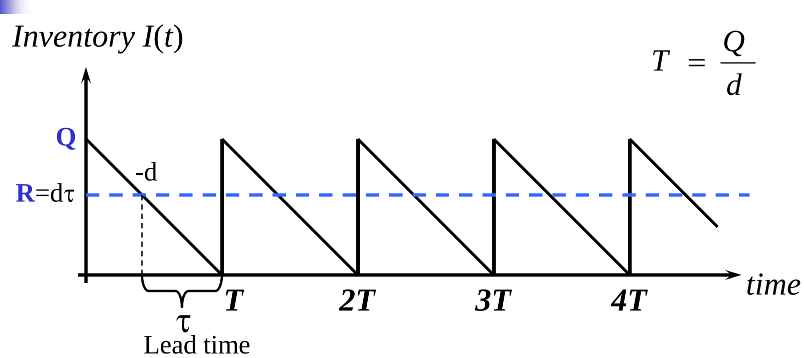
Lot size/Reorder level (Q,R) Models

ISYE 3104 – Fall 2013

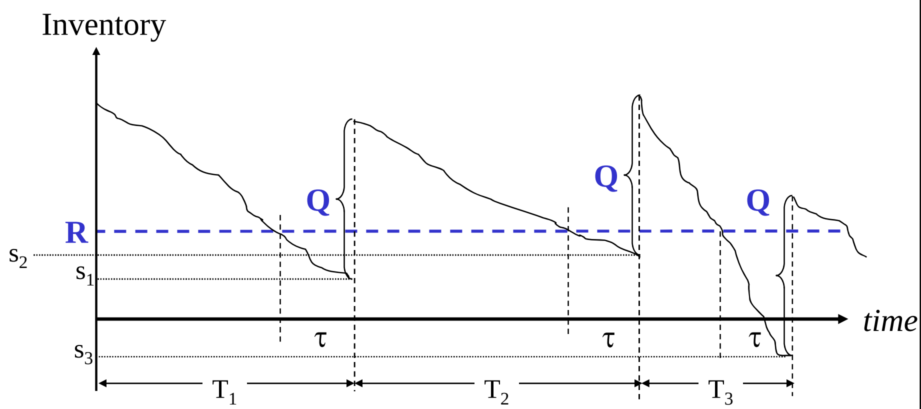
Recap: Basic EOQ



- Place an order when the inventory level is R. The order arrives after τ time periods
  - Q was the only decision variable
  - R could be computed easily because D was deterministic

Uncertain demand

- Both Q and R are decision variables
- Cycle time is no longer constant!



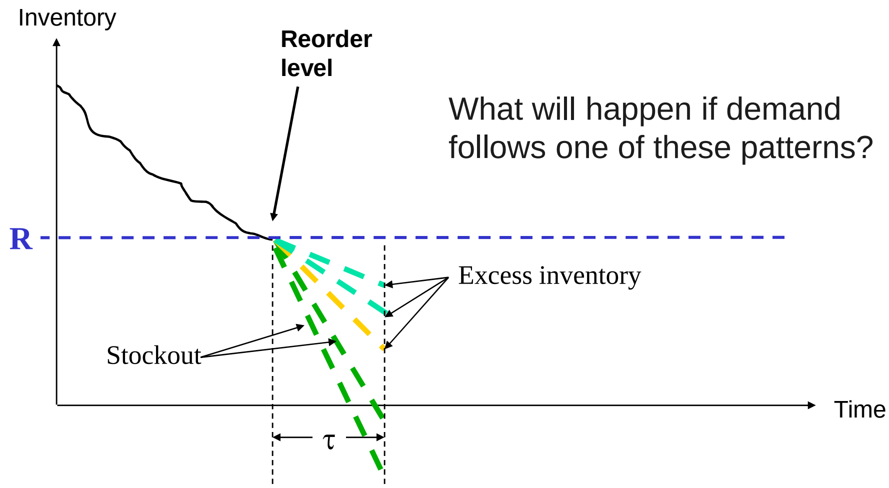
(Q,R) Decisions

- We choose R to meet the demand during lead time
  - Service levels: Protect against uncertainties in demand (or lead time)
  - Balance the costs: stock-outs and inventory
- Tradeoff in Q: Fixed cost versus holding cost

Objective:

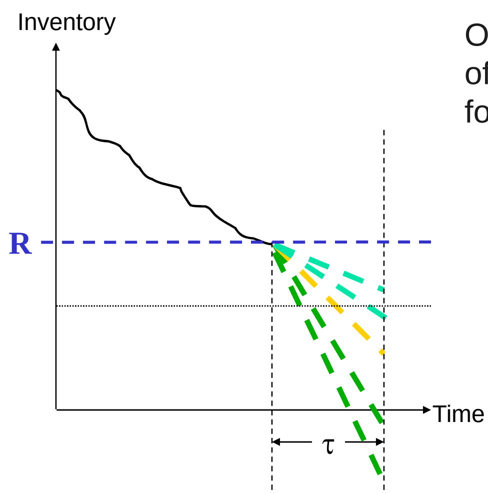
- Minimize
  - fixed cost + holding cost + stockout (backorder) cost

Demand during lead time

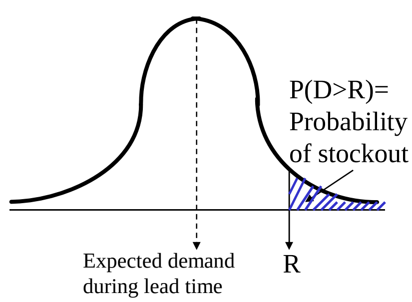


What will happen if demand follows one of these patterns?

Demand during lead time



Often the probability distribution of demand during lead time follows a Normal pattern



(Q,R) Model Assumptions

- Continuous review
- Demand is random and stationary. Expected demand is d per unit time.
- Lead time is τ
- Costs
  - K: Setup cost per order
  - h: Holding cost per unit per unit time
  - c: Purchase price (cost) per unit
  - p: Stockout (backorder) cost per unit
- Demand during lead time is a continuous random variable D with
  - pdf (density function) f(x) and cdf (distribution function) F(x)
  - Mean=μ and standard deviation=σ

(Q,R) Model – Expected total cost per unit time

$$C(Q) = \overbrace{h\left(s + \frac{Q}{2}\right)}^{\text{Holding cost}} + \overbrace{\frac{K}{T}}^{\text{Fixed cost}} + \overbrace{p\frac{n(R)}{T}}^{\text{Shortage cost}} \qquad \text{Recap}: T = \frac{Q}{d}$$

$$\begin{aligned} s &= \text{Average inventory level before an order arrives} \\ &= \text{(Reorder level)} - \text{(expected demand during leadtime)} = R\text{-}\mu \end{aligned}$$

$$n(R) = \text{Expected shortage per cycle}$$

$$D > R \Rightarrow \text{shortage} = D\text{-}R$$

$$D < R \Rightarrow \text{shortage} = 0$$

Standard loss function

$$n(R) = \int_0^R 0\, f(x)dx + \int_R^\infty (x-R) f(x)dx = \int_R^\infty (x-R) f(x)dx = \sigma L(z)$$

(Q,R) Model – Expected total cost per unit time

$$C(Q) = \overbrace{h\left(s+\frac{Q}{2}\right)}^{\text{Holding cost}} + \overbrace{\frac{K}{T}}^{\text{Fixed cost}} + \overbrace{p\frac{n(R)}{T}}^{\text{Shortage cost}} \qquad \text{Recap}: T = \frac{Q}{d}$$

s = Average inventory level before an order arrives

= (Reorder level) – (expected demand during leadtime) = R-μ

$n(R)$ = Expected shortage per cycle

$D > R \Rightarrow$ shortage $= D$-R

$D < R \Rightarrow$ shortage $= 0$

Standard loss function

$$n(R) = \int_0^R 0\, f(x)dx + \int_R^\infty (x-R)\, f(x)dx = \boxed{\int_R^\infty (x-R)\, f(x)dx} = \sigma L(z)$$

(Q,R) Model – Expected total cost per unit time

$$C(Q) = \overbrace{h\left(s+\frac{Q}{2}\right)}^{\text{Holding cost}} + \overbrace{\frac{K}{T}}^{\text{Fixed cost}} + \overbrace{p\frac{n(R)}{T}}^{\text{Shortage cost}} \qquad \text{Recap}: T = \frac{Q}{d}$$

s = Average inventory level before an order arrives

= (Reorder level) – (expected demand during leadtime) = R-μ

$n(R)$ = Expected shortage per cycle

Same expression as the “expected number of stockouts” in the newsvendor model (Q replaced by R)

