Sequence Independent Lifting in Mixed Integer Programming [1]

Zonghao Gu
George L. Nemhauser
Martin W.P. Savelsbergh

Georgia Institute of Technology
School of Industrial and Systems Engineering
Atlanta, GA 30332-0205

Abstract

We investigate lifting, i.e., the process of taking a valid inequality for a polyhedron and extending it to a valid inequality in a higher dimensional space. Lifting is usually applied sequentially, that is, variables in a set are lifted one after the other. This may be computationally unattractive since it involves the solution of an optimization problem to compute a lifting coefficient for each variable. To relieve this computational burden, we study sequence independent lifting, which only involves the solution of one optimization problem. We show that if a certain lifting function is superadditive, then the lifting coefficients are independent of the lifting sequence. We introduce the idea of valid superadditive lifting functions to obtain good aproximations to maximum lifting. We apply these results to strengthen Balas' lifting theorem for cover inequalities and to produce lifted flow cover inequalities for a single node flow problem.

1 Introduction

This paper investigates a general principle, called *lifting*, which is the process of constructing, from a given valid inequality for a low dimensional polydedron, a valid inequality for a higher dimensional polyhedron. When the lifting coefficients are maximum, low dimensional facets are turned into higher dimensional facets. The idea of lifting was introduced by Gomory [1969] in the context of the group problem. Its computational possibilities were emphasized in Padberg [1973], and the approach was generalized by Wolsey [1976], Zemel [1978], and Balas and Zemel [1978].

Lifting is usually applied sequentially; variables in a set are lifted one after the other and a separate optimization problem has to be solved to determine each lifting coefficient. The resulting inequality depends on the order in which the variables are lifted. A better lifting coefficient for a given variable is obtained if the variable is lifted earlier in the

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sequence. In sequence independent lifting, all of the coefficients can be obtained by solving a single optimization problem, and the resulting inequality is independent of the order in which the variables are lifted.

Sequential lifting has been instrumental to the success of branch-and-cut algorithms for 0-1 integer programs (BIPs) based on cover inequalities, see e.g., Crowder, Johnson, and Padberg [1983] and Gu, Nemhauser, and Savelsbergh [1998]. The idea is to find simple valid inequalities (cover inequalities) from individual rows (knapsack inequalities) of the problem that are violated by LP optimal solutions, and then to strengthen these cuts by lifting.

Balas [1975] gave a specific sequence independent formula for the lifting coefficients for a cover inequality. The coefficients obtained from the formula do not always yield maximum lifting coefficients, but they can be computed very fast and yield valid inequalities.

For BIPs, Wolsey [1977] proved that if a lifting function is superadditive, then the maximum lifting coefficients are independent of the lifting order. Gu [Gu 1994] generalized Wolsey's result to mixed 0-1 integer programs (MBIPs), and to lifting several variables simultaneously. For MBIPs the computation of lifting coefficients is typically much more complex than for BIPs. Therefore, the role of sequence independent lifting as a way to relieve the computational burden is especially important.

Gu, Nemhauser, and Savelsbergh [1996] show that lifting can be done effectively for MBIPs. To benefit from the computational advantages of superadditive lifting functions, they relax the true lifting function to an approximate lifting function that is superadditive and use this approximate lifting function to compute lifting coefficients. Marchand and Wolsey [1997] take a slightly different approach. They group variables to be lifted in such a way that the lifting function associated with each group is superadditive. Both of these papers have shown that sequence independent lifting is an important tool in the solutions of MBIPs.

In this paper, we present the fundamental concepts and we prove the basic theorems related to sequence independent lifting. Some of these theorems have been stated in Gu, Nemhauser, and Savelsbergh [1996], but no proofs were presented. In addition, we use the theory of superadditive lifting to strengthen Balas' [1975] result on sequence independent lifting of cover inequalities, and we give a new derivation of a result by Pochet [1993] which gives a complete characterization of all facet inducing lifted flow cover inequalities for single node flow models with only outflow arcs.

In Section 2, we discuss the essential ideas of lifting. In Section 3, we present the fundamental result for MBIPs that superadditive lifting functions lead to sequence independent lifting. We also show how this result can be exploited by working with relaxed lifting. Applications of sequence independent lifting results are given in Sections 4 and 5.

2 Lifting

Consider the set of feasible points for a mixed 0-1 integer program given by

$$X = \{x \in R_+^{|N|} : \sum_{j \in N} a_j x_j \le d,$$
$$\sum_{j \in C_k} w_j x_j \le r_k,\ k = 0, \ldots, t,$$
$$x_j \in \{0,1\}, j \in I \subseteq N\}.$$

Here $\{C_k : k = 0, ..., t\}$ is a partition of N, $a_j,\ j \in N$, and d are $m \times 1$ and $w_j, j \in N$, and r_k are $m_k \times 1$. We assume that a_j, d, and r_k, but not necessarily w_j, are nonnegative.

Initially, we consider the subset of X with $x_j = 0$ for $j \in N \setminus C_0$ given by

$$X^0 = \{x \in R_+^{|C_0|} : \sum_{j \in C_0} a_j x_j \le d,$$
$$\sum_{j \in C_0} w_j x_j \le r_0,$$
$$x_j \in \{0,1\},\ j \in I \cap C_0\}.$$

Let

$$0 \le \alpha_0 - \sum_{j \in C_0} \alpha_j x_j \tag{1}$$

be an arbitrary valid inequality for X^0. We want to construct a valid inequality for X of the form

$$0 \le \alpha_0 - \sum_{0 \le k \le t} \sum_{j \in C_k} \alpha_j x_j. \tag{2}$$

To construct such an inequality, we start with (1) and lift the variables in $N \setminus C_0$. Without loss of generality, we assume that the sets of variables $C_1, \ldots, C_t$ are lifted sequentially in that order and that the variables within the sets $C_1, \ldots, C_t$ are lifted simultaneously. Note that this contains as special cases simultaneous lifting of all variables and sequential lifting of all variables. The intermediate sets of feasible points X^i for $i = 1, ..., t$ are defined by

$$X^i = \{x \in R_+^{\sum_{0 \le k \le i} |C_k|} : \sum_{0 \le k \le i} \sum_{j \in C_k} a_j x_j \le d,$$
$$\sum_{j \in C_k} w_j x_j \le r_k,\ k = 0, \ldots, i,$$
$$x_j \in \{0,1\},\ j \in I \cap (\cup_{k=0}^{i} C_k)\}.$$

Note that if we extend X^i to X by setting $x_j = 0$ for $j \in \cup_{k=i+1}^t C_k$, then $X^{i-1} \subseteq X^i$ for $i = 1, ..., t$ and $X^t = X$.

The **lifting problem** associated with C_i, given a valid inequality

$$0 \leq \alpha_0 - \sum_{0 \leq k < i} \sum_{j \in C_k} \alpha_j x_j \tag{3}$$

for X^{i-1}, is to find α_j for $j \in C_i$ such that

$$\sum_{j \in C_i} \alpha_j x_j \leq \alpha_0 - \sum_{0 \leq k < i} \sum_{j \in C_k} \alpha_j x_j \tag{4}$$

is a valid inequality for X^i.

Let $Z = [0, d]$. Furthermore, for $z \in Z$ let

$$\begin{aligned} h_i(z) = \quad & \max \quad \sum_{j \in C_i} \alpha_j x_j \\ & \text{s.t.} \quad \sum_{j \in C_i} a_j x_j = z \\ & \qquad \sum_{j \in C_i} w_j x_j \leq r_i \\ & \qquad x_j \in \{0,1\},\ j \in C_i \cap I,\ x \in R_+^{|C_i|} \end{aligned} \tag{5}$$

and let

$$\begin{aligned} f_i(z) = \quad & \min \alpha_0 - \sum_{0 \leq k < i} \sum_{j \in C_k} \alpha_j x_j \\ & \text{s.t.} \quad \sum_{0 \leq k < i} \sum_{j \in C_k} a_j x_j \leq d - z \\ & \qquad \sum_{j \in C_k} w_j x_j \leq r_k,\ k = 0, \ldots, i-1 \\ & \qquad x_j \in \{0,1\},\ j \in I \cap (\cup_{k=0}^{i-1} C_k),\ x \in R_+^{\sum_{k=0}^{i-1} |C_k|}. \end{aligned} \tag{6}$$

Note that (6) is always feasible since $x = 0$ is a feasible point. However, (5) may be infeasible, in which case we set $h_i(z) = -\infty$.

