# Information-Based Branching Schemes for Binary Linear Mixed Integer Problems

Fatma Kılınç Karzan, George L. Nemhauser, Martin W.P. Savelsbergh
H. Milton Stewart Industrial and Systems Engineering, Georgia Institute of Technology, 765 Ferst Drive, NW, Atlanta, Georgia 30332-0205, USA, {fkilinc@gatech.edu, gnemhaus@isye.gatech.edu, mwps@isye.gatech.edu }

Branching variable selection can greatly affect the effectiveness and efficiency of a branch-and-bound algorithm. Traditional approaches to branching variable selection rely on estimating the effect of the candidate variables on the objective function. We propose an approach which is empowered by exploiting the information contained in a family of fathomed subproblems, collected beforehand from an incomplete branch-and-bound tree. In particular, we use this information to define new branching rules that reduce the risk of incurring inappropriate branchings. We provide computational results that demonstrate the effectiveness of the new branching rules on various benchmark instances.

*Key words:* branch and bound; variable selection; computational algorithms

---

## 1. Introduction

Branch-and-bound (B&B) empowered with advanced features such as presolve, cuts, heuristics, and strong branching is the preferred algorithm for solving Mixed-Integer Linear Programming (MILP) problems. Branching, that is, partitioning the feasible region of a MILP into two or more subproblems, is at the heart of all B&B algorithms and thus the importance of good branching strategies was recognized early in the development of MILP algorithms (Bénichou et al. (1971)). A subproblem corresponding to a node in a B&B tree is *fathomed* when we are able to certify that it has been fully explored, that is, all of its feasible solutions have been explicitly or implicitly visited; otherwise, it's referred to as an *open* subproblem. The ultimate theoretical goal in branching is to find a strategy that minimizes the total number of B&B nodes that need to be evaluated. Typically the feasible region is divided by branching on a variable, that is, identifying an integer variable $x_j$ which has a fractional value $\bar{x}_j$ at the current node and creating two new subproblems by enforcing $x_j \leq \lfloor \bar{x}_j \rfloor$ in one of them and $x_j \geq \lceil \bar{x}_j \rceil$ in the other. Thus, every open subproblem is restricted to be identical to a previously examined subproblem except for its chronologically last branched

variable. Therefore, inappropriate branchings performed at the beginning can jeopardize the effectiveness of B&B (Forrest et al. (1974)).

Changing the rule for branching variable selection can have a dramatic effect on the overall time needed to solve a problem. With only a few exceptions, existing rules for branching variable selection are based on the impact of a candidate variable on the objective function values of the Linear Programming (LP) relaxation in the child nodes. The candidate variable having the greatest estimated impact is chosen for branching. The motivation for such a rule for branching variable selection is that maximizing the degradation of the objective function values of the LP relaxation at the child nodes gives tighter bounds on the open nodes. For a comprehensive study of early rules for branching variable selection see Linderoth and Savelsbergh (1999). *Pseudocosts*, introduced by Bénichou et al. (1971), estimate the degradation in the objective function value of the LP relaxation of the child node per unit change in the value of a candidate branching variable. *Strong branching*, introduced by Applegate et al. (1998) to solve large-scale traveling salesman problems, tentatively branches on a candidate branching variable and performs a fixed, limited number of dual simplex pivots to get a lower bound on the degradation of the objective function value of the LP relaxation of the child node. Achterberg et al. (2005) examine various combinations of these rules and suggest *reliability branching*, a hybrid between pseudocost and strong branching in which strong branching is performed for candidate variables whose pseudocost values have been updated fewer than a fixed number of times. Glankwamdee and Linderoth (2006) suggest *lookahead branching*, which estimates and uses the impact of a branching decision on the values of the objective function of the LP relaxation of the child nodes two levels deeper than the current node. Gilpin and Sandholm (2007) propose branching based on the entropy (remaining uncertainty) of a variable. The entropy is estimated by treating the fractional portions of integer variables in the LP solution as probabilities, indicating the probability with which the variable is expected to be greater than its current value in the optimal solution. Patel and Chinneck (2007) show that one can reach feasibility for MILPs much faster by selecting the branching variable based on its impact on the active constraints in the parent LP relaxation, instead of using the traditional approach of selecting the branching variable based on its impact on the objective function.

Historically, variable disjunctions have been preferred as a mechanism for partitioning the solution space due to the computational benefits of the dual simplex algorithm when solving the resulting subproblems. However, the partitioning of the solution space can be

based on any disjunction. Beale and Tomlin (1970) suggests branching on disjunctions determined by "special ordered sets." More recently, partitioning the solution space using general disjunctions has been investigated (see Owen and Mehrotra (2001), Karamanov and Cornuéjols (2005), Cornuéjols et al. (2008), Mahajan and Ralphs (2008a), and Mahajan and Ralphs (2008b)).

The information-based variable selection methods we propose are quite different from the existing approaches as they rely on collecting information up-front (as opposed to during B&B) and they estimate the impact of a candidate branching variable on the fathoming decision of child nodes (as opposed to the objective function value of the LP relaxation of child nodes). The methods recognize that the branching decisions made at the top of the B&B tree are the most important ones and use two ideas:

1. A *restart* strategy. To make more informed decisions at the top of the B&B tree, we perform an incomplete B&B search and collect information on fathomed nodes and exploit this information in the actual B&B search.

2. Fathoming-based branching variable selection. To generate small B&B trees, we choose branching variables that are likely to lead to fathoming of child nodes.

Note that using information derived from nodes that are fathomed early in the search, as in back-jumping or learning B&B (see Stallman and Sussman (1977), Marques-Silva and Sakallah (1999)), may be ineffective as this information may already be tainted by inappropriate branchings. Therefore, in our approach, we strengthen the information on fathomed subproblems by eliminating unnecessary branching decisions that had no effect on their fathoming. In addition to using the derived information to guide branching decisions, we use the information to generate valid inequalities and to fix variables.

Chvátal (1997) also suggests obtaining a minimal cardinality clause for each fathomed node, but this clause information is used in resolution search which is quite different than B&B .

The concepts of learning and restarts have been used in artificial intelligence to solve, for example, SAT problems (see Gomes et al. (2008) and Marques-Silva and Sakallah (1999)). We use some of the ideas from the SAT domain in a B&B MILP solver. As such, this work is part of an increasing effort in the integration of approaches from mathematical logic and mathematical programming (see Hooker (1998), Hooker (2000), Dixon and Gins-

berg (2000), Achterberg (2007a), Achterberg (2007b), Sandholm and Shields (2006), Avenali (2007), Achterberg (2009) and Achterberg and Berthold (2009)).

Since Davey et al. (2002), Achterberg (2007a) and Sandholm and Shields (2006) investigate obtaining clauses from fathomed nodes, although in a heuristic fashion and without employing restarts, these papers are most closely related to our research.

More recently, Achterberg (2007b, 2009) proposed *inference branching*, inspired by satisfiability (SAT) solvers, in which inference values, i.e., the number of domain reductions triggered by a branching decision, are computed and used to guide branching decisions. In order to initialize inference values, he suggests collecting historical information in a fashion similar to pseudocosts and using implication graph and clique tables. Compared to our branching rules, this idea is of the "one-step look-a-head" type as it only considers and counts the immediate propagations, whereas our rules are based on the entire active segment of a clause and looks for the overall probability of fathoming a node based on the information collected in the collection phase. Motivated, in part, by a conference presentation of our results (Kılınç Karzan et al. (2008)), Achterberg and Berthold (2009) suggest *Hybrid branching*, a strategy that combines pseudocosts, inference values, and conflict values (related to the *variable independent decaying sum* strategy of SAT solvers (Moskewicz et al. (2001))) with clause length information.

Restarts have become a standard component of modern SAT solvers. The most common form of a restart in SAT solvers is to restart whenever a certain number of fathomed nodes has been reached. Recently, restarts have been introduced in integer programming solvers as well. The latest releases of CPLEX (11.1), SCIP (1.1.0) and MINTO (3.1) include an automatic restart at the root node if a certain amount of preprocessing reductions is achieved. Moreover, SCIP (1.1.0) offers the possibility of a restart based on a specified set of criteria, including the number of clauses generated.

The main contribution of our research is assessing the value of restarts in a binary MILP framework using clause information obtained from fathomed nodes. To strengthen the information obtained from the incomplete B&B search, we develop a MILP that derives a minimum cardinality clause for each fathomed node. (An extension of that model can be used to generate clauses from scratch.) We also provide various branching alternatives that use this information in the restart framework. Our computation study shows that the restart framework is effective for moderate and large-size instances as node counts and solution times are reduced significantly.

The remainder of the paper is organized as follows. In Section 2, we present some basic concepts and results. Our approach to using restart with advanced branching based on information collected from the previous incomplete tree is detailed in Section 3. In Section 3.1, we present a model to improve the information obtained from the fathomed nodes of a partial B&B tree. This model is quite general and can be used to generate generalized cover inequalities for binary MILP problems. Our computational results are provided in Section 4. We give conclusions and further research directions in Section 5.

# 2. Preliminaries and Notation

We consider the binary MILP problem

$$\min \left\{ c^{\mathrm{T}} x + d^{\mathrm{T}} y \;\; | \;\; Ax + By \geq b,\; x \in \{0,1\}^n,\; y \in \mathbb{R}^k_+ \right\} \tag{P}$$

where $c \in \mathbb{R}^n$, $d \in \mathbb{R}^k$, $b \in \mathbb{R}^m$, $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{m \times k}$. Its LP relaxation is obtained by replacing $x_i \in \{0,1\}$ by $0 \leq x_i \leq 1$ for $i = 1, \ldots, n$.

Recall that in a B&B tree, or for short a tree, a node can be fathomed in three ways: (i) the node LP-relaxation is infeasible, (ii) the optimal solution to the node LP-relaxation is integer, and (iii) the optimum value of the node LP-relaxation is no better than the objective value of the currently best known integer feasible solution. To formalize the notion of fathoming, we use the following definitions and notation.

**Definition 1.** *Let $\upsilon^*$ be an upper bound on the objective value of the optimum solution to (P). A partial assignment $C = (C^0, C^1)$ is called a clause if*

$$\{(x,y) \in [0,1]^n \times \mathbb{R}^k_+ :\; c^{\mathrm{T}} x + d^{\mathrm{T}} y \leq \upsilon^*,\; Ax + By \geq b,\; x_j = 0 \;\forall j \in C^0,\; x_j = 1 \;\forall j \in C^1\}$$

*is empty.*

Let $N_0$ denote the root node, $N_i$ denote a node in the tree and $C_i = (C_i^0, C_i^1)$ be the set of binary variables fixed at node $N_i$, where $C_i^0$ ($C_i^1$) denotes the indices of binary variables fixed to 0 (1). Without loss of generality, we can assume that $C_0 = \emptyset$. Therefore for any given fathomed node $N_i$ in a B&B tree, the corresponding pair of index sets $(C_i^0, C_i^1)$, gives a clause. Let $f_j^l$ denote *fixing* the binary variable $x_j$ to the value $l \in \{0,1\}$, i.e. setting $x_j = l$.

**Definition 2.** *A clause $C$ is called minimal if for all $f_j^l \in C$, the partial assignment $C \setminus \{f_j^l\}$ is not a clause.*

If $C$ is a minimal clause, no node $N_i$ in the tree with $C_i \subset C$ can be fathomed by any of the fathoming rules. Moreover any node $N_i$ with $C_i \supseteq C$ can be fathomed together with the subtree rooted at $N_i$. Note that there exists a minimal clause (not necessarily unique) associated with each fathomed node in any binary B&B tree.

Given a clause $C = (C^0, C^1)$, the inequality

$$\sum_{i \in C^0} x_i + \sum_{i \in C^1} (1 - x_i) \geq 1 \tag{1}$$

eliminates all solutions that do not comply with it. Clearly (1) might cut off some feasible solutions. However, the region cut off by (1) is guaranteed not to contain any feasible solution with an objective value better than the current best objective value, $\upsilon^*$. Furthermore (1) is a generalized cover inequality and hence, by obtaining minimal clauses, we actually derive a generalized cover inequality for (P).

Since shorter clauses yield fathoming more quickly, we are particularly interested in finding a minimal clause of *minimum cardinality*. Identifying a minimum cardinality clause is closely related to identifying a minimally infeasible subsystem of inequalities (IIS) (Gleeson and Ryan (1990), Amaldi et al. (2003)). Achterberg (2007a) shows that the minimal cardinality bound IIS problem is NP-Hard, which implies the following result:

**Corollary 1.** *Finding a minimum cardinality clause associated with a fathomed node in a tree is NP-Hard.*

As opposed to the heuristic approach for reducing the size of the clauses in Achterberg (2007a), we choose to obtain a minimum cardinality clause for each clause corresponding to a fathomed node.

