ISyE 6761 Stochastic Processes I

Fall 2005

Exam 1

October 13, 2005

1. No books or notes may be used during the exam.

2. Carefully justify any interchanges of limits or series with results given in class.

Problem 1 [10 points]
Give an example of two integer valued random variables X and Y such that X and Y are not independent, and $E[XY] = E[X]E[Y]$.
Answer: Let $X \in \{-1, 0, 1\}$ and $Y \in \{0, 1\}$ have the following joint distribution:

$$P_{i,j} = \mathbb{P}[X = i, Y = j] = \begin{array}{c} \begin{matrix} Y=0 & Y=1 \end{matrix} \\ \begin{bmatrix} 1/4 & 0 \\ 0 & 1/2 \\ 1/4 & 0 \end{bmatrix} \end{array} \begin{array}{l} \\ X=-1 \\ X=0 \\ X=1 \end{array}$$

Then $E[XY] = E[X]E[Y] = 0$. However, $\mathbb{P}[X = 0, Y = 0] = 0 \neq \mathbb{P}[X = 0]\mathbb{P}[Y = 0] = (1/2)(1/2) = 1/4$, and thus X and Y are not independent.

Problem 2
Let X and Y be nonnegative integer valued random variables on the same probability space. For $s, t \in [-1, 1]$, the marginal generating functions are given by

$$P_X(s) := \sum_{i=0}^{\infty} s^i P[X = i]$$
$$P_Y(t) := \sum_{j=0}^{\infty} t^j P[Y = j]$$

Let

$$P_{X+Y}(s) := \sum_{i=0}^{\infty} s^i P[X + Y = i]$$

denote the generating function of $X+Y$. Define the joint generating function $P_{X,Y}$ as follows:

$$P_{X,Y}(s,t) := \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} s^i t^j P[X = i, Y = j]$$

1. Show that $P_{X+Y}(s) = P_{X,Y}(s,s)$, whether X and Y are independent or not. [20 points]
Answer:

$$
\begin{aligned}
P_{X+Y}(s) &:= \sum_{k=0}^{\infty} \mathbb{P}[X+Y=k]s^k \\
&= \sum_{k=0}^{\infty} \left(\sum_{i=0}^{k} \mathbb{P}[X=i, Y=k-i] \right) s^k \\
&= \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \mathbb{I}_{\{i \le k\}} \mathbb{P}[X=i, Y=k-i] s^k
\end{aligned}
$$

Note that, for $s \in [-1, 1]$,

$$
\begin{aligned}
\sum_{i=0}^{\infty} \left| \mathbb{I}_{\{i \le k\}} \mathbb{P}[X=i, Y=k-i] s^k \right| &= \sum_{i=0}^{\infty} \mathbb{I}_{\{i \le k\}} \mathbb{P}[X=i, Y=k-i] \left| s \right|^k \\
&\le \sum_{i=0}^{k} \mathbb{P}[X=i, Y=k-i] \\
&= \mathbb{P}[X+Y=k] \qquad (1)
\end{aligned}
$$

and

$$
\sum_{k=0}^{\infty} \mathbb{P}[X+Y=k] = 1 < \infty \qquad (2)
$$

and thus it follows from a result given in class (Rudin, Theorem 8.3) as well as (1) and (2) above that the two series can be interchanged. Thus,

$$
\begin{aligned}
P_{X+Y}(s) &= \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \mathbb{I}_{\{i \le k\}} \mathbb{P}[X=i, Y=k-i] s^k \\
&= \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \mathbb{I}_{\{i \le k\}} \mathbb{P}[X=i, Y=k-i] s^k \\
&= \sum_{i=0}^{\infty} \sum_{k=i}^{\infty} \mathbb{P}[X=i, Y=k-i] s^i s^{k-i} \\
&= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \mathbb{P}[X=i, Y=j] s^i s^j \\
&= P_{X,Y}(s,s)
\end{aligned}
$$

2. Consider two binary random variables X and Y on the same probability space. Suppose that $\mathbb{P}[X=0, Y=0] = \mathbb{P}[X=0]\mathbb{P}[Y=0]$. Prove that X and Y are independent. [10 points]