Proposition 1 *Inequality (4) is valid for X^i for any choice of α_j for $j \in C_i$ such that $h_i(z) \leq f_i(z)$ for all $z \in Z$.*

Proof. The proposition is an immediate consequence of the definition of $h_i(z)$ and $f_i(z)$. $\square$

When α_j for $j \in C_i$ are such that $h_i(z) = f_i(z)$ has $|C_i|$ solutions $x^1, x^2, \ldots, x^{|C_i|}$ such that the components in C_i of $x^1, x^2, \ldots, x^{|C_i|}$ are linearly independent, we say that the lifting is *maximal*.

Theorem 1 *If $conv(X^{i-1})$ and $conv(X^i)$ are full dimensional, (3) defines a facet of $conv(X^{i-1})$ and $\alpha_0 \neq 0$, then (4) defines a facet of $conv(X^i)$ if and only if the lifting is maximal.*

Proof. For notational convenience, let $x \in X^i$ be denoted by $x = (y, w)$, where y refers to the variables in $\cup_{0 \le k < i} C_k$ and w refers to the variables in C_i. Since (3) defines a facet of $conv(X^{i-1})$, there are $p = \sum_{0 \le k < i} |C_k|$ points $\bar{y}^1, \ldots \bar{y}^p \in X^{i-1}$ which satisfy (3) at equality and are affinely independent. Let $\bar{x}^j = (\bar{y}^j, 0)$ for $j = 1, ..., p$, then $\bar{x}^j \in X^i$ and $\bar{x}^j$ satisfies (4) at equality for $j = 1, ..., p$. Let $q = |C_i|$. Since the lifting is maximal $h_i(z) = f_i(z)$ has q solutions $x^1, x^2, \ldots, x^q$ such that the components in C_i of $x^1, x^2, \ldots, x^q$ are linearly independent. Then $x^1 - \bar{x}^p, \ldots, x^q - \bar{x}^p, \bar{x}^1 - \bar{x}^p, \ldots, \bar{x}^{p-1} - \bar{x}^p$, i.e. $(y^1 - \bar{y}^p, w^1), \ldots, (y^q - \bar{y}^p, w^q), (\bar{y}^1 - \bar{y}^p, 0), \ldots, (\bar{y}^{p-1} - \bar{y}^p, 0)$ are linearly independent, since $w^1, \ldots, w^q$ are linearly independent and $y^1 - \bar{y}^p, \ldots, \bar{y}^{p-1} - \bar{y}^p$ are linearly independent. Thus X^i has $p + q$ affinely independent points $x^1, \ldots, x^q, \bar{x}^1, \ldots, \bar{x}^p$ satisfying (4) at equality. Hence (4) defines a facet of $conv(X^i)$.

Now suppose that (4) defines a facet of $conv(X^i)$. Since $\alpha_0 \neq 0$, any set of affinely independent points satisfying (3) or (4) at equality are linearly independent. Hence, there are $r = p + q$ linearly independent points $x^1, \ldots, x^r \in X^i$ satisfying (4) at equality. Therefore

$$\begin{vmatrix} y_1^1 & \cdots & y_p^1 & w_1^1 & \cdots & w_q^1 \\ y_1^2 & \cdots & y_p^2 & w_1^2 & \cdots & w_q^2 \\ & & \cdots & \cdots & & \\ y_1^r & \cdots & y_p^r & w_1^r & \cdots & w_q^r \end{vmatrix} \neq 0.$$

Thus there exist q rows $i_1, \ldots, i_q$ such that

$$\begin{vmatrix} w_1^{i_1} & \cdots & w_q^{i_1} \\ w_1^{i_2} & \cdots & w_q^{i_2} \\ & \cdots & \\ w_1^{i_q} & \cdots & w_q^{i_q} \end{vmatrix} \neq 0.$$

Hence we have found q points such that $w^{i_1}, \ldots, w^{i_q}$ are linearly independent. $\square$

Corollary 1 *Given an arbitrary valid inequality (1) for X^0, we can construct a valid inequality (2) for X by sequentially lifting sets C_i for $i = 1, ..., t$. At each step i, the lifting coefficients have to be such that $h_i(z) \le f_i(z)$ for $z \in Z$. If (1) defines a facet of $conv(X^0)$, $conv(X^i)$ is full dimensional for $i = 0, ..., t-1$, and at each step i the lifting is maximal, then (2) defines a facet of $conv(X)$.*

It should be clear that lifting coefficients are, in general, dependent on the lifting sequence $C_1, C_2, \ldots, C_t$.

We conclude this section with some useful properties of the functions $f_i(z)$.

Proposition 2 *$f_i(0) \geq 0$, for $i = 1, ..., t$.*

Proof. This is an immediate consequence of the validity of the inequality (3). □

Proposition 3 *The functions f_i for $i = 1, ..., t$ are nondecreasing and for any $z \in Z$*

$$f_1(z) \geq f_2(z) \geq \ldots \geq f_t(z).$$

Proof. This is an immediate consequence of the definition of f_i for $i = 1, ..., t$. □

Proposition 4 *If x^* is an optimal solution to (6) and $u_l^* = \sum_{l \leq k < i} \sum_{j \in C_k} a_j x_j^*$, then $f_i(z) \geq f_l(z + u_l^*) - f_l(u_l^*)$ for $l = 1, ..., i-1$.*

Proof. First, we show that $f_l(u_l^*) \geq \sum_{l \leq k < i} \sum_{j \in C_k} \alpha_j x_j^*$.

$$\begin{aligned}
0 &\leq f_i(0) \\
&\leq \min \alpha_0 - \sum_{0 \leq k < l} \sum_{j \in C_k} \alpha_j x_j - \sum_{l \leq k < i} \sum_{j \in C_k} \alpha_j x_j^* \\
&\quad \text{s.t.} \sum_{0 \leq k < l} \sum_{j \in C_k} a_j x_j \leq d - u_l^* \\
&= f_l(u_l^*) - \sum_{l \leq k < i} \sum_{j \in C_k} \alpha_j x_j^*.
\end{aligned}$$

Now, we prove the proposition.

$$\begin{aligned}
f_i(z) &= \alpha_0 - \sum_{0 \leq k < l} \sum_{j \in C_k} \alpha_j x_j^* - \sum_{l \leq k < i} \sum_{j \in C_k} \alpha_j x_j^* \\
&= \min \alpha_0 - \sum_{0 \leq k < l} \sum_{j \in C_k} \alpha_j x_j - \sum_{l \leq k < i} \sum_{j \in C_k} \alpha_j x_j^* \\
&\quad \text{s.t.} \sum_{0 \leq k < l} \sum_{j \in C_k} a_j x_j \leq d - z - u_l^* \\
&= f_l(z + u_l^*) - \sum_{l \leq k < i} \sum_{j \in C_k} \alpha_j x_j^* \\
&\geq f_l(z + u_l^*) - f_l(u_l^*). \quad \square
\end{aligned}$$

3 Sequence independent lifting

3.1 Superadditive functions

Definition. A real-valued function f is superadditive on Z if f is bounded for all $z \in Z$ and $f(z_1) + f(z_2) \leq f(z_1 + z_2)$ for all z_1, z_2 and $z_1 + z_2$ in Z.