# 3. An Information-Based Approach

We wish to efficiently identify the most useful clauses and use them effectively in B&B in a restart framework. Our approach is based on deriving information in the form of clauses from the fathomed nodes of an incomplete tree. We employ this information in guiding the search through designing advanced propagation and branching schemes as well as in generating valid inequalities of the form (1).

It is quite likely that the clauses obtained from an incomplete tree are not minimal. We therefore strengthen the information on fathomed nodes by identifying the branching

decisions that are essential to the fathoming of the node. We do this by solving a MILP model that obtains a minimum cardinality clause from a given basic clause.

## 3.1. Identifying Minimum Cardinality Clauses

We present a model that can be used to generate a clause of minimum cardinality. Without loss of generality, we assume that the upper and lower bounds of the binary variables are included in the original formulation as constraints, i.e., $Ax+By \geq b$ includes $0 \leq x_i \leq 1\ \forall i \in \{1,\ldots,n\}$. Let $v^*$ be an upper bound on the objective value of the optimum solution to (P). We can fathom any node of the tree which is either infeasible or has an objective value greater than $v^*$. Fathoming based on the integrality of the LP solution is quite infrequent and given $v^*$, we can simply fathom all nodes with objective value greater than or equal to $v^*$ by bound, which includes fathoming of all integer solutions that are no better than the current best bound as well.

Consider a leaf node of the tree that is fathomed by bound and denote the corresponding set of variable fixings by $C = (C^0, C^1)$. The LP relaxation of this leaf node is

$$\min c^T x + d^T y \tag{2}$$

$$\text{s.t. } Ax + By \geq b \tag{3}$$

$$x_i \geq 1 \qquad \forall i \in C^1 \tag{4}$$

$$-x_i \geq 0 \qquad \forall i \in C^0 \tag{5}$$

$$x \in \mathbb{R}^n_+, \quad y \in \mathbb{R}^k_+ \tag{6}$$

Define the following variables $z_i^0$ ($z_i^1$) for $i \in C^0$ ($i \in C^1$):

$$z_i^0 = \begin{cases} 1, & \text{if inequality } -x_i \geq 0 \text{ is added to the LP relaxation;} \\ 0, & \text{otherwise,} \end{cases}$$

and

$$z_i^1 = \begin{cases} 1, & \text{if inequality } x_i \geq 1 \text{ is added to the LP relaxation;} \\ 0, & \text{otherwise.} \end{cases}$$

Using these new variables, the Mixed Integer Quadratically Constrained Program (MIQCP)

$$\min c^T x + d^T y \tag{7}$$

$$\text{s.t. } Ax + By \geq b \tag{8}$$

$$z_i^1 x_i \geq z_i^1 \qquad \forall i \in C^1 \tag{9}$$

$$-z_i^0 x_i \geq 0 \qquad \forall i \in C^0 \tag{10}$$

$$x \in \mathbb{R}^n_+, \quad y \in \mathbb{R}^k_+ \tag{11}$$

$$z_i^0 \in \{0,1\} \quad \forall i \in C^0, \qquad z_i^1 \in \{0,1\} \quad \forall i \in C^1. \tag{12}$$

is equivalent to the node LP when all of the binaries in it are set to 1.

We want to find a minimum number of bound inequalities such that their inclusion in the original linear programming relaxation of (P) will still lead to a fathoming, i.e., either the LP relaxation is infeasible or the objective value exceeds the lower bound value, $\upsilon^*$. Since the original root LP relaxation is assumed to be feasible and bounded from below, its dual is also feasible. Thus we know that in the case of infeasibility of the node LP, the dual of the node LP is unbounded since when we add new bound inequalities to the primal, we are just adding new columns to the dual of the node LP. Therefore when a node is fathomed by infeasibility, we will be able to find a dual solution with an objective value strictly greater than $\upsilon^*$. So now we consider the dual of the above formulation by treating the variables $z_i^0$ and $z_i^1$ as parameters and we obtain

$$\max \ \lambda^T b + \sum_{i \in C^1} \gamma_i^1 z_i^1 \tag{13}$$

$$\text{s.t. } \lambda^T A_i + \gamma_i^1 z_i^1 \leq c_i \qquad \forall i \in C^1 \tag{14}$$

$$\lambda^T A_i - \gamma_i^0 z_i^0 \leq c_i \qquad \forall i \in C^0 \tag{15}$$

$$\lambda^T A_i \leq c_i \qquad \forall i \in \{1, \ldots, n\} \setminus (C^0 \cup C^1) \tag{16}$$

$$\lambda^T B_j \leq d_j \qquad \forall j \in \{1, \ldots, k\} \tag{17}$$

$$\lambda_l \geq 0 \qquad \forall l \in \{1, \ldots, m\} \tag{18}$$

$$\gamma_i^0 \geq 0 \quad \forall i \in C^0, \qquad \gamma_i^1 \geq 0 \quad \forall i \in C^1. \tag{19}$$

Since we are only interested in the existence of dual solutions with objective value greater than or equal to $\upsilon^*$, we can equivalently turn the objective into a constraint. By considering $z_i^0$ and $z_i^1$ as binary variables with the condition that at most one of them can be set to 1 for

each $i$ (since $C^0 \cap C^1 = \emptyset$, we don't need to include these constraints explicitly), we obtain the following MIQCP where we minimize the sum of $z_i^0$ and $z_i^1$:

$$\min \sum_{i \in C^0} z_i^0 + \sum_{i \in C^1} z_i^1 \tag{20}$$

$$\text{s.t. } \lambda^T b + \sum_{i \in C^1} \gamma_i^1 z_i^1 \geq \upsilon^* \tag{21}$$

$$\lambda^T A_i + \gamma_i^1 z_i^1 \leq c_i \qquad \forall i \in C^1 \tag{22}$$

$$\lambda^T A_i - \gamma_i^0 z_i^0 \leq c_i \qquad \forall i \in C^0 \tag{23}$$

$$\lambda^T A_i \leq c_i \qquad \forall i \in \{1, \ldots, n\} \setminus (C^0 \cup C^1) \tag{24}$$

$$\lambda^T B_j \leq d_j \qquad \forall j \in \{1, \ldots, k\} \tag{25}$$

$$\lambda_l \geq 0 \qquad \forall l \in \{1, \ldots, m\} \tag{26}$$

$$z_i^0 \in \{0,1\}, \quad \gamma_i^0 \geq 0 \quad \forall i \in C^0 \tag{27}$$

$$z_i^1 \in \{0,1\}, \quad \gamma_i^1 \geq 0 \quad \forall i \in C^1. \tag{28}$$

In order to bound the dual variables $\gamma$ and $\lambda$ from above by 1, we introduce another variable $\alpha$, we relabel $\alpha\lambda$ and $\alpha\gamma$ as $\lambda$ and $\gamma$ respectively and to linearize the model, we introduce $u_i^0$ and $u_i^1$ for the nonlinear terms $\gamma_i^0 z_i^0$ and $\gamma_i^1 z_i^1$ respectively. This yields the MILP formulation:

$$\min \sum_{i \in C^0} z_i^0 + \sum_{i \in C^1} z_i^1 \tag{29}$$

$$\text{s.t. } \lambda^T b + \sum_{i \in C^1} u_i^1 - \alpha \upsilon^* \geq 0 \tag{30}$$

$$\lambda^T A_i + u_i^1 - \alpha c_i \leq 0 \qquad \forall i \in C^1 \tag{31}$$

$$\lambda^T A_i - u_i^0 - \alpha c_i \leq 0 \qquad \forall i \in C^0 \tag{32}$$

$$\lambda^T A_i - \alpha c_i \leq 0 \qquad \forall i \in \{1, \ldots, n\} \setminus (C^0 \cup C^1) \tag{33}$$

$$\lambda^T B_j - \alpha d_j \leq 0 \qquad \forall j \in \{1, \ldots, k\} \tag{34}$$

$$u_i^0 \leq \gamma_i^0, \quad u_i^0 \leq z_i^0, \quad u_i^0 - \gamma_i^0 - z_i^0 \geq -1 \qquad \forall i \in C^0 \tag{35}$$

$$u_i^1 \leq \gamma_i^1, \quad u_i^1 \leq z_i^1, \quad u_i^1 - \gamma_i^1 - z_i^1 \geq -1 \qquad \forall i \in C^1 \tag{36}$$

$$0 \leq \lambda_l \leq 1 \qquad \forall l \in \{1, \ldots, m\} \tag{37}$$

$$z_i^0 \in \{0,1\}, \quad 0 \leq u_i^0, \gamma_i^0 \leq 1 \quad \forall i \in C^0 \tag{38}$$

$$z_i^1 \in \{0,1\}, \quad 0 \leq u_i^1, \gamma_i^1 \leq 1 \quad \forall i \in C^1 \tag{39}$$

$$0 < \alpha. \tag{40}$$

Note that the fixing of the last variable on the path from the root node to a leaf node is required to fathom the leaf node. Hence in any feasible solution to the above formulation, we will always have the corresponding binary variable ($z_i^0$ or $z_i^1$) fixed to 1. In our computations, we take advantage of this fact by actually fixing the value of $z_i$ corresponding to the last branched variable to 1. We also replace $\alpha > 0$ with $\alpha \geq 10^{-5}$, which is equivalent to a big-$M$ formulation where the big-$M$ value can be chosen as $\frac{1}{\alpha} = 10^5$. We are aware of the fact that this lower bound on $\alpha$ value might lead to non-optimal solutions for some instances, i.e. for some instances using a smaller bound will lead to a clause with smaller cardinality. A more careful analysis using the inverses of the maximum upper bound values for the dual variables can be applied to find better bounds in this regard. However, in order to avoid possible numerical difficulties, we simply choose to use $10^{-5}$. This issue can also be handled using the indicator-variable feature provided in some integer programming solvers.

As an aside, an auxiliary problem can be solved to obtain a clause (i.e., a set of binary variable fixings that leads to a fathoming) without any knowledge of a fathomed node. Furthermore, when $\upsilon^*$ is the optimal objective value, the optimal value to this problem will give a lower bound on the length of any path in the B&B tree. This problem is given by:

$$\min \ \sum_{i=1}^{n}(z_i^0 + z_i^1) \tag{41}$$

$$\text{s.t. } \lambda^T b + \sum_{i=1}^{n} u_i^1 - \alpha\upsilon^* \geq 0 \tag{42}$$

$$\lambda^T A_i + u_i^1 - u_i^0 - \alpha c_i \leq 0 \qquad \forall i \in \{1,\ldots,n\} \tag{43}$$

$$\lambda^T B_j - \alpha d_j \leq 0 \qquad \forall j \in \{1,\ldots,k\} \tag{44}$$

$$u_i^0 \leq \gamma_i^0, \quad u_i^0 \leq z_i^0, \quad u_i^0 - \gamma_i^0 - z_i^0 \geq -1 \qquad \forall i \in \{1,\ldots,n\} \tag{45}$$

$$u_i^1 \leq \gamma_i^1, \quad u_i^1 \leq z_i^1, \quad u_i^1 - \gamma_i^1 - z_i^1 \geq -1 \qquad \forall i \in \{1,\ldots,n\} \tag{46}$$

$$z_i^0 + z_i^1 \leq 1 \qquad \forall i \in \{1,\ldots,n\} \tag{47}$$

$$0 \leq \lambda_l \leq 1 \qquad \forall l \in \{1,\ldots,m\} \tag{48}$$

$$z_i^0, z_i^1 \in \{0,1\}, \quad 0 \leq u_i^0, u_i^1, \gamma_i^0, \gamma_i^1 \leq 1 \qquad \forall i \in \{1,\ldots,n\} \tag{49}$$

$$0 < \alpha. \tag{50}$$

If an upper bound on the objective value is not available, then we fathom a node only when the corresponding LP relaxation is infeasible or the corresponding optimum solution is integral. In such a case, our models can still be used to identify clauses corresponding to infeasible node LPs by setting the objective function to zero, i.e. $c = 0,\ d = 0$ and $\upsilon^* = 1$.

## 3.2. Guiding Search with Dynamic Branching

Once the clause information is collected, there is still the question of how to use this information to aid the search in the restart phase. Let $\mathcal{C} = \{C_1, ..., C_K\}$ be a set of clauses. Consider a node of the tree $\tilde{N}$ other than the root node, and let $\tilde{C}^0(\tilde{C}^1)$ denote the set of binary variables fixed to 0 (1). We say a clause $C = (C^0, C^1)$ is *active* at $\tilde{N}$ if $C^0 \cap \tilde{C}^1 = \emptyset$ and $C^1 \cap \tilde{C}^0 = \emptyset$, i.e. if it is possible to obtain a descendent node (not necessarily a child) from the current node that can be fathomed by the clause $C$. Whenever a clause becomes inactive for a particular node, it will remain inactive for all the descendent nodes of that node. Since some variables are already fixed at $\tilde{N}$, the active clauses can be updated using this information, i.e., the clause $C = (C^0 \setminus \tilde{C}^0, C^1 \setminus \tilde{C}^1)$ indicates the possible extension of the current node to a child node which can be fathomed by clause $C$. In the rest of the text whenever we refer to an active clause at a node, we actually mean the updated version of the clause.