Answer:

$$\begin{aligned}
\mathbb{P}[X=0,Y=1] &= \mathbb{P}[X=0]-\mathbb{P}[X=0,Y=0] \\
&= \mathbb{P}[X=0]-\mathbb{P}[X=0]\mathbb{P}[Y=0] \\
&= \mathbb{P}[X=0]\left(1-\mathbb{P}[Y=0]\right) \\
&= \mathbb{P}[X=0]\mathbb{P}[Y=1]
\end{aligned}$$

Similarly, $\mathbb{P}[X=1,Y=0]=\mathbb{P}[X=1]\mathbb{P}[Y=0]$. Then

$$\begin{aligned}
\mathbb{P}[X=1,Y=1] &= \mathbb{P}[X=1]-\mathbb{P}[X=1,Y=0] \\
&= \mathbb{P}[X=1]-\mathbb{P}[X=1]\mathbb{P}[Y=0] \\
&= \mathbb{P}[X=1]\left(1-\mathbb{P}[Y=0]\right) \\
&= \mathbb{P}[X=1]\mathbb{P}[Y=1]
\end{aligned}$$

Thus, $\mathbb{P}[X=i,Y=j]=\mathbb{P}[X=i]\mathbb{P}[Y=j]$ for all i,j, and hence X and Y are independent.

3. It follows from the result in the first part above that if X and Y are independent, then $P_{X+Y}(s)=P_{X,Y}(s,s)=P_X(s)P_Y(s)$ for all $s\in[-1,1]$, that is, the generating function of $X+Y$ is equal to the product of the marginal generating functions of X and Y. Give an example of jointly distributed nonnegative integer valued random variables X and Y on the same probability space, which are not independent, but for which

$$P_{X+Y}(s) \;=\; P_{X,Y}(s,s) \;=\; P_X(s)P_Y(s)$$

for all $s\in[-1,1]$. (You have to show that the random variables are not independent and that the equality above holds.) That is, even if $P_{X+Y}(s)=P_{X,Y}(s,s)=P_X(s)P_Y(s)$ for all $s\in[-1,1]$, it does not imply that X and Y are independent. (Hint: One purpose of the previous part with the binary random variables is to give you an idea what is the minimum number of values that the random variables should take in your example.) [10 points]

Answer: Let $X,Y\in\{0,1,2\}$ have the following joint distribution:

$$P_{i,j} \;=\; \mathbb{P}[X=i,Y=j] \;=\; \begin{array}{c} \begin{array}{ccc} Y=0 & Y=1 & Y=2 \end{array} \\ \left[\begin{array}{ccc} 1/9 & 2/9 & 0 \\ 0 & 1/9 & 2/9 \\ 2/9 & 0 & 1/9 \end{array}\right] \end{array} \begin{array}{c} X=0 \\ X=1 \\ X=2 \end{array}$$

Then $P_X(s)=P_Y(s)=1/3+1/3s+1/3s^2$. Thus $P_X(s)P_Y(s)=(1/3+1/3s+1/3s^2)(1/3+1/3s+1/3s^2)=1/9+2/9s+1/3s^2+2/9s^3+1/9s^4$. Also, $\mathbb{P}[X+Y=k]=(1/9,2/9,1/3,2/9,1/9)$ for $k=0,1,2,3,4$. Thus $P_{X+Y}(s)=1/9+2/9s+1/3s^2+2/9s^3+1/9s^4=P_X(s)P_Y(s)$. However, $\mathbb{P}[X=0,Y=2]=0\neq\mathbb{P}[X=0]\mathbb{P}[Y=2]=(1/3)(1/3)=1/9$, and thus X and Y are not independent.