Now, we introduce a superadditive function that will be used later in the paper. Given

$$l > 0, v_1 > 0$$

$$u_i \geq 0, u_i \geq u_{i+1},\ for\ i = 1, 2, \ldots$$

$$v_i \geq 0, v_i \geq v_{i+1},\ for\ i = 1, 2, \ldots$$

$$w_i = u_i + v_i > 0,\ for\ i = 1, 2, \ldots$$

let

$$M_0 = 0, \text{ and } M_h = \sum_{i=1}^{h} w_i,\ for\ h = 1, 2, \ldots, \infty.$$

We define

$$\overline{g}(z) = \begin{cases} 0 & \text{if } z = 0 \\ hl & \text{if } M_h < z \leq M_h + u_{h+1}, h = 0, 1, \ldots \\ l[h + 1 - (M_{h+1} - z)/v_1] & \text{if } M_h + u_{h+1} < z \leq M_{h+1}, h = 0, 1, \ldots. \end{cases}$$

The function $\overline{g}$ is defined on $Z = [0, \infty)$ and shown in Figure 1. It is easy to see that it is piecewise linear and nondecreasing.

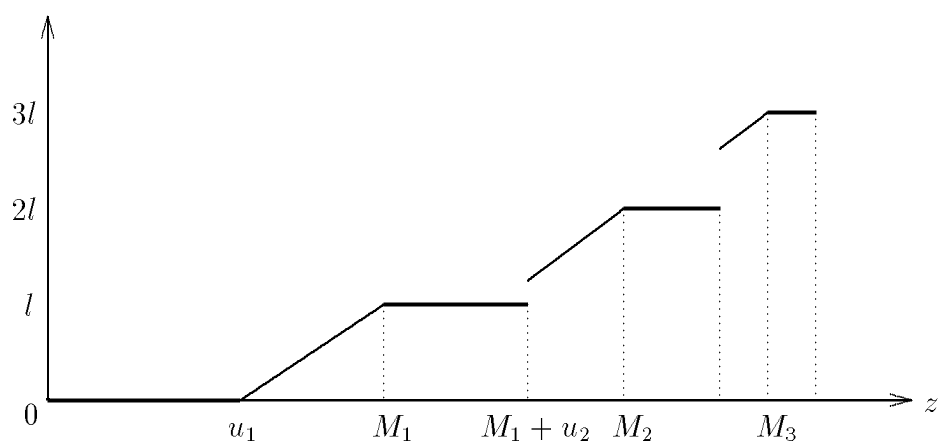

Figure 1: Function $\overline{g}(z)$

Lemma 1 *The function $\overline{g}$ is superadditive on Z.*

Proof. Let

$$D = \max\{\overline{g}(z_1) + \overline{g}(z_2) - \overline{g}(z_1 + z_2) : z_1, z_2 \in Z\}.$$

Then $\overline{g}$ is superadditive if and only if $D \leq 0$. We consider three cases depending on the domain of z_1 and z_2.

Case 1: $M_{h_1} + u_{h_1+1} < z_1 \leq M_{h_1+1}$ and $M_{h_2} + u_{h_2+1} < z_2 \leq M_{h_2+1}$.
We have

$$\begin{aligned} z_1 + z_2 &> M_{h_1} + u_{h_1+1} + M_{h_2} + u_{h_2+1} \\ &= \sum_{i=1}^{h_1} w_i + \sum_{i=1}^{h_2} w_i + u_{h_1+1} + u_{h_2+1} \\ &\geq \sum_{i=1}^{h_1} w_i + \sum_{i=h_1+1}^{h_1+h_2} w_i + u_{h_1+1} + u_{h_2+1} \\ &= M_{h_1+h_2} + u_{h_1+1} + u_{h_2+1} \\ &\geq M_{h_1+h_2} + u_{h_1+h_2+1}. \end{aligned}$$

Case 1.1: $z_1 + z_2 \leq M_{h_1+h_2+1}$.
Then

$$\begin{aligned} &\overline{g}(z_1) + \overline{g}(z_2) - \overline{g}(z_1 + z_2) \\ &\quad = l[h_1 + 1 - (M_{h_1+1} - z_1)/v_1] + l[h_2 + 1 - (M_{h_2+1} - z_2)/v_1] \\ &\qquad - l[(h_1 + h_2 + 1) - (M_{h_1+h_2+1} - z_1 - z_2)/v_1] \\ &\quad = l - (l/v_1)(M_{h_1+1} + M_{h_2+1} - M_{h_1+h_2+1}) \\ &\quad = l - (l/v_1)(u_1 + v_1) - (l/v_1)(\sum_{i=2}^{h_1+1} w_i + \sum_{i=1}^{h_2+1} w_i - \sum_{i=1}^{h_1+h_2+1} w_i) \\ &\quad = -(l/v_1)(u_1 + \sum_{i=2}^{h_1+1} (w_i - w_{i+h_2})) \\ &\quad \leq 0. \end{aligned}$$

Case 1.2: $M_{h_1+h_2+1} < z_1 + z_2 \leq M_{h_1+h_2+1} + u_{h_1+h_2+2}$.
Then

$$\begin{aligned} &\overline{g}(z_1) + \overline{g}(z_2) - \overline{g}(z_1 + z_2) \\ &\quad = l[h_1 + 1 - (M_{h_1+1} - z_1)/v_1] + l[h_2 + 1 - (M_{h_2+1} - z_2)/v_1] - (h_1 + h_2 + 1)l \\ &\quad = l - (l/v_1)(M_{h_1+1} + M_{h_2+1} - z_1 - z_2) \\ &\quad \leq l - (l/v_1)(M_{h_1+1} + M_{h_2+1} - M_{h_1+h_2+1} - u_{h_1+h_2+2}) \end{aligned}$$

$$= l - (l/v_1)(v_1 + u_1 - u_{h_1+h_2+1}) - (l/v_1)(\sum_{i=2}^{h_1+1} w_i + \sum_{i=1}^{h_2+1} w_i - \sum_{i=1}^{h_1+h_2+1} w_i)$$

$$\le -(l/v_1) \sum_{i=2}^{h_1+1} (w_i - w_{i+h_2})$$

$$\le 0.$$

Case 1.3: $M_{h_1+h_2+1} + u_{h_1+h_2+2} < z_1 + z_2 \le M_{h_1+h_2+2}$.
Then

$$\begin{aligned}
&\overline{g}(z_1) + \overline{g}(z_2) - \overline{g}(z_1 + z_2) \\
&\quad = l[h_1 + 1 - (M_{h_1+1} - z_1)/v_1] + l[h_2 + 1 - (M_{h_2+1} - z_2)/v_1] \\
&\quad\; -l[(h_1 + h_2 + 2) - (M_{h_1+h_2+2} - z_1 - z_2)/v_1] \\
&\quad = -(l/v_1)(M_{h_1+1} + M_{h_2+1} - M_{h_1+h_2+2}) \\
&\quad \le 0.
\end{aligned}$$

Case 1.4: $z_1 + z_2 > M_{h_1+h_2+2}$.
Then

$$\overline{g}(z_1 + z_2) \ge (h_1 + h_2 + 2)l,$$

$$\overline{g}(z_1) \le (h_1 + 1)l,$$

and

$$\overline{g}(z_2) \le (h_2 + 1)l.$$

So

$$\overline{g}(z_1) + \overline{g}(z_2) - \overline{g}(z_1 + z_2) \le 0.$$

Case 2: $M_{h_1} < z_1 \le M_{h_1} + u_{h_1+1}$ and $M_{h_2} < z_2 \le M_{h_2} + u_{h_2+1}$.
We have $z_1 + z_2 > M_{h_1} + M_{h_2} \ge M_{h_1+h_2}$. Therefore

$$\overline{g}(z_1 + z_2) \ge (h_1 + h_2)l.$$

Thus

$$\overline{g}(z_1) + \overline{g}(z_2) - \overline{g}(z_1 + z_2) \le h_1 l + h_2 l - (h_1 + h_2)l = 0.$$