Whenever an active clause $C$ has only one variable, i.e., $|C| = 1$, we can immediately fix the value of that variable. Suppose $C = C^0 = \{j\}$, then we can set $x_j = 1$ and create only a single child node, as the other branch will automatically be fathomed by the clause. We refer to this as *propagation*.

Given a node and a set of active clauses at that node, there are several ways to use this information in determining a branching variable. Let $\tilde{x}$ be the vector of current LP relaxation values of variables at the node. We first weight each active clause, denoted by $\omega(C_i)$, to estimate its importance in fathoming. We have tested four different alternatives to estimate $\omega(C_i)$:

0. $\omega(C_i) = 1$ for all clauses $C_i$ (i.e. all clauses are of equal importance),
1. $\omega(C_i) = \frac{1}{|C_i|}$ where $|C_i|$ is the number of variables included in clause $C_i$ (i.e. short clauses are preferred),
2. $\omega(C_i) = 2^{-|C_i|}$ (i.e. exponentially higher preference is given to the shorter clauses),
3. $\omega(C_i) = \frac{1}{\max\{\sum_{j \in C_i^0} \tilde{x}_j + \sum_{j \in C_i^1} (1 - \tilde{x}_j) - 1, 10^{-10}\}}$ where we use the closeness to violation of the clause inequality (1).

Then using the weights of the clauses, we estimate the effectiveness of fixing the binary variable $x_j$ to 0 (1), denoted by $\beta_j^0$ ($\beta_j^1$). We have tested two alternatives to estimate the overall effect of creating a branch with $x_j = 0$ (1):

0. $\beta_j^0 = \max\{\omega(C_i) : j \in C_i^0\}$ and $\beta_j^1 = \max\{\omega(C_i) : j \in C_i^1\}$,

1. $\beta_j^0 = \sum_{i:j \in C_i^0} \omega(C_i)$ and $\beta_j^1 = \sum_{i:j \in C_i^1} \omega(C_i)$.

Let $\beta_j$ denote the overall effect of branching on variable $x_j$. To estimate $\beta_j$, we combine $\beta_j^0$ and $\beta_j^1$ using rules that have been derived to combine pseudocosts (see Linderoth and Savelsbergh (1999)). We suggest and test the following alternatives:

0. $\beta_j = \min\{\tilde{x}_j, 1 - \tilde{x}_j\} \cdot (\beta_j^0 + \beta_j^1)$,

1. $\beta_j = \beta_j^0 + \beta_j^1$,

2. $\beta_j = \max\{\beta_j^0, \beta_j^1\} + 10 \cdot \min\{\beta_j^0, \beta_j^1\}$,

3. $\beta_j = \max\{\beta_j^0, 10^{-6}\} \cdot \max\{\beta_j^1, 10^{-6}\}$.

Note that the first alternative considers the fractionality of the variable, whereas the second one simply adds the individual effects, the third one is similar to the weights used in strong branching and the last one is a multiplicative rule inspired by the one in Achterberg (2007b).

We denote a specific rule for determining a branching variable as $a - b - c$ where $a \in \{0, 1, 2, 3\}$, $b \in \{0, 1\}$ and $c \in \{0, 1, 2, 3\}$.

# 4. Computational Results

The purpose of our computational study is to assess the benefits of learning and restart with both *basic* information directly obtained from the incomplete tree and *improved* information obtained through the minimum clause identification models. We investigate the impact of propagation, of branching, and of adding clause inequalities (1).

## 4.1. Experimental Setup

All experiments are done on a Dual 2.4GHz Xeon Linux workstation with 2GB of RAM. CPLEX 11.1 is used both as a representative state-of-the-art commercial MILP solver with which we compare our new methods, and as the basic MILP solver framework upon which our new methods are built. A C++ program manages the interface between CPLEX and our new routines, especially the process of obtaining necessary information via the callback routines in the CPLEX callable library.

In all of the experiments, including the information collection and restart phases, whenever the branching rule is not overwritten by one of our rules, we use strong branching.

In order to make the comparison among the different branching rules as fair as possible, we provide the value of the optimal solution as a cutoff value. Thus in all of the restart experiments, node counts and solution times indicate the effort required to prove optimality. As heuristics have no effect in this setting, they are turned off. Advanced features such as presolve and cut generation are applied at the root node. The root node relaxation thus obtained is used in the minimum clause identification model. We allow CPLEX to perform local preprocessing and cut generation at nodes both in the information collection and restart phases. Although including local information in the auxiliary problems for obtaining minimum cardinality clauses may lead to significant improvements, we still only use the root node LP to construct the auxiliary problems because retrieving local information is cumbersome and can be very time consuming. Moreover, in the clause improvement phase, we provide the value of the optimal solution as $\upsilon^*$ and use default CPLEX to solve the minimum clause identification models while enforcing a time limit of 5 seconds for each solve.

To collect information from a diverse set of nodes in the B&B tree, we use best-first search in the information collection phase. As we provide the optimum value to the problems in all of the experiments, the node selection strategy does not have a significant effect on the number of nodes. As the easy retrieval of the basis provides some benefits in terms of time, depth-first search is used in the restart solves as well as comparison runs with CPLEX B&B and CPLEX dynamic search.

In the information collection phase, we terminate the search based on a fixed number of fathomed nodes detected. We use a limit of 200 fathomed nodes as a base, but also investigate the impact of this limit.

We use instances from MIPLIB 3.0 (Bixby et al. (1996)), MIBLIB 2003 (Achterberg et al. (2006)), and Cor@l (2009) libraries and from the test set of Mittelmann (2009) for our computational experiments. In order to give results on a meaningful set of problems, we eliminate all instances that meet at least one of the following criteria:

- the instance cannot be solved with CPLEX B&B under these settings in 7200 seconds (the instance is too hard);
- the instance solves in less than 60 seconds or the B&B tree has fewer than 1000 nodes with CPLEX B&B under these settings (the instance is too easy);

- the clause collection phase terminates with fewer than 200 clauses (note that many instances that might have been eliminated by this criterion would have already been eliminated under the "too easy" criterion. However the main reason for eliminating instances by this rule that were neither too hard nor too easy is that the instance had general integer variables as well as binary variables and we could not obtain 200 fathomed clauses that only contained binary variables.);

- the clause collection phase for the instance cannot be completed in a time limit of 7200 seconds or due to memory limits. (Note that the time spent on the collection phase strictly depends on the instance as the collection process is halted whenever we collect 200 clauses or the whole B&B process is finished.)

In addition to this, we divide the set of instances into three groups based on their CPLEX B&B node counts as *easy, medium* and *hard* where *easy* instances require between 1000 and 5000 nodes, *medium* instances require between 5000 and 50,000 nodes, and *hard* ones require at least 50,000 nodes. This yields 13 *hard*, 23 *medium* and 15 *easy* instances with a total of 51 instances. We present and discuss summary results for instances from these three groups in the main text. (Detailed results for all of the instances are given in the **Appendix**.) In the summary tables, we provide summary statistics, i.e. arithmetic and shifted geometric means with a shift of 1, for all of the instances as well as the instances within each group. For a set of nonnegative values $T = \{t_1, \ldots, t_n\}$, the shifted geometric mean with a shift of $\delta$, is given by the formula $\left(\prod_{i=1}^{n}(t_i + \delta)\right)^{1/n} - \delta$.

In addition, we provide various performance profiles (see Dolan and Moré (2002) for more information on performance profiles) to summarize our findings on the behavior of the algorithms in the restart phase. In a performance profile for the number of nodes (solution time), for a given method $f$ and a given point $(\theta, \gamma)$ on the plot of $f$, the value $\gamma$ indicates the fraction $\gamma$ of the instances that were solved by that particular method $f$, within $\theta$ times the number of nodes (time) required by the method with fewest nodes (fastest method) for each instance. In particular, the intercepts of the plot (if any) with the vertical axis to the left indicate the fraction of the instances for which the method performed best respectively.

## 4.2. The Value of the Restart Strategy with Fathoming-based Branching

In Table 1, we report a summary of the computational results for our test set of instances. (Detailed results for all instances are given in Table 6 in the Appendix.) In the first part of Table 1, we provide information about the *Collection phase*, namely the number of nodes in the incomplete tree and the time (in seconds) required to construct the incomplete tree, and the number of clauses collected and their average size, i.e., the average number of binary variables involved in the clauses. In the second part, we provide information about the *Improvement phase*, namely the minimum, the maximum, and the total number of nodes as well as the corresponding solution times required for solving the minimum clause identification problems, the number of clauses improved together with the average size of the improved set of clauses. It is somewhat surprising that the easy instances took more time than the others in the collection and improvement phases. We do not have a good explanation for this. However it is possible that although these instances are easy in the MIP sense of having small trees, they have large or hard LP relaxations.

In the third part, we provide information about the *Restart phase*, namely the node count and solution time statistics for *B-Cuts&Prop&Branch* and *I-Cuts&Prop&Branch*. In *B-Cuts&Prop&Branch* we use basic clause information, i.e., information obtained directly from the incomplete tree, whereas in *I-Cuts&Prop&Branch* we use improved clauses, i.e., information resulting from solving a minimum clause identification problem for each basic clause obtained from the incomplete tree. In both cases, clause inequalities are added to the cut pool and, whenever we have clause information on a candidate branching variable at a node, we perform branching based on the 3-1-1 combination as well as propagation. The "3-1-1" (a-b-c) choice for "weight-effect-combination" is shown to be effective in subsequent experiments.

Finally, for comparison, we provide the node counts and solution times of *CPLEX* with and without dynamic search enabled, *CPLEX-DS* and *CPLEX-BB*, respectively, as well as information on the *total* effort involved with our algorithms. For *B-Cuts&Prop&Branch* this is the sum of the efforts in collection and restart phases and for *I-Cuts&Prop&Branch* it is the sum of the efforts in collection, improvement and restart phases.

Table 1: Value of the restart strategy with fathoming-based branching

| | | Collection phase | | | Improvement phase | | | | | Restart phase *Cuts&Prop&Branch* | | Total | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | # | size | min | max | total | # | (%) size | *B* | *I* | *CPLEX-DS* | *CPLEX-BB* | *Cuts&Prop&Branch* *B* | *I* |
| Arith.Mean | # nodes | 795.8 | 200 | 23.08 | 0 | 84.3 | 653.6 | 70.57 (35.3) | 16.61 | 74498.6 | 76399.6 | 81421.9 | 104274.6 | 75294.4 | 77849 |
| (All: 51 models) | time | 88 | | | 0.7 | 2.6 | 249.3 | | | 861.4 | 752.6 | 1026.7 | 1116.2 | 949.4 | 1089.9 |
| Geo.Mean | # nodes | 691.2 | 200 | 21.07 | 0 | 13.7 | 56.8 | 24.56 (12.3) | 15.14 | 11688.6 | 10095.2 | 11277.7 | 15729.7 | 13418.5 | 12804.8 |
| (All: 51 models) | time | 41.9 | | | 0.5 | 1.9 | 86.9 | | | 494.2 | 417.2 | 524.3 | 677.3 | 581.3 | 712.6 |
| Arith.Mean | # nodes | 948.3 | 200 | 26.33 | 0 | 121.4 | 962.1 | 101.15 (50.6) | 16.58 | 265678.8 | 275569.9 | 287286.2 | 376527 | 266627.1 | 277480.3 |
| (Hard: 13 models) | time | 21.3 | | | 0.1 | 1.3 | 73.6 | | | 1741 | 1574.7 | 1851.2 | 2166.5 | 1762.3 | 1669.6 |
| Geo.Mean | # nodes | 866.7 | 200 | 24.8 | 0 | 35.4 | 183.9 | 77.72 (38.9) | 15.37 | 139124.4 | 119371.1 | 125247.4 | 187595.9 | 140568.2 | 121816.5 |
| (Hard: 13 models) | time | 12.8 | | | 0.1 | 0.9 | 27.1 | | | 1185.5 | 965.6 | 1083.2 | 1486.4 | 1203.8 | 1026 |
| Arith.Mean | # nodes | 789 | 200 | 22.52 | 0 | 77.3 | 648.2 | 73.87 (36.9) | 15.55 | 13540 | 12210.6 | 16108 | 16689.9 | 14329 | 13647.8 |
| (Medium: 23 models) | time | 71.3 | | | 0.6 | 2.8 | 228.9 | | | 608.5 | 500.5 | 776.8 | 866.1 | 679.8 | 800.7 |
| Geo.Mean | # nodes | 710.3 | 200 | 20.31 | 0 | 12 | 56.9 | 21.32 (10.7) | 14.13 | 9465.2 | 7512.4 | 7686.5 | 13503.9 | 10685.6 | 9730.4 |
| (Medium: 23 models) | time | 46.3 | | | 0.4 | 2.1 | 96.4 | | | 428.3 | 342.3 | 383.3 | 654.8 | 498.5 | 595.5 |
| Arith.Mean | # nodes | 673.8 | 200 | 21.12 | 0 | 62.9 | 394.5 | 39.00 (19.5) | 18.27 | 2278.7 | 2208.4 | 3154.1 | 2619.1 | 2952.5 | 3276.7 |
| (Easy: 15 models) | time | 171.5 | | | 1.3 | 3.5 | 433 | | | 486.9 | 426.9 | 695.3 | 589.3 | 658.4 | 1031.4 |
| Geo.Mean | # nodes | 544.7 | 200 | 19.34 | 0 | 7.1 | 20 | 10.86 (5.4) | 16.62 | 1887.4 | 1866.3 | 2519.1 | 2318.5 | 2483.5 | 2768.3 |
| (Easy: 15 models) | time | 97.6 | | | 0.8 | 2.7 | 200.6 | | | 288.1 | 272.9 | 451.8 | 360.8 | 391.3 | 684.1 |