$D > R \Rightarrow$ shortage $= D$-R

$D < R \Rightarrow$ shortage $= 0$

Standard loss function

$$n(R) = \int_0^R 0\, f(x)dx + \int_R^\infty (x-R)\, f(x)dx = \boxed{\int_R^\infty (x-R)\, f(x)dx} = \sigma L(z)$$

(Q,R) Model – Expected total cost per unit time

$$G(Q) = \text{Holding cost} + \text{Fixed cost} + \text{Shortage cost}$$

$$= h\left(\frac{Q}{2} + R - d\tau\right) + K\frac{d}{Q} + p\frac{d\,n(R)}{Q}$$

$$\frac{\partial G}{\partial Q} = \frac{h}{2} - \frac{Kd}{Q^2} - \frac{pdn(R)}{Q^2} = 0 \Rightarrow \frac{h}{2} = \frac{d[K + pn(R)]}{Q^2} \Rightarrow$$

$$Q = \sqrt{\frac{2d[K + pn(R)]}{h}}$$

$$\frac{\partial G}{\partial R} = h - \frac{pd(1 - F(R))}{Q} \Rightarrow \quad 1 - F(R) = \frac{Qh}{pd}$$

(Q,R) Model – Expected total cost per unit time

$$C(Q) = \text{Holding cost} + \text{Fixed cost} + \text{Shortage cost}$$

$$= h\left(\frac{Q}{2} + R - d\tau\right) + K\frac{d}{Q} + p\frac{d\,n(R)}{Q}$$

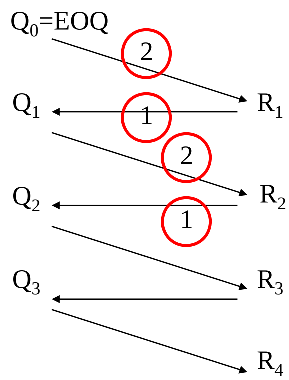
Optimal solution:

1 $\boxed{Q = \sqrt{\frac{2d[K + p\,n(R)]}{h}}}$ $\boxed{F(R) = 1 - \frac{Qh}{pd}}$ 2

How do we pull Q and R from these equations? → Solve iteratively!!

Solving for optimal Q and R

- Start with a Q_0 value and iterate until the Q values converge



Remember: To find *Q*, you need $n(R) = \sigma L(z)$ Lookup for *z* in the Normal tables

Example – Rainbow Colors

- Rainbow Colors paint store uses a (Q,R) inventory system to control its stock levels. For a popular eggshell latex paint, historical data show that the distribution of monthly demand is approximately Normal, with mean 28 and standard deviation 8. Replenishment lead time for this paint is about 14 weeks. Each can of paint costs the store \$6. Although excess demands are backordered, each unit of stockout costs about \$10 due to bookkeeping and loss-of-goodwill. Fixed cost of replenishment is \$15 per order and holding costs are based on a 30% annual interest rate.
  - What is the optimal lot size (order quantity) and reorder level?
  - What is the expected inventory level (safety stock) just before an order arrives?

Example – Rainbow Colors

- Input
  - Monthly demand Normal mean=28 std.dev.=8
  - τ=14 weeks
  - c=\$6, p=\$10, K=\$15
  - h=ic=(0.3)(6)=\$1.8/unit/year
- Computed input
  - d= ? (Expected annual demand)
  - Expected demand during lead time

$$\mu = ?$$

  - Variance of demand during lead time

$$\sigma^2 = ?$$

Example – Rainbow Colors

- Input
  - Monthly demand Normal mean=28 std.dev.=8
  - τ=14 weeks
  - c=\$6, p=\$10, K=\$15
  - h=ic=(0.3)(6)=\$1.8/unit/year
- Computed input
  - d=(28)(12)=336 units/year (Expected annual demand)
  - Expected demand during lead time

$$\boxed{\mu} = \frac{336 \text{ units/year}}{52 \text{ weeks/year}} \times (14 \text{ weeks}) = \boxed{90} \text{ units}$$

  - Variance of demand during lead time

$$\text{Annual variance} = (12)(8^2) = 768$$

$$\text{Variance of lead time demand} = 768 \times \frac{14}{52} = 206.77 \Rightarrow \boxed{\sigma = 14.38}$$