Case 3: $M_{h_1} < z_1 \le M_{h_1} + u_{h_1+1}$ and $M_{h_2} + u_{h_2+1} < z_2 \le M_{h_2+1}$.
Case 3.1: $v_{h_1+1} = 0$.
We have $\overline{g}(z_2) \le (h_2 + 1)l$ and $v_{h_1+h_2+1} = 0$, since $v_{h_1+h_2+1} \le v_{h_1+1} = 0$. Therefore

$$\begin{aligned}
z_1 + z_2 &> M_{h_1} + M_{h_2} + u_{h_2+1} \\
&\ge M_{h_1+h_2} + u_{h_2+1} \\
&\ge M_{h_1+h_2} + u_{h_1+h_2+1} \\
&= M_{h_1+h_2+1}.
\end{aligned}$$

Consequently, $\overline{g}(z_1 + z_2) \geq (h_1 + h_2 + 1)l$, which implies $\overline{g}(z_1) + \overline{g}(z_2) - \overline{g}(z_1 + z_2) \leq h_1 l + (h_2 + 1)l - (h_1 + h_2 + 1)l = 0$.
Case 3.2: $v_{h_1+1} > 0$.
We have $\overline{g}(M_{h_1}) = h_1 l$ and, since $\overline{g}$ is nondecreasing, $\overline{g}(z_1 + z_2) \geq \overline{g}(M_{h_1} + z_2)$. Hence

$$\begin{aligned}\overline{g}(z_1) + \overline{g}(z_2) - \overline{g}(z_1 + z_2) &= h_1 l + \overline{g}(z_2) - \overline{g}(z_1 + z_2) \\ &\leq \overline{g}(M_{h_1}) + \overline{g}(z_2) - \overline{g}(M_{h_1} + z_2) \\ &\leq 0,\end{aligned}$$

since $z_1 = M_{h_1}$ and $M_{h_2} + u_{h_2+1} < z_2 \leq M_{h_2+1}$ falls under Case 1. □

3.2 Lifting

We now develop the concept of sequence independent lifting and its relation to superadditive functions. Wolsey [1977] developed a similar theory for 0-1 integer programs. Among other things, our results extend Wolsey's to mixed 0-1 integer programs with nonnegative coefficients.

Definition. The *lifting function* f with respect to valid inequality (1) for X^0 is defined to be $f(z) = f_1(z)$ for all $z \in Z$.

Note that $f(z)$ is independent of the lifting order and the constraint coefficients for the elements of $C_1, C_2, \ldots, C_t$. Therefore, if $f(z) = f_i(z)$ for $z \in Z$ and $i = 2, ..., t$, then the lifting coefficients in a sequential lifting of the elements of $C_1, C_2, \ldots, C_t$ are independent of the ordering of the set $\{C_1, C_2, \ldots, C_t\}$.

Definition. If $f(z) = f_i(z)$ for $z \in Z$, $i = 2, ..., t$, and all lifting sequences, then the lifting is said to be *sequence independent.*

Now we give a sufficient condition for sequence independent lifting.

Theorem 2 *If f is superadditive on Z, then lifting is sequence independent.*

Proof. Let $z \in Z$, x' be an optimal solution to (6) and $u' = \sum_{j \in C_{i-1}} a_j x'_j$. Now by Proposition 4

$$f_i(z) \geq f_{i-1}(z + u') - f_{i-1}(u'),$$

and by superadditivity

$$f_{i-1}(z + u') \geq f_{i-1}(z) + f_{i-1}(u').$$

Hence $f_i(z) \geq f_{i-1}(z)$. Proposition 3 gives

$$f_i(z) \leq f_{i-1}(z).$$

Consequently,

$$f_i(z) = f_{i-1}(z). \ \square$$

Obviously, a superadditive lifting function greatly reduces the computational burden of the lifting process. Instead of having to compute lifting functions f_i for all i, we only have to compute f. Unfortunately, most lifting functions are not superadditive. To be able to profit from the computational advantages of sequence independent lifting, we consider the class of superadditive valid lifting functions.

Definition. A superadditive function g is called a *superadditive valid lifting function* for f, if $g(z) \leq f(z)$ for all $z \in Z$.

Theorem 3 *If g is a superadditive valid lifting function and if α_j for $j \in C_i$ are such that $h_i(z) \leq g(z)$ for $z \in Z$ and for $i = 1, \ldots, t$, then the lifted inequality (2) is valid for X.*

Proof. By hypothesis $g(z) \leq f_1(z)$ for $z \in Z$. As an induction hypothesis suppose $g(z) \leq f_{i-1}(z)$ for $z \in Z$. We will prove that $g(z) \leq f_i(z)$ for $z \in Z$. Hence $h_t(z) \leq g(z) \leq f_t(z)$ for $z \in Z$ and the result follows.

Let x^* an optimal solution to (6) and $u^* = \sum_{j \in C_{i-1}} a_j x_j^*$. Then, as in the proof of Proposition 4,

$$\begin{aligned} f_i(z) &= \alpha_0 - \sum_{0 \leq k \leq i-1} \sum_{j \in C_k} \alpha_j x_j^* \\ &= f_{i-1}(z + u^*) - \sum_{j \in C_{i-1}} \alpha_j x_j^* \\ &\geq f_{i-1}(z + u^*) - h_{i-1}(u^*) \\ &\geq g(z + u^*) - h_{i-1}(u^*) \\ &\geq g(z + u^*) - g(u^*) \\ &\geq g(z). \ \square \end{aligned}$$

Next, we will show the existence of a superadditive valid lifting function.

Definition. $\gamma(z) = \min_{u \in Z} \{f(z + u) - f(u) : z + u \in Z\}$ for all $z \in Z$.

Theorem 4 *The function γ is a superadditive valid lifting function.*

Proof. To prove validity, we show $\gamma(z) \leq f_t(z) \leq f(z)$ for all $z \in Z$. Let $z \in Z$, x' be an optimal solution to (6) with $i = t$, and $u' = \sum_{1 \leq k < t} \sum_{j \in C_k} a_j x'_j$. Proposition 4 gives

$$\begin{aligned} f_t(z) &\geq f(z + u') - f(u') \\ &\geq \min_{u \in Z} \{f(z+u) - f(u)\} \\ &= \gamma(z). \end{aligned}$$

Next, we show that $\gamma(z)$ is superadditive. Let

$$u^* = \text{argmin}_{u \in Z} \{f(z+u) - f(u)\},$$

so that

$$\gamma(z) = f(z + u^*) - f(u^*).$$

Let $z = z_1 + z_2$. Since $\gamma(z_i) = \min_{u \in Z}\{f(z_i + u) - f(u)\}$ for $i = 1, 2$, we have

$$\gamma(z_1) + \gamma(z_2) \leq \{f(z_1 + u^*) - f(u^*)\} + \{f(z_1 + z_2 + u^*) - f(z_1 + u^*)\} = \gamma(z_1 + z_2). \square$$

Next, we address the problem of choosing a 'good' superadditive valid lifting function. A desirable property is that g should not be *dominated* by another superadditive valid lifting function g', i.e., there is no superadditive g' with $g(z) \leq g'(z)$ for all $z \in Z$ and $g(z') < g'(z')$ for some $z' \in Z$.

Another interesting property is maximality. Let $E = \{z \in Z : f_i(z) = f(z)$ for $i = 1, ..., t$ for all $C_1, C_2, \ldots, C_t$ and all lifting orders$\}$. We say that g is a *maximal* superadditive valid lifting function if $g(z) = f(z)$ for all $z \in E$. Note that if f is superadditive, then $E = Z$, and that if $\gamma(z) = f(z)$, then $z \in E$.

We will illustrate the following maximality property in the next section.