Table 2: Effect of providing information in different forms: Cuts, Propagation, Branching

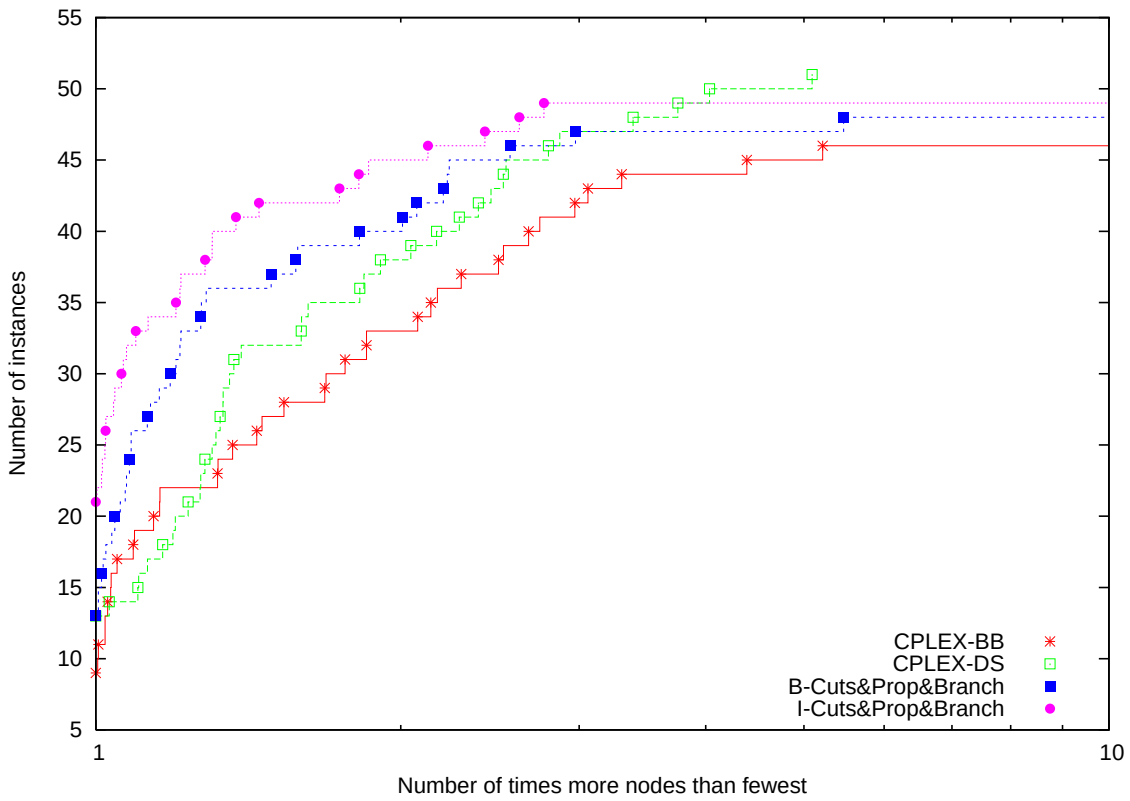
| | | *CPLEX* | | *Cuts* | | *Prop* | | *Prop&Branch* | | *Cuts&Prop* | | *Cuts&Prop&Branch* | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | DS | B&B | B | I | B | I | B | I | B | I | B | I |
| Arith.Mean | # nodes | 81421.9 | 104274.6 | 98267.5 | 78190.6 | 99286 | 71424.2 | 78554.3 | 76915.4 | 96766.1 | 69657.1 | 74498.6 | 76399.6 |
| (All: 51 models) | time | 1026.7 | 1116.2 | 1059.8 | 926.2 | 1158.1 | 949.7 | 877.8 | 740.1 | 1059.8 | 920.9 | 861.4 | 752.6 |
| Geo.Mean | # nodes | 11277.7 | 15729.7 | 13826.9 | 12353.0 | 15774.6 | 13243.6 | 11782.9 | 10050.7 | 13971.1 | 12277.0 | 11688.6 | 10095.2 |
| (All: 51 models) | time | 524.3 | 677.3 | 607.8 | 554.8 | 694.4 | 588.9 | 488.3 | 402.8 | 626.7 | 560.6 | 494.2 | 417.2 |
| Arith.Mean | # nodes | 287286.2 | 376527 | 356392.8 | 278861.0 | 356427 | 249255.9 | 281874.2 | 277031.2 | 350429.3 | 244892.1 | 265678.8 | 275569.9 |
| (Hard: 13 models) | time | 1851.2 | 2166.5 | 2226.8 | 1842.9 | 2241.3 | 1756.6 | 1834.8 | 1548.5 | 2177.9 | 1721.8 | 1741 | 1574.7 |
| Geo.Mean | # nodes | 125247.4 | 187595.9 | 188082.9 | 157273.9 | 190045 | 140600.5 | 149230.1 | 121213.8 | 180646.8 | 136568.1 | 139124.4 | 119371.1 |
| (Hard: 13 models) | time | 1083.2 | 1486.4 | 1510.9 | 1295.2 | 1523.4 | 1162.6 | 1229.7 | 949.6 | 1495.5 | 1153.6 | 1185.5 | 965.6 |
| Arith.Mean | # nodes | 16108.0 | 16689.9 | 15046.4 | 14350.5 | 16563.9 | 15801.2 | 13402.4 | 12580.3 | 15070.9 | 14628.5 | 13540 | 12210.6 |
| (Medium: 23 models) | time | 776.8 | 866.1 | 767.9 | 682.9 | 869.6 | 753.4 | 594.4 | 498.8 | 797.9 | 729.8 | 608.5 | 500.5 |
| Geo.Mean | # nodes | 7686.5 | 13503.9 | 11602.5 | 10202.5 | 13212 | 11311.5 | 9169.7 | 7460.1 | 12146.8 | 10987.0 | 9465.2 | 7512.4 |
| (Medium: 23 models) | time | 383.3 | 654.8 | 574.9 | 513.1 | 651.2 | 564.5 | 408.4 | 332.1 | 620.9 | 560.3 | 428.3 | 342.3 |
| Arith.Mean | # nodes | 3154.1 | 2619.1 | 2164.7 | 2164.2 | 3271 | 2592.1 | 2243.4 | 2129.0 | 2190.7 | 2164.1 | 2278.7 | 2208.4 |
| (Easy: 15 models) | time | 695.3 | 589.3 | 496 | 504.9 | 661.7 | 551.2 | 482.9 | 409.4 | 492.4 | 519.7 | 486.9 | 426.9 |
| Geo.Mean | # nodes | 2519.1 | 2318.5 | 1883.1 | 1825.5 | 2393.8 | 2176.1 | 1916.3 | 1833.7 | 1883 | 1803.3 | 1887.4 | 1866.3 |
| (Easy: 15 models) | time | 451.8 | 360.8 | 300.4 | 299.7 | 387.5 | 348.3 | 288.1 | 257.3 | 298.8 | 299.9 | 288.1 | 272.9 |

Comparing the performance of *B-Cuts&Prop&Branch* with the performance of CPLEX B&B, we see that the node counts and solution time improve on average (even when the statistics are added from the collection phase and restart phase and compared with that of CPLEX B&B ). The node count improvements are substantial for some instances: for `neos-1480121` the node count goes from 1,711,005 to 484,311, and for `neos-1228986` from 137,818 to 36,581 (see Table 6 in the Appendix for details). Although the computation times improve too on average, the changes are not as significant due to the overhead of our branching strategy at each node. Note that this overhead can be reduced with advanced data structures and the use of internal CPLEX structures. Comparing the performance of *I-Cuts&Prop&Branch* with the performance of *B-Cuts&Prop&Branch*, we see from the performance profile of Figure 1 that for the majority of instances the B&B trees get even smaller. This improved behavior obtained is much more significant for the instances from the *hard* set. Note that although our methods are built on top of CPLEX B&B, the overall performance of our approach is comparable with CPLEX dynamic search. By simply adding up the overall average number of nodes from the collection, improvement and restart phases for *I-Cuts&Prop&Branch* we obtain 77849.0 whereas average CPLEX B&B node count is 104274.6 and CPLEX DS node count is 81421.9. The same procedure for overall solution times leads to 1089.9 seconds for *I-Cuts&Prop&Branch*, whereas CPLEX B&B is 1116.2 seconds and CPLEX DS is 1026.7 seconds. We want to emphasize that the improvements over CPLEX are clearer when we restrict ourselves to *hard* instances.

The benefits of our restart strategy with fathoming-based branching can also be seen in Figures 1 and 2, where we provide performance profiles in which we compare *CPLEX*, *CPLEX-DS*, *B-Cuts&Prop&Branch* and *I-Cuts&Prop&Branch* in terms of node counts and solution times in the restart phase.

Table 1 also shows that on average about 35% of the clauses collected are improved by solving the minimum clause identification problem, with an average reduction in size of about 28%. Yet there are instances where none of the collected clauses can be improved, e.g., `harp2`, `neos-598183` and `stein45` (see Table 6 in the Appendix). For most instances, the time to improve clauses is a fraction of the solution time of the original instance. However, some easy and medium size instances, such as `neos22`, `neos-631694`, `lrn` and `seymour-1`, are exceptions since they require quite significant improvement times, yet not so many of the clauses are improved. Usually, this behavior is due to numerical difficulties in the solution of the LP-relaxations of the minimum clause identification problems. In fact, frequently in

Figure 1: Comparison of # of nodes in the restart phase

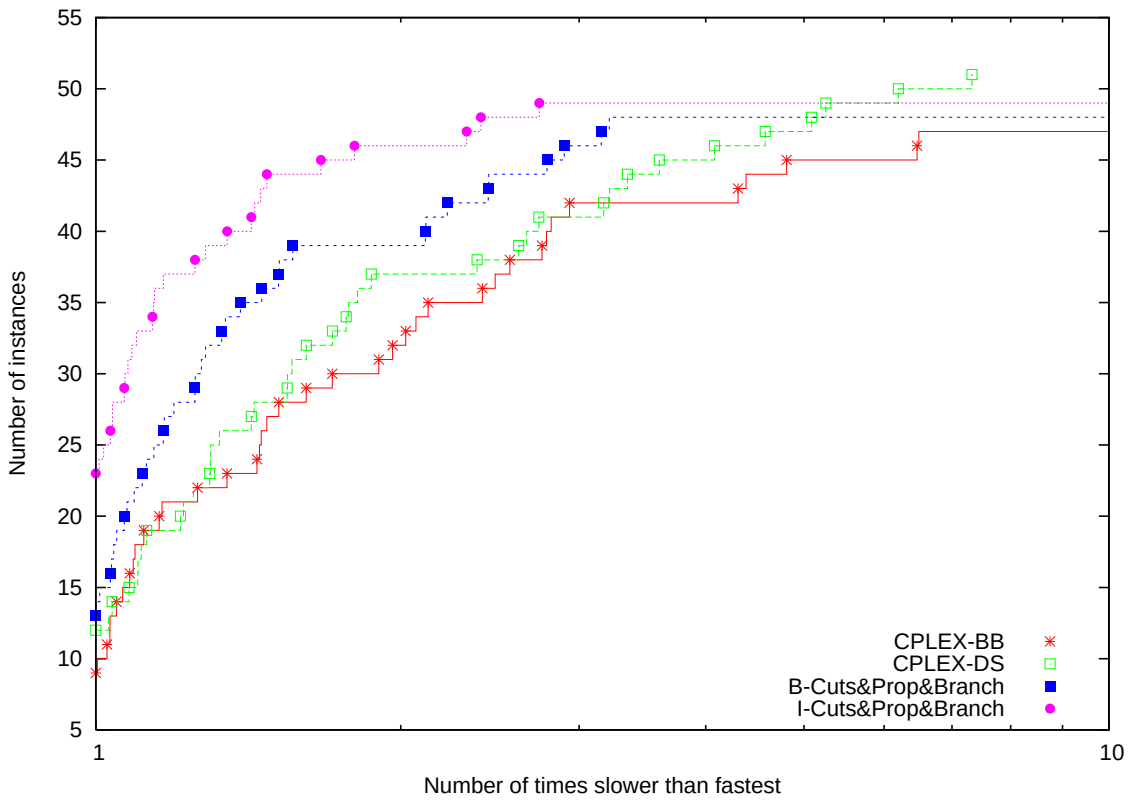



these cases, the LP relaxations of the minimum clause improvement problems could not be solved within the time limit of 5 seconds.

