ISyE 6663 Optimization III

Spring 2011

Assignment 1 (Revision)

Issued: January 18, 2011

Due: January 27, 2011

Problem 1
Consider $\mathcal{S}^1, \mathcal{S}^2 \subset \mathbb{R} \cup \{+\infty\}$. If you need additional assumptions for a result to hold, then state those assumptions, and motivate why the assumptions are needed. Show that:

(1)
$$\inf\{\mathcal{S}^1 + \mathcal{S}^2\} \;=\; \inf \mathcal{S}^1 + \inf \mathcal{S}^2$$

(2)
$$\inf\{\mathcal{S}^1 \cup \mathcal{S}^2\} \;=\; \min\{\inf \mathcal{S}^1, \inf \mathcal{S}^2\}$$

(3)
$$\inf\{\mathcal{S}^1 \cap \mathcal{S}^2\} \;\geq\; \max\{\inf \mathcal{S}^1, \inf \mathcal{S}^2\}$$

Give an example where strict inequality holds.

Problem 2
If you need additional assumptions for a result to hold, then state those assumptions, and motivate why the assumptions are needed.

(1) Consider $f, g : \mathcal{X} \mapsto \mathbb{R}$. Show that

$$\inf_{x\in\mathcal{X}} f(x) + \inf_{x\in\mathcal{X}} g(x) \;\leq\; \inf_{x\in\mathcal{X}}\{f(x) + g(x)\}$$

Give an example where strict inequality holds.

(2) Consider $f : \mathcal{X} \mapsto \mathbb{R}$ and $g : \mathcal{Y} \mapsto \mathbb{R}$. Show that

$$\inf_{x\in\mathcal{X}} f(x) + \inf_{y\in\mathcal{Y}} g(y) \;=\; \inf_{x\in\mathcal{X}, y\in\mathcal{Y}}\{f(x) + g(y)\}$$

(3) Consider $f : \mathcal{X} \times \mathcal{Y} \mapsto \mathbb{R}$. Show that

$$\inf_{x\in\mathcal{X}}\{\inf_{y\in\mathcal{Y}} f(x,y)\} \;=\; \inf_{y\in\mathcal{Y}}\{\inf_{x\in\mathcal{X}} f(x,y)\} \;=\; \inf_{x\in\mathcal{X}, y\in\mathcal{Y}} f(x,y)$$

(4) Consider $f : \mathcal{X} \times \mathcal{Y} \mapsto \mathbb{R}$. Show that

$$\sup_{x\in\mathcal{X}}\{\inf_{y\in\mathcal{Y}} f(x,y)\} \;\leq\; \inf_{y\in\mathcal{Y}}\{\sup_{x\in\mathcal{X}} f(x,y)\}$$

Give an example where strict inequality holds.

Problem 3
Order notation:

(1) Let $f(n)$ and $g(n)$ be asymptotically nonnegative functions. Use the definition of the Ω-notation in the text (beware: other texts often use Θ for the same concept). Show that $\max\{f(n), g(n)\} = \Omega(f(n) + g(n))$.

(2) Show that for any constants a and b, with $b > 0$, $(n + a)^b = \Omega(n^b)$.

(3) Is $2^{n+1} = O(2^n)$? Is $2^{2n} = O(2^n)$? Why or why not?

Problem 4
Assume that $\mathcal{L} \subset \mathbb{R}^n$ is a linear subspace, and that $\mathcal{B}^1$ and $\mathcal{B}^2$ are two bases for $\mathcal{L}$.

(1) Show that $|\mathcal{B}^1| = |\mathcal{B}^2|$, where $|\mathcal{B}|$ denotes the number of elements (cardinality) of set $\mathcal{B}$.

(2) Show that the expression of any element of $\mathcal{L}$ as a linear combination of the elements of $\mathcal{B}^1$ is unique.

(3) Represent a linear combination by its vector of coefficients. Show that the function that maps the expression of an element of $\mathcal{L}$ as a linear combination of the elements of $\mathcal{B}^1$ to the expression of the same element as a linear combination of the elements of $\mathcal{B}^2$ is an invertible linear function. Show how to construct the matrix representation of the abovementioned function and its inverse.

Problem 5
Equivalence of norms in $\mathbb{R}^n$:

(1) Let $\|\cdot\|$ and $|||\cdot|||$ be any two norms on $\mathbb{R}^n$. Show that $\|\cdot\|$ and $|||\cdot|||$ are equivalent, in the sense that there exists a constant $c > 0$ such that $\|x\| \leq c|||x|||$ for all $x \in \mathbb{R}^n$. (Compare this result with the result (A.35) of the text.)

(2) Let $\|\cdot\|$ be any norm on $\mathbb{R}^n$. Consider the topology on $\mathbb{R}^n$ generated by the balls $B(x, r) := \{y \in \mathbb{R}^n : \|y - x\| < r\}$ for $x \in \mathbb{R}^n$ and $r > 0$. Show that any norm $|||\cdot|||$ is a continuous function from $\mathbb{R}^n$ to $\mathbb{R}$.

Problem 6
Consider a symmetric $n \times n$ matrix A with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$. Let $R : \mathbb{R}^n \setminus \{0\} \mapsto \mathbb{R}$ be defined by $R(x) := x^T Ax/x^T x$.

(1) Show that $\max_{x \neq 0} R(x) = \lambda_1$, and $\min_{x \neq 0} R(x) = \lambda_n$.

(2) Show that the maximum is attained by any eigenvector of A corresponding to λ_1, and the minimum is attained by any eigenvector of A corresponding to λ_n.

(3) Show that

$$\lambda_n \|x\|^2 \quad \leq \quad x^T Ax \quad \leq \quad \lambda_1 \|x\|^2$$

for all $x \in \mathbb{R}^n$. (Compare this result with the result on p.599 of the text.)

Problem 7
Let A be an $m \times n$ matrix with Euclidean norm

$$\|A\|_2 \quad := \quad \max_{x \in \mathbb{R}^n \setminus \{0\}} \frac{\|Ax\|_2}{\|x\|_2}$$

(see (A.38) in the text).

(1) Show that all the eigenvalues of $A^T A$ are nonnegative.

(2) Show that $\|A\|_2^2 = \lambda_1(A^T A)$, where $\lambda_1(A^T A)$ denotes the largest eigenvalue of $A^T A$. (Compare this result with the result (A.39b) of the text.)

(3) Suppose that A is a symmetric $n \times n$ matrix with eigenvalues $\lambda_1(A), \ldots, \lambda_n(A)$. Show that

$$\|A\|_2 \quad = \quad \max_{i \in \{1,\ldots,n\}} |\lambda_i(A)|$$

(4) Suppose that A is a symmetric and nonsingular $n \times n$ matrix with eigenvalues $\lambda_1(A), \ldots, \lambda_n(A)$. Show that

$$\|A^{-1}\|_2 \quad = \quad \frac{1}{\min_{i \in \{1,\ldots,n\}} |\lambda_i(A)|}$$

(5) Let the condition number $\kappa(A)$ of A be defined by $\kappa(A) := \|A\|_2 \|A^{-1}\|_2$ (see (A.42) in the text). Suppose again that A is a symmetric and nonsingular $n \times n$ matrix with eigenvalues $\lambda_1(A), \ldots, \lambda_n(A)$. Show that

$$\kappa(A) \quad = \quad \frac{\max_{i \in \{1,\ldots,n\}} |\lambda_i(A)|}{\min_{i \in \{1,\ldots,n\}} |\lambda_i(A)|}$$