Problem 3 [20 points]

For a discrete time Markov chain $X : \Omega \mapsto \mathcal{S}^\infty$, use only basic identities in probability and the Markov property

$$\mathbb{P}[X_{n+1} = j | X_0 = i_0, X_1 = i_1, \ldots, X_{n-1} = i_{n-1}, X_n = i] \quad = \quad p_{i,j}$$

for all histories $(i_0, i_1, \ldots, i_{n-1}, i)$ and all j, to prove that

$$\begin{aligned} & \mathbb{P}[X_{n+1} = k_1, \ldots, X_{n+m} = k_m | X_0 = i_0, \ldots, X_{n-1} = i_{n-1}, X_n = i] \\ = \; & \mathbb{P}[X_{n+1} = k_1, \ldots, X_{n+m} = k_m | X_n = i] \\ = \; & \mathbb{P}[X_1 = k_1, \ldots, X_m = k_m | X_0 = i] \end{aligned}$$

for all states $i_0, \ldots, i_{n-1}, i, k_1, \ldots, k_m$.

Answer:

$$\begin{aligned} & \mathbb{P}[X_1 = k_1, \ldots, X_m = k_m | X_0 = i] \\ = \; & \mathbb{P}[X_1 = k_1 | X_0 = i]\mathbb{P}[X_2 = k_2 | X_0 = i, X_1 = k_1] \cdots \mathbb{P}[X_m = k_m | X_0 = i, X_1 = k_1, \ldots, X_{m-1} = k_{m-1}] \\ = \; & p_{i,k_1} p_{k_1,k_2} \cdots p_{k_{m-1},k_m} \end{aligned}$$

$$\begin{aligned} & \mathbb{P}[X_{n+1} = k_1, \ldots, X_{n+m} = k_m | X_0 = i_0, \ldots, X_{n-1} = i_{n-1}, X_n = i] \\ = \; & \mathbb{P}[X_{n+1} = k_1 | X_0 = i_0, \ldots, X_{n-1} = i_{n-1}, X_n = i] \\ & \times \mathbb{P}[X_{n+2} = k_2 | X_0 = i_0, \ldots, X_{n-1} = i_{n-1}, X_n = i, X_{n+1} = k_1] \\ & \times \cdots \times \mathbb{P}[X_{n+m} = k_m | X_0 = i_0, \ldots, X_{n-1} = i_{n-1}, X_n = i, X_{n+1} = k_1, \ldots, X_{n+m-1} = k_{m-1}] \\ = \; & p_{i,k_1} p_{k_1,k_2} \cdots p_{k_{m-1},k_m} \end{aligned}$$

$$\begin{aligned} & \mathbb{P}[X_{n+1} = k_1, \ldots, X_{n+m} = k_m | X_n = i] \\ = \; & \frac{\mathbb{P}[X_n = i, X_{n+1} = k_1, \ldots, X_{n+m} = k_m]}{\mathbb{P}[X_n = i]} \\ = \; & \sum_{i_0, \ldots, i_{n-1} \in \mathcal{S}} \frac{\mathbb{P}[X_0 = i_0, \ldots, X_{n-1} = i_{n-1}, X_n = i, X_{n+1} = k_1, \ldots, X_{n+m} = k_m]}{\mathbb{P}[X_n = i]} \\ = \; & \sum_{i_0, \ldots, i_{n-1} \in \mathcal{S}} \mathbb{P}[X_0 = i_0, \ldots, X_{n-1} = i_{n-1}, X_n = i]\mathbb{P}[X_{n+1} = k_1 | X_0 = i_0, \ldots, X_{n-1} = i_{n-1}, X_n = i] \\ & \frac{\times \cdots \times \mathbb{P}[X_{n+m} = k_m | X_0 = i_0, \ldots, X_{n-1} = i_{n-1}, X_n = i, X_{n+1} = k_1, \ldots, X_{n+m-1} = k_{m-1}]}{\mathbb{P}[X_n = i]} \\ = \; & \sum_{i_0, \ldots, i_{n-1} \in \mathcal{S}} \frac{\mathbb{P}[X_0 = i_0, \ldots, X_{n-1} = i_{n-1}, X_n = i]}{\mathbb{P}[X_n = i]} p_{i,k_1} p_{k_1,k_2} \cdots p_{k_{m-1},k_m} \\ = \; & \frac{\mathbb{P}[X_n = i]}{\mathbb{P}[X_n = i]} p_{i,k_1} p_{k_1,k_2} \cdots p_{k_{m-1},k_m} \\ = \; & p_{i,k_1} p_{k_1,k_2} \cdots p_{k_{m-1},k_m} \end{aligned}$$