Proposition 5 *If $f_i(z)$ is integral for all z and $i = 1, ..., t$, $g(z)$ is a superadditive valid lifting function such that $g(z) > f(z) - 1$ for all z, and $g(z') = f(z')$, then $z' \in E$.*

Proof. Let $u' = \text{argmin}_{u \in Z}\{f(z' + u) - f(z')\}$, then $\gamma(z') = f(z' + u') - f(u')$. If $g(z') = f(z')$, then

$$\begin{aligned} f_i(z') &\geq \gamma(z') \\ &= f(z' + u') - f(u') \\ &\geq g(z' + u') - f(u') \\ &> g(z' + u') - g(u') - 1 \\ &\geq g(z') - 1 \\ &= f(z') - 1. \end{aligned}$$

Because $f_i(z') \leq f(z')$, $f_i(z') = f(z')$. Hence $z' \in E$. $\square$

In the next two sections, we will illustrate the benefits of sequence independent lifting for two specific polytopes and valid inequalities.

4 Lifted knapsack cover inequalities

One of the most successful approaches for solving general 0-1 integer programs is a branch-and-cut algorithm based on lifted knapsack cover inequalities. In this section, we study sequence independent lifting of knapsack cover inequalities.

Consider the set of feasible solutions to a 0-1 knapsack problems given by

$$X = \{x \in B^{|N|} : \sum_{j \in N} a_j x_j \leq b\},$$

where we assume $a_j > 0$ (since 0-1 variables can be complemented) and integral, and $a_j \leq b$ (since $a_j > b$ implies $x_j = 0$).

The set $C \subseteq N$ is called a *cover* if $\sum_{j \in C} a_j > b$. The cover is *minimal* if C is minimal with respect to the property. For any cover C,

$$\sum_{j \in C} x_j \leq |C| - 1, \tag{7}$$

is called a *cover inequality*. It is valid for the knapsack polytope, and defines a facet of $conv(X^0)$, where

$$X^0 = \{x \in B^{|C|} : \sum_{j \in C} a_j x_j \leq b\}.$$

A sequential *lifted cover inequality* (LCI) is an inequality of the form

$$\sum_{j \in C} x_j + \sum_{j \in N \setminus C} \alpha_j x_j \leq |C| - 1, \tag{8}$$

where C is a minimal cover. An LCI is obtained by starting from cover inequality (7) and lifting all variables in $N \setminus C$ sequentially. Let $C_1 = \{j_1\}, C_2 = \{j_2\}, \ldots, C_t = \{j_t\}$ be the lifting sequence of all the variables in $N \setminus C$.

The functions $h_i(z)$ for $i = 1, ..., t$ are given by

$$\begin{aligned} h_i(z) = \quad & \max \alpha_{j_i} x_{j_i} \\ & a_{j_i} x_{j_i} = z \\ & x_{j_i} \in \{0, 1\} \end{aligned}$$

and the functions $f_i(z)$ for $i = 1, ..., t$ are given by

$$\begin{aligned} f_i(z) = \quad \min \quad & |C| - 1 - \sum_{j \in C} x_j + \sum_{1 \leq k < i} \alpha_{j_k} x_{j_k} \\ \text{s.t.} \quad & \sum_{j \in C} a_j x_j + \sum_{1 \leq k < i} a_{j_k} x_{j_k} \leq b - z \\ & x_j \in \{0, 1\}, \; j \in C \cup \{j_1, \ldots, j_{i-1}\}. \end{aligned}$$

Observe that $h_i(z)$ is feasible only for $z \in \{0, a_{j_i}\}$ and that $h_i(0) = 0$ and $h_i(a_{j_i}) = \alpha_{j_i}$. Therefore, to obtain a facet inducing lifted cover inequality for X, the lifting coefficient α_{j_i} has to be equal to $f_i(a_{j_i})$, since it is the unique feasible solution to $h_i(z) = f_i(z)$.

Let $Z = [0, b]$. The lifting function f is given by

$$\begin{aligned} f(z) = \quad & \min \quad |C| - 1 - \sum_{j \in C} x_j \\ & \text{s.t.} \quad \sum_{j \in C} a_j x_j \leq b - z \\ & \qquad\quad x_j \in \{0, 1\},\ j \in C. \end{aligned}$$

For the remainder we assume, without loss of generality, that $C = \{1, 2, \ldots, r\}$ with $a_1 \geq a_2 \geq \ldots \geq a_r$. Let $\lambda = \sum_{j \in C} a_j - b$ and $\mu_0 = 0$, $\mu_i = \sum_{1 \leq h \leq i} a_h$ for $i = 1, \ldots, r$. Since $a_i \geq a_{i+1}$ for $i = 1, ..., r-1$, there is always an optimal solution x for $f(z)$ such that $x_1 \leq x_2 \leq \ldots \leq x_r$. So we have

$$f(z) = \begin{cases} 0 & \text{if } 0 \leq z \leq \mu_1 - \lambda \\ h & \text{if } \mu_h - \lambda < z \leq \mu_{h+1} - \lambda, \qquad h = 1, \ldots, r-1. \end{cases}$$

The function f is not superadditive on Z in general. Therefore, we define the superadditive valid lifting function g for f given by

$$g(z) = \begin{cases} 0 & \text{for } z = 0 \\ h & \text{if } \mu_h - \lambda + \rho_h < z \leq \mu_{h+1} - \lambda, h = 0, \ldots, r-1 \\ h - (\mu_h - \lambda + \rho_h - z)/\rho_1 & \text{if } \mu_h - \lambda < z \leq \mu_h - \lambda + \rho_h, h = 1, \ldots, r-1 \end{cases}$$

where $\rho_h = \max\{0, a_{h+1} - (a_1 - \lambda)\}$ for $h = 0, \ldots, r-1$.

Example. Let $C = \{1, 2, 3, 4\}$, $a_1 = 8$, $a_2 = 7$, $a_3 = 6$, $a_4 = 4$, and $b = 22$. Then $\lambda = \sum_{j=1}^{4} a_j - b = 25 - 22 = 3$, $\rho_0 = \lambda = 3$, $\rho_1 = a_2 - (a_1 - \lambda) = 2$, $\rho_2 = 1$, $\rho_3 = 0$. Both $f(z)$ and $g(z)$ are shown in Figure 2. Note that $f(6) = 1 > g(6)$ and that f is not superadditive since $f(6) + f(6) = 2 > f(12) = 1$. Also, it is easy to see that $g(z) > f(z) - 1$ for all z and for $1 \leq z \leq 22$ and integral, $f(z) = g(z)$ if $z \neq 6$. Therefore, by Proposition 5, if $a_j \neq 6$ the lifting coefficient of x_j is $f(a_j)$ independent of the lifting sequence. However, it is easy to see that $f_2(6) = 0 < f_1(6)$, if the variable lifted first is x_5 and $a_5 = 6$.