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- Input
  - Monthly demand Normal mean=28 std.dev.=8
  - τ=14 weeks
  - c=\$6, p=\$10, K=\$15
  - h=ic=(0.3)(6)=\$1.8/unit/year
- Computed input
  - d=(28)(12)=336 units/year (Expected annual demand)
  - Expected demand during lead time

$$\boxed{\mu}=\frac{(28)(12)\text{ units/year}}{52\text{ weeks/year}}\times(14\text{ weeks})=\boxed{90}\text{ units}$$

  - Variance of demand during lead time

$$\text{Annual variance}=(12)(8^2)=768$$

$$\text{Variance of lead time demand}=768\times\frac{14}{52}=206.77\Rightarrow\boxed{\sigma=14.38}$$

Example – Rainbow Colors

- Input
  - Monthly demand Normal mean=28 std.dev.=8

As the lead time increases, so does the mean and variance of demand during lead time

Shorter lead times ⇔ Less variability of demand during lead time

$$\boxed{\mu}=\frac{(28)(12)\text{ units/year}}{52\text{ weeks/year}}\times(14\text{ weeks})=\boxed{90}\text{ units}$$

  - Variance of demand during lead time

$$\text{Annual variance}=(12)(8^2)=768$$

$$\text{Variance of lead time demand}=768\times\frac{14}{52}=206.77\Rightarrow\boxed{\sigma=14.38}$$

Example – Rainbow Colors

- Iteration 0: Compute EOQ

$$\boxed{Q_0} = \sqrt{\frac{2Kd}{h}} = \sqrt{\frac{(2)(15)(336)}{1.8}} = \boxed{75}$$

Example – Rainbow Colors

- Iteration 1: Compute R_1 (given Q_0) and then compute Q_1 (given R_1)

$$F(R_1) = 1 - \frac{Q_0 h}{pd} = 1 - \frac{(75)(1.8)}{(10)(336)} = 0.96 = \Phi(z) \Rightarrow z = 1.75 \leftarrow$$

From standard Normal table

$$\text{Recap: } F(R) = P(D \le R) = P\left(\underbrace{\frac{D-\mu}{\sigma}}_{Z} \le \underbrace{\frac{R-\mu}{\sigma}}_{z}\right) = P(Z \le z) = \Phi(z)$$

Standard Normal

$$z = \frac{R-\mu}{\sigma} \Rightarrow R = \sigma z + \mu \Rightarrow \boxed{R_1} = (14.38)(1.75) + 90 \approx \boxed{115}$$

Safety Stock

Expected Demand during Lead time

Q_0=75

2

R_1=115

Q_1

1

Example – Rainbow Colors

- Iteration 1 (continued): Compute Q_1 (given R_1)

$$Q = \sqrt{\frac{2d[K + p\,n(R)]}{h}} \quad 1$$

$$n(R_1) = \sigma L(z) = (14.38)(0.0162) = 0.233$$

$$\boxed{Q_1} = \sqrt{\frac{(2)(336)[15 + (10)(0.233)]}{1.8}} \approx \boxed{80}$$

Q_0=75 → 2 → R_1=115

Q_1=80 ← 1 ← R_1=115

Q_1=80 → 2 → R_2

Q_0 and Q_1 not close, continue iterations

Example – Rainbow Colors

- Iteration 2: Compute R_2 (given Q_1) and then compute Q_2 (given R_2)

$$F(R_2) = 1 - \frac{Q_1 h}{pd} = 1 - \frac{(80)(1.8)}{(10)(336)} = 0.957 = \Phi(z) \;\Rightarrow\; z = 1.72$$

$$R = \sigma z + \mu \Rightarrow \boxed{R_2} = (14.38)(1.72) + 90 \approx \boxed{115}$$