In Table 2, we report a summary of the computational results for our test set of instances where we focus on the contribution of each of the components of our approach. (Detailed results for all instances are given in Table 7 in the Appendix.) We report results for only adding cuts (*B-Cuts*, *I-Cuts*), only applying propagation (*B-Prop*, *I-Prop*), applying propagation and performing branching (*B-Prop&Branch*, *I-Prop&Branch*), applying propagation and adding cuts (*B-Cuts&Prop*, *I-Cuts&Prop*) and the use of all components (*B-Cuts&Prop&Branch*, *I-Cuts&Prop&Branch*).

The performance improvement from the addition of clause inequalities alone as cuts is limited, particularly when only basic clause information is used. Propagation alone gives comparable performance, although with improved clauses its performance is better than that of cuts especially on the hard instances. Adding cuts and applying propagation together gives a slight improvement to just applying propagation or cuts in both basic and improved cases. Using our branching in addition to applying propagation gives significantly

Figure 2: Comparison of solution times in the restart phase

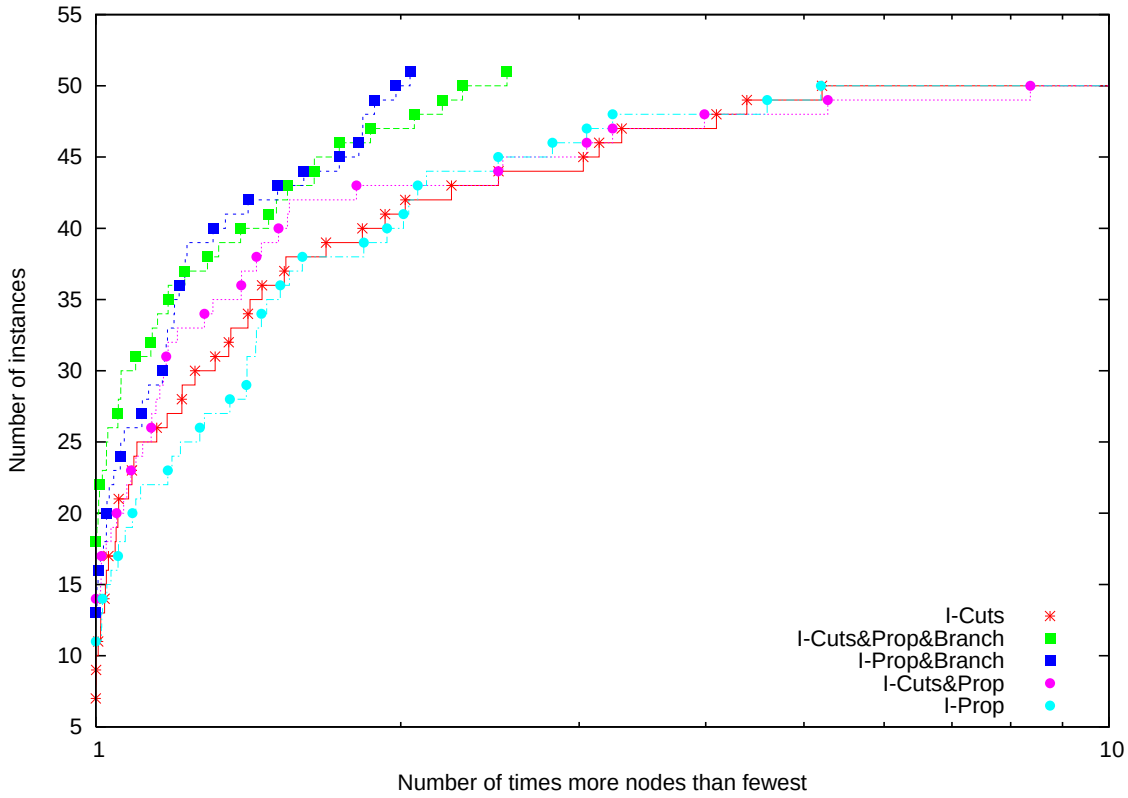



greater performance. Finally, using all the components is only slightly better than just using branching and propagation. This clearly suggests that the value of adding clause inequalities as cuts is rather limited when this information is already provided in another form, as has been already reported by Achterberg (2007a). The most noteworthy finding is the performance of the improved clauses. While using only the basic clauses can reduce time and node counts, the results using improved clauses together with branching and propagation are by far the most significant. We attribute this to the fact that as the clauses get shorter, they are more likely to be active at a node and their effectiveness increases.

The relative performance of the different variants of our approach (when using improved clause information) can also be seen in Figures 3 and 4, where we provide performance profiles in terms of node counts and solution times in the restart phase. (See Figures 5 and 6 in the Appendix for the ones with basic clause information.)

Recall that we make branching and propagation decisions only when there are fractional variables in the solution to the LP relaxation at a node for which we have clause information, otherwise we let CPLEX decide on the branching variable. Table 3 provides summary

Figure 3: Comparison of # of nodes in the restart phase for solvers with improved information




statistics on the number of nodes in the tree where we actually make a decision on the branching variable, both in terms of propagation and branching. The percentages shown represent the number of times we make a branching decision as a percentage of the total number of branching decisions. (Individual results for the instances are given in Table 8 in the Appendix.) The results show that when using improved information, we make many more propagation and branching decisions. However, the number of branching decisions increases more significantly. For example, on average, nearly 50,000 branching decisions are made for the hard instances although we only collect and use 200 clauses.

## 4.3. Effect of Collecting More Information

An important question to address is how many fathomed nodes (and hence clauses) to collect from an incomplete tree. Given that collecting information from the complete B&B tree is unrealistic, we did experiments with a few of the harder instances to see the effect of collecting more fathomed nodes from an incomplete tree. We varied the number of clauses

Figure 4: Comparison of solution times in the restart phase for solvers with improved information

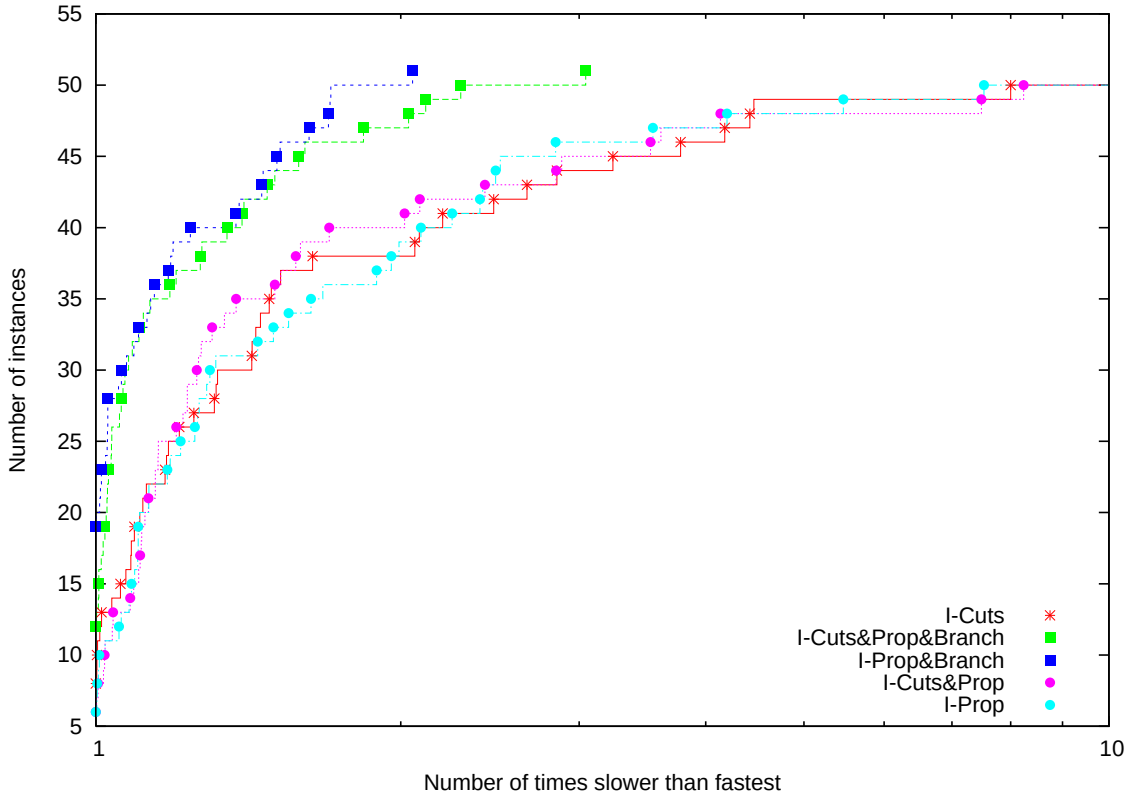



Table 3: Number of times the CPLEX branching is overwritten

| | # decisions in *B-Prop* | *I-Prop* | # decisions in *B-Prop&Branch* | | | # decisions in *I-Prop&Branch* | | |
|---|---|---|---|---|---|---|---|---|
| | Prop | Prop | Prop | Branch | (%) | Prop | Branch | (%) |
| `Arith.Mean(All)` | 91.78 | 1113.37 | 74.47 | 625.53 | 0.80 | 1999.16 | 13409.57 | 17.43 |
| `Geo.Mean(All)` | 26.94 | 151.36 | 41.67 | 375.12 | 3.18 | 140.90 | 966.61 | 9.62 |
| `Arith.Mean(Hard)` | 172.85 | 3828.77 | 96.15 | 994.00 | 0.35 | 7401.54 | 49572.54 | 17.89 |
| `Geo.Mean(Hard)` | 42.09 | 561.49 | 35.68 | 816.67 | 0.55 | 494.70 | 7304.12 | 6.03 |
| `Arith.Mean(Medium)` | 74.91 | 241.30 | 55.74 | 622.96 | 4.65 | 158.30 | 1383.83 | 11.00 |
| `Geo.Mean(Medium)` | 21.99 | 134.48 | 31.06 | 332.80 | 3.63 | 83.51 | 586.21 | 7.86 |
| `Arith.Mean(Easy)` | 47.40 | 97.20 | 84.40 | 310.13 | 13.82 | 139.73 | 507.80 | 23.85 |
| `Geo.Mean(Easy)` | 24.88 | 57.81 | 74.38 | 229.42 | 11.97 | 105.22 | 359.92 | 19.63 |

collected from 10 to 1000 and used the *I-Cuts&Prop&Branch* algorithm with the branching rule combination 3-1-1 in these experiments. The summary results are given in Table 4 and the detailed results are provided in Table 9 in the Appendix.

We observe that the number of improved clauses increases linearly with the number of collected clauses, whereas the average sizes of the collected and improved clauses do not vary much, except for `neos-1480121` and `neos14` (see Table 9 in the Appendix). As

Table 4: Percentage change with respect to *CPLEX-BB* as the number of clauses collected increases

| | **Collection** | | **Improvement** | | **Restart** | | **Total (Arith.)** | | **Total (Geo.)** | |
|---|---|---|---|---|---|---|---|---|---|---|
| **# Clauses** | # nodes | time | # nodes | time | # nodes | time | # nodes | time | # nodes | time |
| *CPLEX-BB* | | | | | | | 100.00 | 100.00 | 100.00 | 100.00 |
| *CPLEX-DS* | | | | | | | 85.67 | 90.91 | 66.76 | 72.86 |
| 10 | 0.10 | 0.15 | 0.02 | 0.21 | 75.74 | 85.68 | 75.86 | 86.04 | 69.37 | 78.34 |
| 100 | 0.42 | 0.66 | 0.32 | 1.76 | 79.92 | 85.25 | 80.66 | 87.68 | 69.40 | 78.64 |
| 200 | 0.78 | 1.16 | 0.66 | 3.20 | 76.79 | 75.99 | 78.22 | 80.35 | 64.94 | 69.02 |
| 500 | 1.68 | 3.05 | 2.38 | 8.82 | 77.36 | 88.38 | 81.42 | 100.26 | 65.49 | 86.86 |
| 1000 | 3.00 | 7.03 | 4.13 | 17.43 | 74.80 | 96.27 | 81.93 | 120.74 | 66.74 | 104.24 |

expected, collecting more information requires more nodes and time spent in the collection and improvement phases. Yet the overall value of using more information in B&B is not clear. For `neos-1228986` and `neos-863472`, there is a significant improvement, however the impact is mixed for the rest of the instances. Note that as the amount of information increases, the time it takes to decide on a branching variable also increases because at each node we have to determine and update the active clauses, calculate the weights for each candidate variable, etc. This observation is very clear in `mas74`, where in the restart phase, the node counts for 100, 200, 500, and 1000 clauses are roughly the same, but the solution times are significantly different.