Theorem 5 *The function g is a superadditive valid lifting function for f that is non-dominated and maximal on $[0, b]$.*

Proof.
1. The function g is superadditive on $[0, b]$. We use the function $\overline{g}$ defined in Section 3

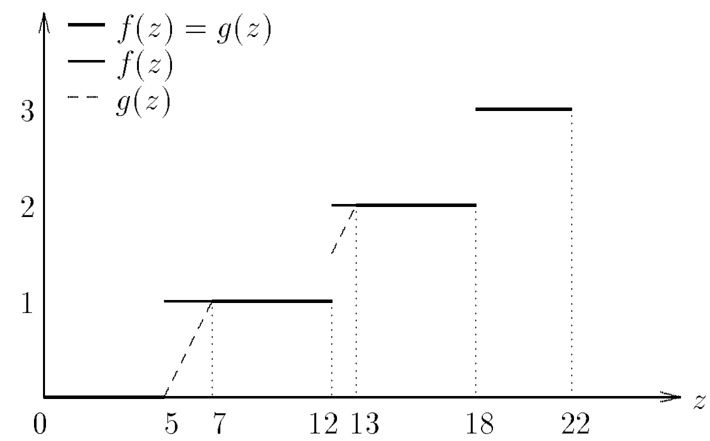

Figure 2: Functions $f(z)$ and $g(z)$ for the example

and Lemma 1 to show the superadditivity of g. Let $l = 1$, $u_i = a_i - \rho_{i-1}$ for $i = 1, \ldots, r$, $v_i = \rho_i$ for $i = 1, \ldots, r-1$. If $\rho_{i-1} > 0$, then $u_i = a_i - a_i + (a_1 - \lambda) = a_1 - \lambda$, and if $\rho_{i-1} = 0$, then $u_i = a_i \leq a_1 - \lambda$. Thus $u_i \geq u_{i+1} > 0$. Since $\rho_i \geq \rho_{i+1}$, $v_i \geq v_{i+1}$. Also $u_i + v_i > 0$. Hence the choices of parameters for $\overline{g}(z)$ are legal. Then

$$M_h = \sum_{i=1}^{h}(u_i + v_i) = \sum_{i=1}^{h}(a_i - \rho_{i-1} + \rho_i) = \sum_{i=1}^{h} a_i - \rho_0 + \rho_h = \mu_h - \lambda + \rho_h$$

and

$$M_h + u_{h+1} = \mu_h - \lambda + \rho_h + a_{h+1} - \rho_h = \mu_{h+1} - \lambda,$$

and thus $g(z) = \overline{g}(z)$ on $[0, b]$. Since $\overline{g}$ is superadditive, g is also superadditive.

2. The function g is maximal. Suppose $g(z) < f(z)$ for some $z \in [0, b]$, then we will show that $f_2(z) < f(z)$. To do so, we first show that if $z \in [0, b]$ and $g(z) < f(z)$, then there is a $z' \in [0, b]$ such that $z + z' \in [0, b]$ and $f(z) + f(z') > f(z + z')$. Let $z \in [0, b]$ and $g(z) < f(z)$. This implies that $z \in (\mu_h - \lambda, \mu_h - \lambda + \rho_h)$ for some h and that $f(z) = h$. Since the interval is nonempty, we have $\rho_h > 0$, which means $\rho_h = a_{h+1} - (a_1 - \lambda)$. Let $z' = a_1 - \lambda + (\mu_h - \lambda + \rho_h - z)$. Then $z' > a_1 - \lambda$ and

$$z + z' = a_1 - \lambda + \mu_h - \lambda + \rho_h = \mu_h + a_{h+1} - \lambda = \mu_{h+1} - \lambda,$$

which implies $f(z') \geq 1$ and $f(z+z') = h$. Hence $f(z) + f(z') > f(z+z')$. Now suppose that x_{j_1} is lifted first and that its coefficient $a_{j_1} = z'$. Then we have that $f_2(z) = \min\{f(z+z') - f(z'), f(z)\} = f(z+z') - f(z') < f(z)$ by the last result.

3. The function g is nondominated. Suppose that there is another superadditive valid lifting function $g'(z)$ such that $g(z) \leq g'(z) \leq f(z)$ for all $z \in [0, b]$ and there exists a $z' \in [0, b]$ such that $g(z') < g'(z')$. Then $z' \in (\mu_h - \lambda, \mu_h - \lambda + \rho_h)$ for some h, which implies that $\rho_h > 0$ and thus $\rho_h = a_{h+1} - (a_1 - \lambda)$. Let $z'' = \mu_{h+1} - \lambda - z'$. Then

$$z'' > \mu_{h+1} - \lambda - (\mu_h - \lambda + \rho_h) = a_{h+1} - \rho_h = a_1 - \lambda + \rho_h - \rho_h = a_1 - \lambda,$$

which implies that $g(z'') > 0$. Since $g(z') > h - 1$,

$$g(z'') \leq g(z' + z'') - g(z') < h - (h-1) = 1.$$

Hence

$$\begin{aligned} g(z') + g(z'') &= h - (\mu_h - \lambda + \rho_h - z')/\rho_1 + 1 - (\mu_1 - \lambda + \rho_1 - z'')/\rho_1 \\ &= h + 1 - [\mu_h - \lambda + a_{h+1} - (a_1 - \lambda) + a_1 - \lambda + \rho_1 - \mu_{h+1} + \lambda]/\rho_1 \\ &= h + 1 - (\mu_h + a_{h+1} - \mu_{h+1} + \rho_1)/\rho_1 \\ &= h. \end{aligned}$$

Since $g'(z' + z'') = g'(\mu_{h+1} - \lambda) = h$, we have

$$g(z'') = h - g(z') = g'(z' + z'') - g(z') \geq g'(z') + g'(z'') - g(z') > g'(z''),$$

which is a contradiction. □

Theorem 5 implies that g is a "good" superadditive valid lifting function in the sense that it is nondominated and maximal. But there may be other functions with these properties. If $\mu_1 - \lambda \geq \rho_1$, then we can construct a large family of such functions as follows

$$g_w(z) = \begin{cases} 0 & \text{for } z = 0 \\ h & \text{if } \mu_h - \lambda + \rho_h < z \leq \mu_{h+1} - \lambda, h = 0, \ldots, r-1 \\ h - w(\mu_h - \lambda + \rho_h - z) & \text{if } \mu_h - \lambda < z \leq \mu_h - \lambda + \rho_h, h = 1, \ldots, r-1, \end{cases}$$

where $w(x)$ is any nondecreasing function of $x \in [0, \rho_1]$ with $w(x_1) + w(\rho_1 - x_1) = 1$.

As a consequence of Theorem 5, we obtain the following strengthening of Balas' lifting theorem for knapsack covers [Balas, 1975].

Theorem 6 *Every valid inequality of the form*

$$\sum_{j \in C} x_j + \sum_{j \in N \setminus C} \alpha_j x_j \leq |C| - 1$$

that represents a facet of the $0-1$ *knapsack polytope satisfies the following conditions:*

i *If* $\mu_h - \lambda + \rho_h \le a_j \le \mu_{h+1} - \lambda$, *then* $\alpha_j = h$.

ii *If* $\mu_{h+1} - \lambda < a_j < \mu_{h+1} - \lambda + \rho_h$, *then (a)* $\alpha_j \in [h, h+1]$ *and (b) there is at least one facet of this form with* $\alpha_j = h + 1$.

We get Balas' theorem by taking $\rho_h = \lambda$ for all h. Note that with our choice of $\rho_h \le \lambda$, for some portion of the interval in case **ii** where Balas' theorem gives $\alpha_j \in [h, h+1]$, our result gives $\alpha_j = h + 1$.

Example (continued). In Figure 3, we compare the intervals given by Balas' theorem and Theorem 6 for the instance defined in the previous example.

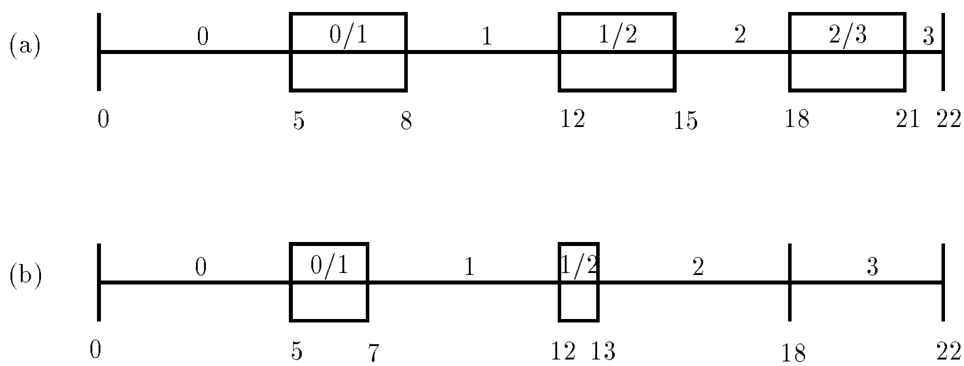

Figure 3: Comparison of Balas' theorem (a) and Theorem 6 (b) for the example.