Q_0=75 → R_1=115

Q_1=80 ← R_1=115

Q_1=80 → R_2=115

STOP! R values converged, optimal $\boxed{(Q,R)=(80,115)}$

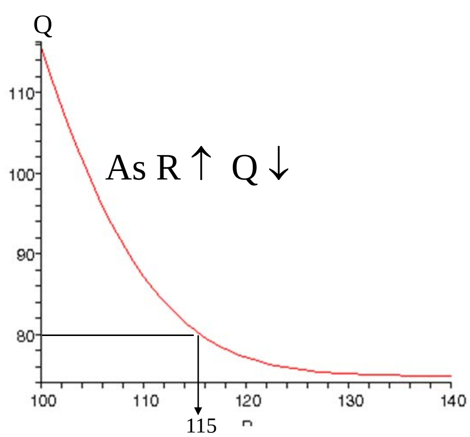
Example – Rainbow Colors

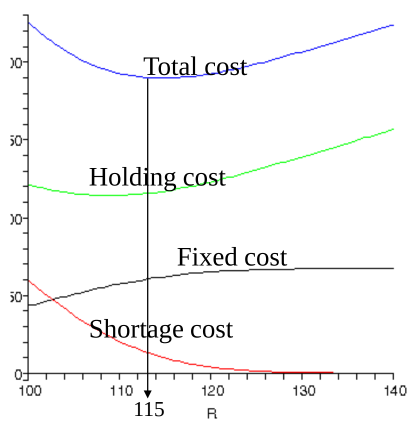
- (Q,R)=(80,115)
  - Reorder level is larger than expected demand during lead time. Why?
  - Optimal order quantity is larger than EOQ. Why?
- Safety stock
  - s=R-μ=115-90=25 units

Weekly demand~5.64
Avg cycle time T=Q/d=80/6.64=12.38 weeks.
Lead time=14 weeks. Cycle time shorter than lead time

Impact of R on the costs and Q

R↑ inventory↑, therefore Q↓





Optimal R as a function of Q

$$F(R) = 1 - \frac{Qh}{pd}$$

As the order quantity increases, the reorder level decreases

Q ↑ holding cost ↑ and setup cost ↓, therefore R ↓ so that we can bring the holding cost ↓, although a lower R means shortage cost ↑

The Impact of Holding Cost on the Optimal (Q,R)

As h goes up, both Q and R go down, but in this example Q drops at a faster rate!

i	Q	R
0.2	97	116
0.3	81	115
0.4	71	114
0.5	64	113
0.6	59	112
0.7	55	111

The Impact of Stockout Cost on the Optimal (Q,R)

As p goes up, Q goes ??? and R goes ???

The Impact of Stockout Cost on the Optimal (Q,R)

As p goes up, Q goes down and R goes up!

p	Q	R
2	84	101
6	81	111
10	81	115
14	81	117
18	80	118
22	80	120

Summary: (Q,R) Models

- Balance between holding cost, setup/fixed cost, and shortage cost
  - To save on the shortage cost, we want large R
  - To save on the holding cost, we want small Q and small R
  - To save on the fixed cost, we want large Q

Choose Q and R to strike a good balance among these three costs!!!

Service Levels in (Q,R) Models

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Service objectives

- **Type I** service level (α)
  - The proportion of cycles in which no stockouts occur
  - Example: 90% Type I service level → There are no stockouts in 9 out of 10 cycles (on average)
- **Type II** service level (fill rate, β)
  - Fraction of demand satisfied on time

Service objectives - Example

Order cycle	Demand	Stock-outs
1	180	0
2	75	0
3	235	45
4	140	0
5	180	0
6	200	10
7	150	0
8	90	0
9	160	0
10	40	0
TOTAL:	1450	55

Fraction of periods with no stock-outs = 8/10
→Type I service = 80%
α = 0.8

Fraction of demand satisfied on time = (1450-55)/1450=0.96
→ Type II service = 96%
β = 0.96

In general, is it easier to achieve an x% Type I service or Type II service level?

Type I service level, α

α : Long-run average proportion of cycles with no stock-outs
α : Probability of having no stock-outs in a cycle
α : Probability of having no stock-outs during lead time
α : Probability that demand during lead time is less than R !!!

$$\alpha = P(D \le R)$$

$$\text{Recap}: \quad P(D \le R) = P\left(\frac{D-\mu}{\sigma} \le \frac{R-\mu}{\sigma}\right) = P(Z \le z) = \Phi(z) = \alpha$$

- Set Q=EOQ
- Find z that satisfies $\Phi(z) = \alpha$
- Set $R = \sigma z + \mu$ (safety stock + expected demand during lead time)

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Type I service level, α

α : Long-run average proportion of cycles with no stock-outs

α : Probability of having no stock-outs in a cycle

α : Probability of having no stock-outs during lead time

α : Pr … R !!!

$\alpha = P$ …

Why is Q=EOQ optimal in this case?