## 4.4. Effect of Different Branching Variable Selection Rules

Branching rules can have a significant impact on the size of the B&B tree. Table 5 summarizes the performance of the different branching rules detailed in Section 3.2 (see Table 11 from Appendix for detailed results on individual instances.). In these experiments, we limit the number of clauses collected to 200 and use the *I-Cuts&Prop&Branch* algorithm while varying the branching rule combination. The best rule with respect to the node count and time is shown in bold face.

We observe significant variability. For example, the first branching policy 0-0-0 performs very poorly for all instances, except for `neos-1480121` where it is the best (see Table 11 in the Appendix). We expected poor performance of this policy as it weights each active clause the same regardless of its size, uses the maximum count of the number of times each variable occurs in these clauses and weights this with the fractionality of the variable. Although there is a clear indication that giving equal weight to each clause (setting $0 - x - x$) is worse

Table 5: Percentage change with respect to *CPLEX-BB* for different branching rules

| | Restart phase | | | | Total | | | |
|---|---|---|---|---|---|---|---|---|
| | Arith. Mean | | Geo. Mean | | Arith. Mean | | Geo. Mean | |
| Rule | # nodes | time | # nodes | time | # nodes | time | # nodes | time |
| *CPLEX-BB* | | | | | 100.00 | 100.00 | 100.00 | 100.00 |
| *CPLEX-DS* | | | | | 85.67 | 90.91 | 66.76 | 72.86 |
| 0 0 0 | 116.80 | 110.48 | 76.88 | 76.09 | 118.23 | 114.85 | 78.20 | 81.32 |
| 0 0 1 | 115.39 | 112.62 | 99.95 | 99.74 | 116.86 | 116.89 | 101.33 | 104.56 |
| 0 0 2 | 115.39 | 112.75 | 99.95 | 100.02 | 116.86 | 117.02 | 101.33 | 104.86 |
| 0 0 3 | 115.19 | 118.11 | 99.57 | 106.82 | 116.65 | 122.38 | 100.95 | 110.94 |
| 0 1 0 | 82.00 | 80.38 | 65.76 | 68.38 | 83.43 | 84.74 | 67.10 | 72.76 |
| 0 1 1 | 80.29 | 79.40 | 64.13 | 65.27 | 81.73 | 83.76 | 65.45 | 69.24 |
| 0 1 2 | 83.09 | 81.95 | 71.97 | 72.04 | 84.53 | 86.32 | 73.40 | 76.83 |
| 0 1 3 | 78.45 | 80.85 | 65.60 | 69.72 | 79.89 | 85.22 | 66.95 | 73.82 |
| 1 0 0 | 87.53 | 84.50 | 66.05 | 65.43 | 88.96 | 88.86 | 67.25 | 69.69 |
| 1 0 1 | 83.31 | 83.39 | 72.90 | 72.76 | 84.75 | 87.75 | 74.26 | 77.25 |
| 1 0 2 | 82.14 | 81.97 | 71.54 | 71.90 | 83.57 | 86.33 | 72.89 | 76.61 |
| 1 0 3 | 82.28 | 86.49 | 72.48 | 76.62 | 83.71 | 90.86 | 73.81 | 80.59 |
| 1 1 0 | 78.98 | 78.49 | 62.82 | 66.05 | 80.41 | 82.86 | 64.12 | 70.57 |
| 1 1 1 | 76.81 | **75.97** | 61.97 | 63.15 | 78.25 | **80.33** | 63.29 | 67.24 |
| 1 1 2 | 77.41 | 76.66 | 64.57 | 65.12 | 78.84 | 81.03 | 65.90 | 69.73 |
| 1 1 3 | 78.82 | 82.22 | 66.22 | 70.68 | 80.26 | 86.59 | 67.55 | 74.70 |
| 2 0 0 | 80.38 | 78.38 | 64.25 | 65.81 | 81.82 | 82.75 | 65.51 | 70.78 |
| 2 0 1 | 81.62 | 81.02 | 68.23 | 67.90 | 83.05 | 85.38 | 69.51 | 72.12 |
| 2 0 2 | 83.15 | 82.48 | 69.09 | 69.27 | 84.58 | 86.85 | 70.39 | 73.66 |
| 2 0 3 | 79.03 | 83.55 | 66.31 | 70.72 | 80.46 | 87.92 | 67.57 | 74.67 |
| 2 1 0 | 80.68 | 80.47 | 66.31 | 69.01 | 82.11 | 84.84 | 67.62 | 73.48 |
| 2 1 1 | 82.17 | 83.55 | 70.05 | 74.29 | 83.60 | 87.92 | 71.40 | 78.44 |
| 2 1 2 | 81.73 | 81.51 | 65.91 | 65.24 | 83.16 | 85.87 | 67.22 | 69.66 |
| 2 1 3 | 79.51 | 83.91 | 67.26 | 71.14 | 80.94 | 88.28 | 68.55 | 75.05 |
| 3 0 0 | 84.69 | 82.47 | 64.19 | 63.45 | 86.12 | 86.83 | 65.38 | 67.41 |
| 3 0 1 | 79.47 | 77.95 | 68.44 | 68.11 | 80.91 | 82.32 | 69.78 | 72.88 |
| 3 0 2 | 81.87 | 80.49 | 70.92 | 70.37 | 83.30 | 84.85 | 72.28 | 75.30 |
| 3 0 3 | 82.97 | 86.92 | 72.57 | 78.62 | 84.41 | 91.29 | 73.94 | 82.95 |
| 3 1 0 | 78.87 | 77.09 | **55.94** | **55.85** | 80.30 | 81.46 | **57.16** | **60.17** |
| 3 1 1 | **76.79** | 75.99 | 63.63 | 64.95 | **78.22** | 80.35 | 64.94 | 69.02 |
| 3 1 2 | 77.39 | 76.87 | 66.04 | 66.30 | 78.83 | 81.23 | 67.37 | 70.91 |
| 3 1 3 | 77.60 | 82.37 | 67.20 | 72.43 | 79.03 | 86.73 | 68.54 | 76.53 |

than weighting schemes that take clause size into account (settings $1-x-x$, $2-x-x$, and $3-x-x$), there is no clear winner among any of these strategies. We further observe that in general, the policies that simply add the weights of the clauses that a variable occurs in (setting $x-1-x$) rather than the policies that use the maximum of the weights of the clauses that a variable occurs in (setting $x-0-x$) lead to better results. The changes observed when varying the estimation method for $\beta_j$ are smaller, and although there is no policy that clearly outperforms the others with respect to the arithmetic means, $x-x-0$

seems to perform slightly better than the $x-x-1$, $x-x-2$ and $x-x-3$ weighting schemes with respect to the geometric means.

# 5. Conclusions and Further Research

In this research, we have integrated restart strategies and learning mechanisms for binary MILPs. Our computational results demonstrate the effectiveness of learning and restart mechanisms in terms of both node counts and solution times. Most of the improved performance can be attributed to branching and propagation and the use of improved clause information. The benefit of collecting more information from a larger incomplete tree diminishes rapidly. Clause-informed branching, especially when the importance of the clauses are weighted according to their size, seems to be especially beneficial.

Some important topics for further investigation are:

- Is it important to generate minimum cardinality clauses exactly, or should it be done heuristically based on minimal clauses?
- Are there benefits of generating more clauses from each leaf node? (There may be many minimum cardinality clauses corresponding to a leaf node, but we only generate one for each fathomed node. Note that generating multiple clauses will require significant modification in the branching rules as the group of clauses generated from the same leaf node will be highly correlated in terms of the variables involved.)
- Would multiple restarts be beneficial?
- Is it possible to use the clause information to do a partial restart and if so how?
- Should information from open nodes in the incomplete tree be used in the learning phase?

# 6. Acknowledgments

This research has been supported in part by the National Science Foundation under grant CMMI-0522485 and by the Air Force Office of Scientific Research under grant FA9550-07-1-0177. We thank the referees for their thorough reviews.

# References


Achterberg, T. 2007a. Conflict analysis in mixed integer programming. *Discrete Optim.* **4** 4–20.

Achterberg, T. 2007b. Constraint integer programming. Ph.D. thesis, Technische Universität Berlin. `http://opus.kobv.de/tuberlin/volltexte/2007/1611/`.

Achterberg, T. 2009. SCIP: Solving constraint integer programs. *Math. Programming Comput.* **1** 1–32.

Achterberg, T., T. Berthold. 2009. Hybrid branching. Willem Jan van Hoeve, John N. Hooker, eds., *CPAIOR*, *Lecture Notes in Computer Science*, vol. 5547. Springer, 309–311.

Achterberg, T., T. Koch, A. Martin. 2005. Branching rules revisited. *Oper. Res. Lett.* **33** 42–54.

Achterberg, T., T. Koch, A. Martin. 2006. Miplib 2003. *Oper. Res. Lett.* **34** 361–372.

Amaldi, E., M.E. Pfetsch, L.E. Trotter. 2003. On the maximum feasible subsystem problem, IISs, and IIS-hypergraphs. *Math. Programming* **95** 533–554.

Applegate, D., R. Bixby, V. Chvatal, W. Cook. 1998. On the solution of traveling salesman problems. *Documenta Mathematica Journal der Deutschen Mathematiker-Vereinigung*. International Congress of Mathematicians, 645–656.

Avenali, A. 2007. Resolution branch and bound and an application: The maximum weighted stable set problem. *Oper. Res.* **55** 932–948.

Beale, E.M.L., J.A. Tomlin. 1970. Special facilities in a general mathematical programming system for nonconvex problems using ordered sets of variables. *Proc. of the fifth internat. conf. on oper. res.*, vol. 55. Tavistock publication, London, UK, 447–454.

Bénichou, M., J.M. Gauthier, P. Girodet, G. Hentges, G. Ribière, O. Vincen. 1971. Experiments in mixed-integer linear programming. *Math. Programming* **1** 76–94.

Bixby, R.E., S. Ceria, C.M. McZeal, M.W.P. Savelsbergh. 1996. An updated mixed integer programming library: MIPLIB 3.0 .

Chvátal, V. 1997. Resolution search. *Discrete Applied Mathematics* **73** 81–99.

Cor@l. 2009. Computational optimization research at lehigh university. `http://coral.ie.lehigh.edu/mip-instances/`.

Cornuéjols, G., L. Liberti, G. Nannicini. 2008. Improved strategies for branching on general disjunction. Optimization online, `http://www.optimization-online.org/DB_FILE/2008/08/2071.pdf`.

CPLEX. 11.1. ILOG. `http://www.ilog.com/products/cplex`.

Davey, B., N. Boland, P.J. Stuckey. 2002. Efficient intelligent backtracking using linear programming. *INFORMS J. on Comput.* **14** 373–386.

Dixon, H.E., M.L. Ginsberg. 2000. Combining satisfiability techniques from AI and OR. *Knowl. Engrg. Rev.* **15** 31–45.

Dolan, E.D., J.J. Moré. 2002. Benchmarking optimization software with performance profiles. *Math. Programming* **91** 201–213.

Forrest, J.J.H., J.P.H. Hirst, J.A Tomlin. 1974. Practical solution of large scale mixed integer programming problems with UMPIRE. *Management Sci.* **20** 736–773.

Gilpin, A., T. Sandholm. 2007. Information-theoretic approaches to branching in search. *Proc. of the internat. joint conf. on artificial intelligence (IJCAI)*. Hyderabad, India.

Glankwamdee, W., J.T. Linderoth. 2006. Lookahead branching for mixed integer programming. Optimization online, `http://www.optimization-online.org/DB_FILE/2006/10/1490.pdf`.

Gleeson, J., J. Ryan. 1990. Identifying minimally infeasible subsystem of inequalities. *ORSA J. on Comput.* **2** 61–63.

Gomes, C., H. Kautz, A. Sabharwal, B. Selman. 2008. Satisfiability solvers. *Handbook of Knowledge Representation*, *Foundations of Artificial Intelligence*, vol. 3. Elsevier, 89–134.

Hooker, J.N. 1998. Constraint satisfaction methods for generating valid cuts. D. L. Woodruff, ed., *Advances in computational and stochastic optimization, logic programming, and heuristic search: interfaces in computer science and operations research*. Kluwer Academic Publishers, Norwell, MA, USA, 1–30.

Hooker, J.N. 2000. *Logic-based methods for optimization: Combining optimization and constraint satisfaction*. John Wiley & Sons, New York, NY.

Karamanov, M., G. Cornuéjols. 2005. Branching on general disjunctions. Technical report, `http://integer.tepper.cmu.edu/webpub/d-branching.pdf`.

Kılınç Karzan, F., G.L. Nemhauser, M.W.P. Savelsbergh. 2008. Information based branching rules in integer programming. *INFORMS Annual Meeting*. Washington, DC, USA.

