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December 14, 2022

Name: _____

ISyE 2027 C
Test 3

Calculators, notes, and books are not allowed. Put your name on both sides of this sheet. Please stop when time is up. You may leave terms like $\binom{52}{5}$ and e^{-2} in your answers.

1. (30 points) Let $k \geq 1$. Suppose $\mathrm{P}\{X = 1\} = \mathrm{P}\{X = -1\} = 1/(2k^2)$ and $\mathrm{P}\{X = 0\} = 1 - 1/k^2$.
   (a) Compute $\mathrm{P}\{X = 1 \mid X \neq 0\}$.
   (b) Compute $\mathrm{P}\{|X| \geq 1\}$.
   (c) Suppose Y is a r.v. with the same mean and variance as X. Obtain a good upper bound on $\mathrm{P}\{|Y| \geq 1\}$.

2. (30 points) An aluminum plant needs 2 rollers. Currently, 2 rollers are in use, and a 3rd roller is being repaired. The length of time until a roller in use fails is exponentially distributed with rate μ. The length of time until the 3rd roller is repaired is exponentially distributed with rate λ. All 3 times are independent.
   (a) What is the probability that it takes t more units of time to repair the 3rd roller given that it has already been worked on for 2 units of time?
   (b) What is the expected time until the first roller failure?
   (c) What is the probability that a roller fails before the 3rd roller is repaired?

3. (30 points) Suppose (X, Y) have joint p.d.f. $f_{X,Y}(s,t) = st/4$ for $0 \leq s \leq 2$ and $0 \leq t \leq 2$.
   (a) Compute the marginal p.d.f. of X.
   (b) Compute $\mathrm{E}[1/(XY)]$.
   (c) Compute $\mathrm{P}\{\max\{X, Y\} \leq 1\}$.

4. (30 points) Suppose $\mathrm{Cov}(X_i, X_i) = 2$ and $\mathrm{Cov}(X_i, X_j) = 1$ for $i \neq j$.
   (a) What is the variance of $3X_1$?
   (b) What is the $\mathrm{Cov}(X_1 - X_2, X_1 + X_2)$?
   (c) What is the variance of $X_1 + X_2 + X_3$?

5. (30 points) Let X_i be the number of people that arrive on day i for $i = 1, 2, \ldots, 25$. Assume that $X_1, \ldots, X_{25}$ are i.i.d. Poisson random variables with mean 4. Let S be the total number of people that arrive on the 25 days.
   (a) Compute $\mathrm{P}\{X_1 + X_2 = 1\}$.
   (b) Accurately approximate the probability that 105 or fewer people arrive during those 25 days.
   (c) Compute $\mathrm{E}[z^{X_1}]$.