Recal… $= \alpha$

- Set Q=EOQ
- Find z that satisfies $\Phi(z) = \alpha$
- Set $R = \sigma z + \mu$ (safety stock + expected demand during lead time)

Example – Rainbow Colors

- Rainbow Colors paint store uses a (Q,R) inventory system to control its stock levels. For a popular eggshell latex paint, historical data show that the distribution of monthly demand is approximately Normal, with mean 28 and standard deviation 8. Replenishment lead time for this paint is about 14 weeks. Each can of paint costs the store $6. Although excess demands are backordered, each unit of stockout costs about $10 due to bookkeeping and loss-of-goodwill. Fixed cost of replenishment is $15 per order and holding costs are based on a 30% annual interest rate.
  - What is the optimal lot size (order quantity) and reorder level?
  - What is the expected inventory level (safety stock) just before an order arrives?

Example – Rainbow Colors

- Rainbow Colors is not sure whether the \$10 estimate for the shortage cost is accurate. Hence, they decided to use a service level approach. What are the optimal (Q,R) values if they want to
  - achieve no stockouts in 90% of the order cycles?
  - satisfy 90% of the demand on time?

Example – Rainbow Colors

- Input
  - Monthly demand Normal mean=28 std.dev.=8
  - τ=14 weeks, c=\$6, K=\$15
  - h=Ic=(0.3)(6)=\$1.8/unit/year
  - α = 0.9 or β = 0.9
- Computed input
  - d=(28)(12)=336 units/year (Expected annual demand)
  - Expected demand during lead time

$$\boxed{\mu} = \frac{(28)(12)\text{ units / year}}{52\text{ weeks / year}} \times (14\text{ weeks}) = \boxed{90}\text{ units}$$

  - Variance of demand during lead time

$$\text{Annual variance} = (12)(8^2) = 768$$

$$\text{Variance of lead time demand} = 768 \times \frac{14}{52} = 206.77 \Rightarrow \boxed{\sigma = 14.38}$$

Rainbow Colors – Type I service

Find (Q,R) to have 90% Type I service level

- Q=EOQ=75
- $\Phi(z) = \alpha = 0.9$ → z=1.28
- R= $\sigma z+\mu$ → R=(14.38)(1.28)+90=108
- For 90% Type I service level (Q,R)=(75,108)

Remember: With unit penalty cost of $10, we found (Q,R)=(80,115). What is the Type I service level that corresponds to (Q,R)=(80,115)?

R= $\sigma z+\mu$ → 115=(14.38)z+90 → z=1.7385

$\Phi(1.7385)=0.96$ →

96% Type I service level when (Q,R)=(80,115)

Type II service level

- β : Fraction of demand met on time
- 1- β : Fraction of demand not met on time (stock-outs)

Recap:

$$\text{Expected \# of stockouts per unit time} = \frac{n(R)}{T} = \frac{d\,n(R)}{Q} \qquad \left(\text{since } T = \frac{Q}{d}\right)$$

$$1-\beta = \frac{\text{Expected \# of stockouts per unit time}}{\text{Expected demand per unit time}} = \frac{n(R)}{Q} \Rightarrow \boxed{1-\beta = \frac{n(R)}{Q}} \quad 4$$

With this information, for a given (Q,R), we can compute β.

Rainbow Colors

- For 90% Type I service level we found (Q,R)=(75,108)
- What is the Type II service level which corresponds to this policy?

The same policy results in 90% Type I service and 99% Type II service!!

$$\frac{R-\mu}{\sigma}=\frac{108-90}{14.38}=1.25=z$$

$$n(R)=\sigma L(z)=\sigma L(1.25)=(14.38)(0.0506)=0.7276$$

$$1-\beta=\frac{n(R)}{Q}=\frac{0.7276}{75}=0.0097 \Rightarrow \beta \approx 0.99$$

Finding the optimal (Q,R) for a desired Type II service level

Remember: Optimal solution when we have stock-out cost p:

1 $\boxed{Q=\sqrt{\frac{2d[K+p\,n(R)]}{h}}}$ $\boxed{F(R)=1-\frac{Qh}{pd}}$ 2

Finding the optimal (Q,R) for a desired Type II service level

Optimal solution when we have stock-out cost p:

1 $\boxed{Q = \sqrt{\frac{2d[K + p\,n(R)]}{h}}}$ $\boxed{F(R) = 1 - \frac{Qh}{pd}}$ 2