Linderoth, J.T., M.W.P. Savelsbergh. 1999. A computational study of search strategies for mixed integer programming. *INFORMS J. on Comput.* **11** 173–187.

Mahajan, A., T.K. Ralphs. 2008a. Experiments with branching using general disjunctions. Optimization online, `http://www.optimization-online.org/DB_FILE/2008/06/2020.pdf`.

Mahajan, A., T.K. Ralphs. 2008b. On the complexity of branching on general hyperplanes for integer programming. Optimization online, `http://www.optimization-online.org/DB_FILE/2008/10/2119.pdf`.

Marques-Silva, J.P., K.A. Sakallah. 1999. Grasp: A search algorithm for propositional satisfiability. *IEEE Transactions on Computers* **48** 506–521.

MINTO. 3.1. MINTO: Mixed INTeger Optimizer. Georgia Institute of Technology, `http://coral.ie.lehigh.edu/minto/`.

Mittelmann, H.D. 2009. Milptestset. `http://plato.asu.edu/ftp/milp/`.

Moskewicz, M.W., C.F. Madigan, Y. Zhao, L. Zhang, S. Malik. 2001. Chaff: Engineering an efficient sat solver. *DAC '01: Proceedings of the 38th annual Design Automation Conference*.

Owen, J.H., S. Mehrotra. 2001. Experimental results on using general disjunctions in branch-and-bound for general-integer linear programs. *Comput. Optim. Appl.* **20** 159–170.

Patel, J., J.W. Chinneck. 2007. Active-constraint variable ordering for faster feasibility of mixed integer linear programs. *Math. Programming* **110** 445–474.

Sandholm, T., R. Shields. 2006. Nogood learning for mixed integer programming. Technical report, CMU-CS-06-155, Carnegie Mellon University, Computer Science Department, http://www.cs.cmu.edu/ sandholm/nogoodsForMip.techReport06.pdf.

SCIP. 1.1.0. SCIP: Solving constraint integer programs. Zuse Institute Berlin, http://scip.zib.de.

Stallman, R.M., G.J. Sussman. 1977. Forward reasoning and dependency-directed backtracking in a system for computer-aided circuit analysis. *Artif. Intell.* **9** 135–196.

# Appendix

Table 6: Value of the restart strategy with fathoming-based branching

| | | Collection phase | | | Improvement phase | | | | | Restart phase *Cuts&Prop&Branch* | | Total | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Problem | | | # | size | min | max | total | # (%) | size | *B* | *I* | *CPLEX-DS* | *CPLEX-BB* | *Cuts&Prop&Branch* *B* | *I* |
| neos-1480121 | # nodes | 798 | 200 | 38.17 | 0 | 0 | 0 | 50 (25.0) | 32.63 | 483513 | 187946 | 88346 | 1711005 | 484311 | 188744 |
| | time | 5.7 | | | 0 | 0 | 3.8 | | | 1243.9 | 409.5 | 394.3 | 2559.4 | 1249.6 | 419 |
| mas74 | # nodes | 1190 | 200 | 26.41 | 0 | 396 | 3836 | 118 (59.0) | 17.68 | 1062473 | 1130859 | 1695810 | 1068579 | 1063663 | 1135885 |
| | time | 6.3 | | | 0 | 0.4 | 15.8 | | | 3378.1 | 3704.5 | 6319.3 | 3488.8 | 3384.4 | 3726.6 |
| neos5 | # nodes | 1212 | 200 | 24.98 | 0 | 42 | 234 | 144 (72.0) | 12.46 | 693074 | 1178505 | 648299 | 707857 | 694286 | 1179951 |
| | time | 8.3 | | | 0 | 0.2 | 10.1 | | | 2116.2 | 3012.8 | 2042.4 | 2223.7 | 2124.5 | 3031.2 |
| markshare_4_0 | # nodes | 336 | 200 | 18.12 | 0 | 102 | 935 | 135 (67.5) | 12.47 | 545488 | 503334 | 566080 | 521166 | 545824 | 504605 |
| | time | 0.2 | | | 0 | 0.1 | 3.5 | | | 215.9 | 199.6 | 148.1 | 199.5 | 216.1 | 203.3 |
| neos23 | # nodes | 1030 | 200 | 21.13 | 0 | 10 | 105 | 46 (23.0) | 17.31 | 104033 | 90175 | 51820 | 158582 | 105063 | 91310 |
| | time | 27.4 | | | 0.1 | 0.5 | 29.1 | | | 1977.3 | 1686.6 | 615.5 | 2959 | 2004.7 | 1743.1 |
| neos-1228986 | # nodes | 1031 | 200 | 16.25 | 0 | 44 | 469 | 106 (53.0) | 11.15 | 35550 | 31388 | 117852 | 137818 | 36581 | 32888 |
| | time | 23.7 | | | 0 | 0.5 | 22.2 | | | 422.9 | 332.6 | 1748.2 | 1458.4 | 446.6 | 378.5 |
| mas76 | # nodes | 1195 | 200 | 22 | 0 | 141 | 1662 | 109 (54.5) | 14.96 | 130850 | 126488 | 159932 | 126188 | 132045 | 129345 |
| | time | 5.6 | | | 0 | 0.2 | 9.3 | | | 347.2 | 314.9 | 552 | 312.5 | 352.8 | 329.8 |
| neos-863472 | # nodes | 1008 | 200 | 26.5 | 0 | 41 | 381 | 124 (62.0) | 13.74 | 85612 | 57466 | 124665 | 106332 | 86620 | 58855 |
| | time | 33.2 | | | 0.1 | 1.2 | 80.1 | | | 3310.4 | 2116.1 | 5532.6 | 4026.2 | 3343.6 | 2229.4 |
| pk1 | # nodes | 1447 | 200 | 29.86 | 0 | 697 | 4255 | 113 (56.5) | 19.39 | 82254 | 87185 | 107129 | 85072 | 83701 | 92887 |
| | time | 7.5 | | | 0 | 0.9 | 17.5 | | | 256.8 | 276.2 | 277 | 257.7 | 264.3 | 301.2 |
| swath2 | # nodes | 656 | 200 | 46.52 | 0 | 1 | 2 | 76 (38.0) | 28.43 | 53890 | 54639 | 41937 | 73907 | 54546 | 55297 |
| | time | 43.8 | | | 1.2 | 4.5 | 465.2 | | | 2431.7 | 2541.6 | 2228.8 | 3288.2 | 2475.5 | 3050.6 |
| neos-1211578 | # nodes | 419 | 200 | 13.69 | 0 | 42 | 463 | 112 (56.0) | 9.65 | 60625 | 29235 | 40683 | 73016 | 61044 | 30117 |
| | time | 6.6 | | | 0 | 0.3 | 18.9 | | | 899.3 | 368.6 | 575.6 | 1026.6 | 905.9 | 394.1 |
| neos3 | # nodes | 538 | 200 | 39.98 | 0 | 51 | 155 | 180 (90.0) | 7.1 | 55418 | 52345 | 21618 | 64223 | 55956 | 53038 |
| | time | 34.9 | | | 0.4 | 5.5 | 195.4 | | | 3608.7 | 3432.5 | 1477.3 | 4158.3 | 3643.6 | 3662.8 |
| neos14 | # nodes | 1468 | 200 | 18.73 | 0 | 11 | 11 | 2 (1.0) | 18.61 | 61044 | 52844 | 70550 | 61106 | 62512 | 54323 |
| | time | 73.6 | | | 0 | 3 | 85.8 | | | 2425.1 | 2075.3 | 2154.1 | 2206.2 | 2498.7 | 2234.7 |
| neos-803220 | # nodes | 748 | 200 | 18.34 | 0 | 0 | 0 | 0 (0.0) | 18.34 | 37720 | 37720 | 69387 | 41066 | 38468 | 38468 |
| | time | 27.9 | | | 0.1 | 0.1 | 14.5 | | | 849.8 | 849.8 | 1313.9 | 917.8 | 877.7 | 892.2 |
| pg5_34 | # nodes | 845 | 200 | 22.3 | 0 | 0 | 0 | 0 (0.0) | 22.3 | 39185 | 39185 | 41295 | 37532 | 40030 | 40030 |
| | time | 46.3 | | | 3.1 | 3.9 | 737.4 | | | 992.8 | 992.8 | 1092.8 | 1025.2 | 1039.1 | 1776.5 |
| neos-1440447 | # nodes | 406 | 200 | 12.86 | 0 | 37 | 943 | 106 (53.0) | 8.98 | 19849 | 22359 | 47358 | 36721 | 20255 | 23708 |
| | time | 13.4 | | | 0.1 | 0.6 | 33.8 | | | 551.3 | 785.1 | 4038.2 | 1082.4 | 564.7 | 832.3 |
| neos22 | # nodes | 782 | 200 | 26.24 | 0 | 0 | 0 | 27 (13.5) | 22.38 | 31632 | 27596 | 493 | 31919 | 32414 | 28378 |
| | time | 67.6 | | | 2.9 | 5.5 | 967.4 | | | 1942.3 | 1724.6 | 30.4 | 1898.8 | 2009.9 | 2759.6 |
| neos-1173026 | # nodes | 428 | 200 | 9.02 | 0 | 21 | 107 | 0 (0.0) | 9.02 | 239 | 239 | 489 | 29768 | 667 | 774 |
| | time | 14.6 | | | 0.1 | 1.4 | 63.8 | | | 7 | 7 | 28.6 | 2279 | 21.6 | 85.4 |
| neos-538867 | # nodes | 994 | 200 | 38.4 | 0 | 340 | 1679 | 137 (68.5) | 17.24 | 16532 | 14704 | 28075 | 24817 | 17526 | 17377 |
| | time | 51.9 | | | 0.1 | 2.3 | 120.2 | | | 531.4 | 382.5 | 1018.6 | 980.3 | 583.3 | 554.6 |
| neos-598183 | # nodes | 457 | 200 | 13.53 | 0 | 0 | 0 | 0 (0.0) | 13.53 | 15708 | 15708 | 571 | 23741 | 16165 | 16165 |
| | time | 22.5 | | | 0.3 | 0.7 | 68.8 | | | 601.8 | 601.8 | 19.4 | 817.2 | 624.3 | 693.1 |
| stein45 | # nodes | 1614 | 200 | 15.46 | 0 | 0 | 0 | 0 (0.0) | 15.46 | 19235 | 19235 | 21882 | 18798 | 20849 | 20849 |
| | time | 16.2 | | | 0 | 0 | 3.5 | | | 99.4 | 99.4 | 105.8 | 96.2 | 115.6 | 119.1 |