By Theorem 3,

$$\sum_{j \in C} x_j + \sum_{j \in N \setminus C} g(a_j) x_j \le |C| - 1 \tag{9}$$

is a valid inequality for $conv(X)$ for any superadditive valid lifting function g. Although (9) does not necessarily define a facet, it may do so. In fact, it may induce a facet that cannot be obtained by sequential lifting.

Suppose the complete knapsack inequality was given by

$$8x_1 + 7x_2 + 6x_3 + 4x_4 + 6x_5 + 6x_6 + 6x_7 \le 22.$$

If we use the superadditive valid lifting function g defined above, then we get $\alpha_5 = \alpha_6 = \alpha_7 = 0.5$ and we obtain the lifted cover inequality

$$x_1 + x_2 + x_3 + x_4 + 0.5x_5 + 0.5x_6 + 0.5x_7 \le 3.$$

It is easy to see that this inequality cannot be obtained by sequential lifting of a cover inequality and that it defines a facet of the 0-1 knapsack polytope. The facet-defining inequalities obtained from maximum sequential lifting are:

$$x_1 + x_2 + x_3 + x_4 + x_{5+k} \le 3, \qquad k = 0, 1, 2$$

5 Lifted flow cover inequalities

One of the most successful approaches for solving general mixed 0-1 integer programs is a branch-and-cut algorithm based on lifted flow cover inequalities. In this section, we study sequence independent lifting of flow cover inequalities. To the best of our knowledge only Pochet [1993] has investigated lifting of flow covers. Although he did not see a link with superadditive functions, his unpublished paper contains a theorem that is very close to the main result of this section (Theorem 9). Lifting of flow covers is considerably more difficult than lifting of knapsack covers since it requires simultaneous lifting of variable pairs and the lifting functions involved are much more complicated.

Consider a single-node flow model with exogenous supply d and n outflow arcs. For each $j \in \{1, ..., n\} = N$ the flow x_j on the jth arc is bounded by the capacity $m_j > 0$ if the arc is open ($y_j = 1$) and 0 otherwise ($y_j = 0$). We assume, without loss of generality, that $m_j \leq d$. Flow models have been studied extensively, see for instance Padberg, Van Roy, and Wolsey [1985] and Van Roy and Wolsey [1986]. The feasible region can be represented by

$$X = \{(x, y) \in R^n_+ \times B^n : \sum_{j=1}^{n} x_j \leq d, x_j \leq m_j y_j, j = 1, \ldots, n\}.$$

Definition. A set S is called a *flow cover* if $\sum_{j \in S} m_j > d$. For convenience, we assume $S = \{1, 2, \ldots, s\}$.

Let $m_j \geq m_{j+1}$ for $j = 1, \ldots, s-1$, $\lambda = \sum_{j \in S} m_j - d$, $S^+ = \{1, 2, \ldots, r\} = \{j \in S, m_j > \lambda\}$, $M_0 = 0$, and $M_j = \sum_{k=1}^{j} m_j$ for $j = 1, \ldots, r$. Assume $S^+ \neq \emptyset$.

Consider the subset of X given by

$$X^0 = \{(x, y) \in X : x_j = y_j = 0 \text{ for } j \in N \setminus S\}.$$

The *flow cover inequality*

$$0 \leq d - \sum_{j \in S} x_j - \sum_{j \in S^+} (m_j - \lambda)(1 - y_j)$$

defines a facet of conv(X^0), see Nemhauser and Wolsey [1988].

Since we have $x_j \leq m_j y_j$, we lift variable pairs (x_j, y_j) for $j \in N \setminus S$ instead of single variables. Suppose the lifting sequence is $(x_{s+1}, y_{s+1}), (x_{s+2}, y_{s+2}), \ldots, (x_n, y_n)$. Let $C_1 = \{s+1\}, C_2 = \{s+2\}, \ldots, C_{n-s} = \{n\}$ be the index sets defining the variable pairs being lifted.

The functions $h_i(z)$ for $i = 1, ..., n-s$ are given by

$$\begin{aligned} h_i(z) = \quad & \max \quad \alpha_i x_{s+i} + \beta_i y_{s+i} \\ & s.t. \quad x_{s+i} = z \\ & \qquad\quad x_{s+i} \leq m_{s+i} y_{s+i} \\ & \qquad\quad y_{s+i} \in \{0, 1\}, \end{aligned}$$

and the functions $f_i(z)$ for $i = 1, ..., n-s$ are given by

$$\begin{aligned} f_i(z) = \quad & \min d - \textstyle\sum_{j \in S} x_j - \sum_{j \in S^+} (m_j - \lambda)(1 - y_j) - \sum_{1 \leq k < i} (\alpha_k x_{s+k} + \beta_k y_{s+k}) \\ & \textstyle\sum_{j \in S \cup (\cup_{1 \leq k < i} C_k)} x_j \leq d - z \\ & x_j \leq m_j y_j, \text{ for } j \in S \cup (\cup_{1 \leq k < i} C_k) \\ & (x, y) \in R_+^{|S| + |\cup_{1 \leq k < i} C_k|} \times B^{|S| + |\cup_{1 \leq k < i} C_k|}. \end{aligned}$$

Observe that $h_i(z) = \alpha_i z + \beta_i$ for $0 < z \leq m_{s+i}$.

Let $Z = [0, d]$. The lifting function f is given by

$$\begin{aligned} f(z) = \quad & \min \quad d - \sum_{j \in S} x_j - \sum_{j \in S^+} (m_j - \lambda)(1 - y_j) \\ & \text{s.t} \quad \sum_{j \in S} x_j \leq d - z \\ & \qquad x_j \leq m_j y_j, \text{ for } j \in S \\ & \qquad (x, y) \in R_+^{|S|} \times B^{|S|}. \end{aligned} \tag{10}$$

Theorem 7

$$f(z) = \begin{cases} 0 & if \quad z = 0 \\ i\lambda & if \quad M_i < z \leq M_{i+1} - \lambda \ for \ i = 0, ..., r-1 \\ z - M_i + i\lambda & if \quad M_i - \lambda < z \leq M_i \ for \ i = 1, ..., r-1 \\ z - M_r + r\lambda & if \quad M_r - \lambda < z \leq d \end{cases}$$

Proof. Observe that for $z = 0$ there exists an optimal solution with $x_j = m_j$ and $y_j = 1$ for $j \in S \setminus S^+$ and either $x_j \geq m_j - \lambda$ and $y_j = 1$ or $x_j = 0$ and $y_j = 0$ for $j \in S^+$. Given this observation, we have that for $z = 0$ an optimal solution is given by $x_1 = y_1 = 0$, and $x_j = m_j$, $y_j = 1$ for $j \in S \setminus \{1\}$, which gives $f(0) = 0$. Moreover, this solution is feasible for $z \leq m_1 - \lambda$ so that $f(z) = 0$ for $0 \leq z \leq m_1 - \lambda$.