From 2 : $\boxed{p = \frac{Qh}{d(1 - F(R))}}$ 5 **Imputed shortage cost**

Substitute p into 1:

$$\boxed{Q = \frac{n(R)}{1 - F(R)} + \sqrt{\frac{2Kd}{h} + \left(\frac{n(R)}{1 - F(R)}\right)^2}} \quad 3$$

To be solved simultaneously with $\boxed{n(R) = (1 - \beta)Q}$ 4

Impact of service level β on R

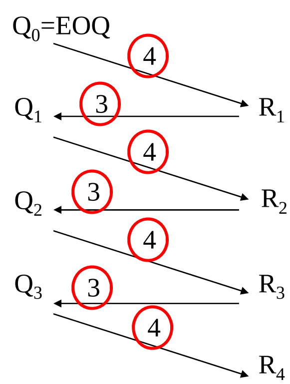
- For a given Q
  - As β↑ n(R)=(1- β)Q↓ i.e., R↑

As the service level ↑, the reorder level ↑ as well

Finding the optimal (Q,R) for a desired Type II service level

$$Q = \frac{n(R)}{1-F(R)} + \sqrt{\frac{2Kd}{h} + \left(\frac{n(R)}{1-F(R)}\right)^2} \quad (3)$$

$$n(R) = (1-\beta)Q \quad (4)$$



- Start with a Q_0 value and iterate until the Q values (or the R values) converge

Example – Rainbow Colors

- Iteration 0: Compute EOQ

$$\boxed{Q_0} = \sqrt{\frac{2Kd}{h}} = \sqrt{\frac{(2)(15)(336)}{1.8}} = \boxed{75}$$

Example – Rainbow Colors

- Iteration 1: Compute R_1 (given Q_0) and then compute Q_1 (given R_1)

$$n(R_1) = (1-\beta)Q_0 = (1-0.9)(75) = 7.5 = \sigma L(z) \Rightarrow$$

4

$$L(z) = 0.5216 \Rightarrow z = -0.22$$

$$R_1 = \sigma z + \mu = (14.38)(-0.22) + 90 = 86.83 \approx 87$$

To find Q_1 we need 1-$F(R_1)$. Look at the Normal table.

$$1 - F(-0.22) = F(0.22) = 0.5871$$

$$Q_1 \approx 89$$

3

Q_0=75

4

Q_1=89 3 R_1=87

Example – Rainbow Colors

- Iteration 2: Compute R_2 (given Q_1) and then compute Q_2 (given R_2)

Example – Rainbow Colors

- Iteration 2: Compute R_2 (given Q_1) and then compute Q_2 (given R_2)

$$n(R_2) = (1-\beta)Q_1 = (1-0.9)(89) = 8.9 = \sigma L(z) \Rightarrow$$

4

$$L(z) = 0.619 \Rightarrow z = -0.38$$

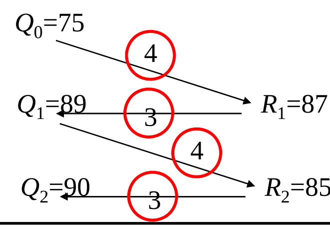
$$R_2 = \sigma z + \mu = (14.38)(-0.38) + 90 = 84.5 \approx 85$$

To find Q_1 we need 1-$F(R_1)$. Look at the Normal table.

$$1 - F(-0.38) = F(0.38) = 0.648$$

$$Q_2 \approx 90$$

3



Example – Rainbow Colors

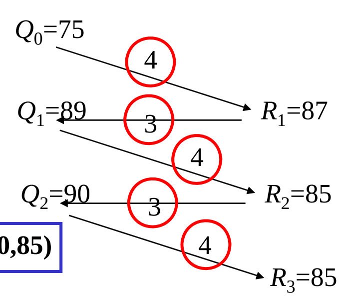
- Iteration 3: Compute R_3 (given Q_2) and then compute Q_3 (given R_3)

$$n(R_3) = (1-\beta)Q_2 = (1-0.9)(85) = 8.5 = \sigma L(z) \Rightarrow$$

4

$$L(z) = 0.591 \Rightarrow z = -0.34$$

$$R_3 = \sigma z + \mu = (14.38)(-0.34) + 90 = 85.1 \approx 85$$



STOP! R values converged, optimal $\boxed{(Q,R)=(90,85)}$