begin{aligned}
\lim_{t\to\infty} Z(t) &= \frac{1}{\mathbb{E}[X_1]} \int_0^\infty z(t)\,dt \\
&= \frac{1}{\mathbb{E}[X_1]} \int_{x+y}^\infty [1 - F(u)]\,du
\end{aligned}$$

4. Recall that if F is non-arithmetic (non-lattice), then

$$\lim_{t\to\infty} \mathbb{P}[A(t) > x] \quad = \quad \lim_{t\to\infty} \mathbb{P}[B(t) > x] \quad = \quad \frac{1}{\mathbb{E}[X_1]} \int_x^\infty [1 - F(s)]ds$$

Determine exact conditions under which $\lim_{t\to\infty} Z(t)$ is a product measure, that is

$$\lim_{t\to\infty} Z(t) \quad = \quad \left(\frac{1}{\mathbb{E}[X_1]} \int_x^\infty [1 - F(s)]ds\right) \left(\frac{1}{\mathbb{E}[X_1]} \int_y^\infty [1 - F(s)]ds\right)$$

for all $x, y \geq 0$.
Answer: For any $x \geq 0$, let

$$\overline{F}_0(x) \quad := \quad \frac{1}{\mathbb{E}[X_1]} \int_x^\infty [1 - F(s)]ds$$

It follows from the previous question that $\lim_{t\to\infty} Z(t) = \overline{F}_0(x + y)$. The question is to determine conditions under which

$$\overline{F}_0(x+y) \quad = \quad \overline{F}_0(x)\overline{F}_0(y)$$

for all $x, y \geq 0$. Note that $\overline{F}_0$ is continuous and

$$\lim_{x\to 0} \overline{F}_0(x) \quad = \quad \frac{1}{\mathbb{E}[X_1]} \int_0^\infty [1 - F(s)]ds \quad = \quad 1$$

Thus there is $\delta > 0$ such that $\overline{F}_0(\delta) > 0$. For $\overline{F}_0$ to satisfy $\overline{F}_0(x + y) = \overline{F}_0(x)\overline{F}_0(y)$ for all $x, y \geq 0$, it has to hold that $\overline{F}_0(n\delta) = \left(\overline{F}_0(\delta)\right)^n > 0$ for all integers $n \geq 0$.

Consider any $x \geq 0$; then there is an integer $n > 0$ such that $x < n\delta$. Note that $\overline{F}_0$ is nonincreasing. Thus $\overline{F}_0(x) \geq \overline{F}_0(n\delta) > 0$; that is, $\overline{F}_0(x) > 0$ for all $x \geq 0$. Thus we can take the logarithm of $\overline{F}_0(x)$ for all $x \geq 0$. Let $\overline{G}_0(x) := \ln\left(\overline{F}_0(x)\right)$ for all $x \geq 0$. Thus

$$\begin{aligned} \overline{F}_0(x+y) &= \overline{F}_0(x)\overline{F}_0(y) \\ \Rightarrow \quad \ln\left(\overline{F}_0(x+y)\right) &= \ln\left(\overline{F}_0(x)\right) + \ln\left(\overline{F}_0(y)\right) \\ \Rightarrow \quad \overline{G}_0(x+y) &= \overline{G}_0(x) + \overline{G}_0(y) \end{aligned}$$

Hence $\overline{G}_0$ is linear, say $\overline{G}_0(x) = -\lambda x$. Thus

$$\begin{aligned} \overline{F}_0(x) &= e^{\overline{G}_0(x)} = e^{-\lambda x} \\ \Rightarrow \quad \frac{1}{\mathbb{E}[X_1]} \int_x^\infty [1 - F(s)]ds &= e^{-\lambda x} \end{aligned}$$

Taking the derivative with respect to x on both sides, it follows that

$$\begin{aligned} -\frac{1}{\mathbb{E}[X_1]}[1 - F(x)] &= -\lambda e^{-\lambda x} \\ \Rightarrow \quad F(x) &= 1 - \lambda \mathbb{E}[X_1] e^{-\lambda x} \end{aligned}$$

For F to be a valid probability distribution function, we need that $F(0) \geq 0$, hence $\lambda \leq 1/\mathbb{E}[X_1]$. In addition, we want $F(0) < 1$, hence $\lambda > 0$. We conclude that if for some $\lambda \in (0, 1/\mathbb{E}[X_1]]$, $F(x) = 1 - \lambda \mathbb{E}[X_1] e^{-\lambda x}$ for all $x \geq 0$, then $\lim_{t\to\infty} Z(t)$ is a product measure. In words, F can be any combination of an exponential distribution and a point mass at 0. As a reality check, verify that $\int_0^\infty [1 - F(x)]dx = \int_0^\infty \lambda \mathbb{E}[X_1] e^{-\lambda x} dx = \mathbb{E}[X_1]$, as it should be.