Now, as z increases, we can maintain optimality by appropriately reducing variables x_j in the order $x_2, ..., x_s$. In particular, we first reduce x_2 and f increases by Δx_2 until $x_2 = m_2 - \lambda$ so that $f(m_1) = \lambda$. At this point, we set $x_2 = y_2 = 0$ so that $f(z) = \lambda$ for $M_1 \leq z \leq M_2 - \lambda$. This process is repeated for all variables in S^+ in the order $3, ..., r$. When $z = M_r - \lambda$, $f(z)$ increases with z as the variables x_j, $j \in S \setminus S^+$ are decreased in any order. $\square$

Theorem 8 *The function f is superadditive on $[0, d]$.*

Proof. We use the superadditive function $\overline{g}$ defined in Section 3 and Lemma 1 to show the superadditivity of f. Let $l = \lambda$, $v_i = \lambda$ for all i, and $u_i = (m_i - \lambda)^+$ for all i. Then

$$\overline{g}(z) = \begin{cases} 0 & \text{if } z = 0 \\ h\lambda & \text{if } M_h < z \leq M_{h+1} - \lambda,\ h = 0, 1, ..., \\ \lambda(h+1) + z - M_{h+1} & \text{if } M_{h+1} - \lambda < z \leq M_{h+1},\ h = 0, 1, ..., \end{cases}$$

and $f(z) = \overline{g}(z)$ for $0 \leq z \leq d$. □

Since f is superadditive, we can apply sequence independent lifting to obtain the lifted flow cover inequality

$$0 \leq d - \sum_{j \in S} x_j - \sum_{j \in S^+} (m_j - \lambda)(1 - y_j) - \sum_{1 \leq i \leq n-s} (\alpha_i x_{s+i} + \beta_i y_{s+i}). \tag{11}$$

Moreover, since we have a closed form expression for the function f, we can compute sets H_i of lifting coefficients (α_i, β_i) for $i \in N \setminus S$ that define facet inducing lifted flow cover inequalities. Recall from Theorem 1 that we obtain a facet if and only if the pairs (α_i, β_i) are such that $h_i(z) = f_i(z)$ has two solutions (x^1, y^1) and (x^2, y^2) such that the ith components of (x^1, y^1) and (x^2, y^2) are linearly independent. Let $l = \text{argmax}_{0 \leq h \leq r}\{m_i \geq M_h - \lambda\}$. Because $m_1 \geq m_2 \geq \ldots \geq m_r$, the points $0, M_1 - \lambda, M_2 - \lambda, \ldots, M_l - \lambda$, and m_i define the lower envelope of the function $f_i(z)$. Furthermore, the pairs of adjacent points on the lower envelope, i.e., $(0, M_1 - \lambda)$, $(M_k - \lambda, M_{k+1} - \lambda)$ for $k = 1, ..., l-1$, and $(M_l - \lambda, m_i)$, define the sets of two values z for which $h_i(z) = f_i(z)$ and for which the associated solutions (x^1, y^1) and (x^2, y^2) have linearly independent ith components. We get the sets H_i by computing the slopes and the intercepts of the lines defining the lower envelope. This leads to the following theorem.

Theorem 9 *For $i \in N \setminus S$, let $l = argmax_{0 \leq h \leq r}\{m_i \geq M_h - \lambda\}$. If $l = 0$, then let $H_i = \{(0,0)\}$. If $l > 0$, then let $H_i = \{(0,0)\} \cup H_i^1 \cup H_i^2$, where*

$$H_i^1 = \{(\frac{\lambda}{m_k}, \lambda[k - 1 - \frac{M_k - \lambda}{m_k}]) : k = 2, \ldots, l\}$$

and

$$H_i^2 = \begin{cases} \emptyset & \text{if } m_i = M_l - \lambda \\ \{(1, l\lambda - M_l)\} & \text{if } M_l - \lambda < m_i \leq M_l \text{ or } m_i > M_r \\ \{(\frac{\lambda}{m_i + \lambda - M_l}, l\lambda - \frac{\lambda m_i}{m_i + \lambda - M_l})\} & \text{if } m_i \leq M_r \text{ and } M_l < m_i < M_{l+1} - \lambda. \end{cases}$$

Then if $(\alpha_i, \beta_i) \in H_i$ for $i \in N \setminus S$, (11) defines a facet of $conv(X)$.

Proof. To compute the slopes and the intercepts of the lines defining the lower envelope of the function $f_i(z)$ we partition the set of lines into three classes. The first class contains the line defined by the $(z, f_i(z))$ points (0,0) and $(M_1 - \lambda, 0)$. The slope and intercept of this line are 0 and 0 respectively. The second class contains the lines defined by the points $(M_1 - \lambda, 0), (M_2 - \lambda, \lambda), \ldots, (M_l - \lambda, (l-1)\lambda)$. The slopes and intercepts of the lines in this class are given by $\frac{\lambda}{m_k}$ and $\lambda(k - 1 - \frac{M_k - \lambda}{m_k})$ respectively, i.e., the set H_i^1. Finally, the last class contains the line defined by the points $(M_l - \lambda, (l-1)\lambda)$ and $(m_i, l\lambda)$. The slope and intercept of this line depends on whether $M_l - \lambda < m_i \leq M_l$ or $M_l < m_i < M_{l+1} - \lambda$. In the former case, we find slope 1 and intercept $l\lambda - M_l$, in the latter case, we find slope $\frac{\lambda}{m_i + \lambda - M_l}$ and intercept $l\lambda - \frac{\lambda m_i}{m_i + \lambda - M_l}$, i.e., the set H_i^2. □

Theorem 9 is illustrated in Figure 4 for a single variable pair (x_i, y_i). The line $h_i^1(z)$ corresponds to lifting coefficients (0,0), the lines $h_i^2(z)$ and $h_i^3(z)$ correspond to lifting coefficients in H_i^1, and the line $h_i^4(z)$ corresponds to lifting coefficients in H_i^2.

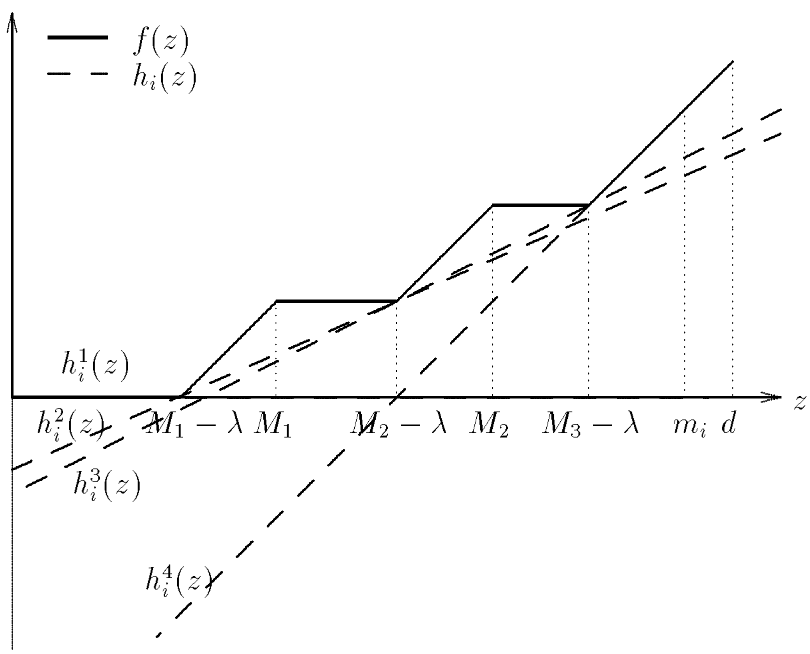

Figure 4: The set H_i for a single variable pair (x_i, y_i)

Example.
Let $m_1 = 9$, $m_2 = 7$, $m_3 = 6$, $m_4 = 10$, $m_5 = 14$, $d = 17$, and $S = \{1, 2, 3\}$. Then

$\lambda = m_1 + m_2 + m_3 - d = 22 - 17 = 5$ and the flow cover inequality is

$$0 \leq 17 - x_1 - x_2 - x_3 - (9-5)(1-y_1) - (7-5)(1-y_2) - (6-5)(1-y_3).$$

We have $r = s = 3$ and $M_0 = 0, M_1 = 9$, $M_2 = 16$, and $M_3 = 22$. For $i = 4$, $l = 1$ and the lower envelope of $f_i(z)$ is defined by the points 0,4, and 10, which gives $H_i = \{(0,0), (\frac{5}{6}, -\frac{20}{6})\}$. For $i = 5$, $l = 2$ and the lower envelope of $f_i(z)$ is defined by the points 0,4,11, and 14, which gives $H_i = \{(0,0), (\frac{5}{7}, -\frac{20}{7}), (1,-6)\}$.

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