Continued on next page

Table 6: Value of the restart strategy with fathoming-based branching

| Problem | | Collection phase | | | Improvement phase | | | | | Restart phase *Cuts&Prop&Branch* | | Total | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | # | size | min | max | total | # (%) | size | *B* | *I* | *CPLEX-DS* | *CPLEX-BB* | *Cuts&Prop&Branch* *B* | *I* |
| harp2 | # nodes | 588 | 200 | 24.65 | 0 | 0 | 0 | 0 (0.0) | 24.65 | 10726 | 10726 | 27245 | 14144 | 11314 | 11314 |
| | time | 31.9 | | | 0 | 0 | 1.7 | | | 583.6 | 583.6 | 1850.9 | 650.9 | 615.5 | 617.2 |
| prod2 | # nodes | 560 | 200 | 14.51 | 0 | 9 | 119 | 126 (63.0) | 8.38 | 14300 | 13526 | 17773 | 13587 | 14860 | 14205 |
| | time | 20 | | | 0.1 | 1.1 | 54.1 | | | 342.9 | 339.4 | 396.2 | 326.9 | 362.9 | 413.5 |
| prod1 | # nodes | 405 | 200 | 15.94 | 0 | 3 | 61 | 160 (80.0) | 6.22 | 12904 | 13175 | 15916 | 13258 | 13309 | 13641 |
| | time | 8.2 | | | 0.1 | 0.3 | 22 | | | 198.3 | 161.7 | 165.2 | 149 | 206.5 | 191.9 |
| neos-631694 | # nodes | 821 | 200 | 35.24 | 0 | 0 | 0 | 4 (2.0) | 34.5 | 16400 | 1429 | 516 | 11988 | 17221 | 2250 |
| | time | 117.5 | | | 2.3 | 5.7 | 592 | | | 1094.2 | 231.1 | 96.3 | 1639.4 | 1211.7 | 940.6 |
| neos-860244 | # nodes | 471 | 200 | 29.45 | 0 | 9 | 49 | 142 (71.0) | 9.64 | 2177 | 2009 | 3204 | 10482 | 2648 | 2529 |
| | time | 80.8 | | | 2.1 | 6.3 | 658.2 | | | 328 | 305.5 | 435.1 | 1975.2 | 408.8 | 1044.5 |
| neos-538916 | # nodes | 850 | 200 | 28.86 | 0 | 272 | 1942 | 130 (65.0) | 14.83 | 11238 | 7101 | 20382 | 10363 | 12088 | 9893 |
| | time | 54 | | | 0.1 | 4.2 | 156.7 | | | 540.6 | 255.6 | 920.1 | 529.2 | 594.6 | 466.3 |
| qiu | # nodes | 1084 | 200 | 13.69 | 0 | 164 | 2446 | 93 (46.5) | 11.4 | 10147 | 3411 | 11568 | 9121 | 11231 | 6941 |
| | time | 250.2 | | | 0.6 | 5.3 | 425.5 | | | 1499.5 | 517.2 | 1662.5 | 1518 | 1749.7 | 1192.9 |
| misc07 | # nodes | 690 | 200 | 31.75 | 0 | 198 | 782 | 100 (50.0) | 18.89 | 8905 | 8250 | 10472 | 8427 | 9595 | 9722 |
| | time | 33.4 | | | 0.1 | 1.3 | 36.3 | | | 331.5 | 264.4 | 408.4 | 305.3 | 364.9 | 334.1 |
| neos-886822 | # nodes | 2427 | 200 | 25.48 | 0 | 3 | 141 | 148 (74.0) | 10.11 | 8232 | 8571 | 8488 | 8408 | 10659 | 11139 |
| | time | 376.4 | | | 0.5 | 5 | 268.1 | | | 976.7 | 1043.4 | 1096.4 | 1002 | 1353.1 | 1687.9 |
| neos-480878 | # nodes | 927 | 200 | 17.43 | 0 | 253 | 1693 | 81 (40.5) | 13.98 | 8434 | 12150 | 15275 | 8384 | 9361 | 14770 |
| | time | 100.4 | | | 0.4 | 3.4 | 149.2 | | | 663 | 942.9 | 1167 | 657.1 | 763.4 | 1192.5 |
| neos-1200887 | # nodes | 591 | 200 | 13.78 | 0 | 78 | 1764 | 129 (64.5) | 8.84 | 4788 | 2154 | 10972 | 7117 | 5379 | 4509 |
| | time | 61.6 | | | 0.1 | 1.6 | 87.1 | | | 329.1 | 118 | 731.5 | 508.1 | 390.7 | 266.7 |
| neos-807639 | # nodes | 547 | 200 | 13.79 | 0 | 0 | 0 | 0 (0.0) | 13.79 | 6006 | 6006 | 5928 | 6858 | 6553 | 6553 |
| | time | 37.4 | | | 0.1 | 0.2 | 34.8 | | | 242.6 | 242.6 | 189 | 272.6 | 280 | 314.8 |
| neos2 | # nodes | 566 | 200 | 64.19 | 0 | 28 | 162 | 129 (64.5) | 31.48 | 6968 | 8283 | 3163 | 6574 | 7534 | 9011 |
| | time | 36 | | | 0.3 | 5 | 125.7 | | | 330.4 | 392.5 | 218.1 | 316.4 | 366.4 | 554.2 |
| bienst1 | # nodes | 609 | 200 | 14.99 | 0 | 61 | 1139 | 140 (70.0) | 7.75 | 5266 | 2358 | 5953 | 5411 | 5875 | 4106 |
| | time | 87.2 | | | 0.3 | 5.2 | 332.7 | | | 546.2 | 257.9 | 334.6 | 521.6 | 633.4 | 677.8 |
| neos-504674 | # nodes | 738 | 200 | 18.01 | 0 | 301 | 1882 | 47 (23.5) | 15.92 | 4830 | 4949 | 4078 | 5384 | 5568 | 7569 |
| | time | 83.9 | | | 0.5 | 5.2 | 311.4 | | | 412.8 | 412.6 | 546.4 | 450.9 | 496.7 | 807.9 |
| rout | # nodes | 3086 | 200 | 19.8 | 0 | 3 | 34 | 55 (27.5) | 15.62 | 6185 | 6337 | 11115 | 4865 | 9271 | 9457 |
| | time | 138.8 | | | 0.1 | 1.1 | 26.8 | | | 384.5 | 366.2 | 1474.3 | 322 | 523.3 | 531.8 |
| neos-1396125 | # nodes | 597 | 200 | 26.98 | 0 | 18 | 232 | 76 (38.0) | 19.62 | 3904 | 2144 | 3474 | 4660 | 4501 | 2973 |
| | time | 195.1 | | | 2.1 | 5.3 | 878.1 | | | 1140.7 | 513.2 | 930.4 | 1414.7 | 1335.8 | 1586.4 |
| neos21 | # nodes | 374 | 200 | 16.48 | 0 | 56 | 65 | 2 (1.0) | 16.43 | 3225 | 3274 | 4370 | 4649 | 3599 | 3713 |
| | time | 83.4 | | | 0.2 | 1.7 | 61.1 | | | 560.8 | 570.8 | 803.5 | 850 | 644.2 | 715.3 |
| neos-892255 | # nodes | 716 | 200 | 24.3 | 0 | 7 | 132 | 97 (48.5) | 15.5 | 4419 | 3989 | 4340 | 3642 | 5135 | 4837 |
| | time | 386.5 | | | 1 | 5.4 | 757.9 | | | 1616.3 | 1416.5 | 1835.4 | 1646.6 | 2002.8 | 2560.9 |
| neos-1109824 | # nodes | 400 | 200 | 21.41 | 0 | 94 | 383 | 104 (52.0) | 14.54 | 2280 | 2242 | 2686 | 3439 | 2680 | 3025 |
| | time | 56.4 | | | 0.4 | 5.1 | 303.6 | | | 221.6 | 212.7 | 582.2 | 452.8 | 278 | 572.7 |
| lrn | # nodes | 282 | 200 | 59.17 | 0 | 0 | 0 | 0 (0.0) | 59.17 | 1331 | 1331 | 1778 | 2850 | 1613 | 1613 |
| | time | 115.6 | | | 5 | 5.3 | 1030.2 | | | 375.2 | 375.2 | 642.5 | 642.2 | 490.8 | 1521 |
| neos-570431 | # nodes | 331 | 200 | 12.18 | 0 | 13 | 49 | 6 (3.0) | 11.98 | 887 | 1137 | 3579 | 2433 | 1218 | 1517 |

Continued on next page

Table 6: Value of the restart strategy with fathoming-based branching

| Problem | | Collection phase time | Collection phase # | Collection phase size | Improvement phase min | Improvement phase max | Improvement phase total | Improvement phase # (%) | Improvement phase size | Restart phase *Cuts&Prop&Branch* B | Restart phase *Cuts&Prop&Branch* I | Total *CPLEX-DS* | Total *CPLEX-BB* | Total *Cuts&Prop&Branch* B | Total *Cuts&Prop&Branch* I |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | time | 74.9 | | | 0.2 | 1.6 | 66.2 | | | 178.9 | 204.4 | 598.9 | 443.3 | 253.8 | 345.5 |
| neos-512201 | # nodes | 576 | 200 | 16.52 | 0 | 228 | 2710 | 66 (33.0) | 13.55 | 1602 | 2203 | 4482 | 2186 | 2178 | 5489 |
| | time | 54.8 | | | 0.5 | 5.2 | 415.7 | | | 117.2 | 170.4 | 596 | 170.7 | 172 | 640.9 |
| neos-806323 | # nodes | 639 | 200 | 14.76 | 0 | 0 | 0 | 0 (0.0) | 14.76 | 2471 | 2471 | 2615 | 2041 | 3110 | 3110 |
| | time | 54.8 | | | 0.1 | 0.2 | 33.6 | | | 191.3 | 191.3 | 186.3 | 152.7 | 246.1 | 279.7 |
| neos-803219 | # nodes | 421 | 200 | 16.95 | 0 | 0 | 0 | 0 (0.0) | 16.95 | 1893 | 1893 | 2591 | 1987 | 2314 | 2314 |
| | time | 17 | | | 0.1 | 0.1 | 14.5 | | | 65.1 | 65.1 | 84.6 | 68.3 | 82.1 | 96.6 |
| neos-504815 | # nodes | 522 | 200 | 14.99 | 0 | 517 | 2306 | 59 (29.5) | 12.65 | 1389 | 1412 | 1381 | 1575 | 1911 | 4240 |
| | time | 36.7 | | | 0.3 | 5.1 | 317.1 | | | 67.5 | 60.1 | 97 | 75.8 | 104.2 | 413.9 |
| neos-820879 | # nodes | 421 | 200 | 20.18 | 0 | 7 | 7 | 76 (38.0) | 15.25 | 1262 | 1489 | 801 | 1348 | 1683 | 1917 |
| | time | 55.9 | | | 1.3 | 5.6 | 738.2 | | | 138.6 | 152.3 | 91.3 | 147.3 | 194.5 | 946.4 |
| neos-807705 | # nodes | 649 | 200 | 11.86 | 0 | 0 | 0 | 0 (0.0) | 11.86 | 1320 | 1320 | 1633 | 1231 | 1969 | 1969 |
| | time | 57.4 | | | 0.1 | 0.2 | 34.2 | | | 107.3 | 107.3 | 125.6 | 100.6 | 164.7 | 198.9 |
| bc1 | # nodes | 378 | 200 | 26.87 | 0 | 0 | 0 | 44 (22.0) | 21.78 | 608 | 479 | 1176 | 1210 | 986 | 857 |
| | time | 257 | | | 2.4 | 5.2 | 806 | | | 554 | 412.7 | 981.7 | 993.7 | 811 | 1475.7 |
| seymour1 | # nodes | 715 | 200 | 14.34 | 0 | 0 | 0 | 0 (0.0) | 14.34 | 1405 | 1405 | 1291 | 1171 | 2120 | 2120 |
| | time | 988 | | | 5 | 5.1 | 1011.9 | | | 1584.5 | 1584.5 | 1399.5 | 1359.3 | 2572.5 | 3584.4 |
| Arith.Mean | # nodes | 795.8 | 200 | 23.08 | 0 | 84.3 | 653.6 | 70.57 (35.3) | 16.61 | 74498.6 | 76399.6 | 81421.9 | 104274.6 | 75294.4 | 77849 |
| (All: 51 models) | time | 88 | | | 0.7 | 2.6 | 249.3 | | | 861.4 | 752.6 | 1026.7 | 1116.2 | 949.4 | 1089.9 |
| Geo.Mean | # nodes | 691.2 | 200 | 21.07 | 0 | 13.7 | 56.8 | 24.56 (12.3) | 15.14 | 11688.6 | 10095.2 | 11277.7 | 15729.7 | 13418.5 | 12804.8 |
| (All: 51 models) | time | 41.9 | | | 0.5 | 1.9 | 86.9 | | | 494.2 | 417.2 | 524.3 | 677.3 | 581.3 | 712.6 |
| Arith.Mean | # nodes | 948.3 | 200 | 26.33 | 0 | 121.4 | 962.1 | 101.15 (50.6) | 16.58 | 265678.8 | 275569.9 | 287286.2 | 376527 | 266627.1 | 277480.3 |
| (Hard: 13 models) | time | 21.3 | | | 0.1 | 1.3 | 73.6 | | | 1741 | 1574.7 | 1851.2 | 2166.5 | 1762.3 | 1669.6 |
| Geo.Mean | # nodes | 866.7 | 200 | 24.8 | 0 | 35.4 | 183.9 | 77.72 (38.9) | 15.37 | 139124.4 | 119371.1 | 125247.4 | 187595.9 | 140568.2 | 121816.5 |
| (Hard: 13 models) | time | 12.8 | | | 0.1 | 0.9 | 27.1 | | | 1185.5 | 965.6 | 1083.2 | 1486.4 | 1203.8 | 1026 |
| Arith.Mean | # nodes | 789 | 200 | 22.52 | 0 | 77.3 | 648.2 | 73.87 (36.9) | 15.55 | 13540 | 12210.6 | 16108 | 16689.9 | 14329 | 13647.8 |
| (Medium: 23 models) | time | 71.3 | | | 0.6 | 2.8 | 228.9 | | | 608.5 | 500.5 | 776.8 | 866.1 | 679.8 | 800.7 |
| Geo.Mean | # nodes | 710.3 | 200 | 20.31 | 0 | 12 | 56.9 | 21.32 (10.7) | 14.13 | 9465.2 | 7512.4 | 7686.5 | 13503.9 | 10685.6 | 9730.4 |
| (Medium: 23 models) | time | 46.3 | | | 0.4 | 2.1 | 96.4 | | | 428.3 | 342.3 | 383.3 | 654.8 | 498.5 | 595.5 |
| Arith.Mean | # nodes | 673.8 | 200 | 21.12 | 0 | 62.9 | 394.5 | 39.00 (19.5) | 18.27 | 2278.7 | 2208.4 | 3154.1 | 2619.1 | 2952.5 | 3276.7 |
| (Easy: 15 models) | time | 171.5 | | | 1.3 | 3.5 | 433 | | | 486.9 | 426.9 | 695.3 | 589.3 | 658.4 | 1031.4 |
| Geo.Mean | # nodes | 544.7 | 200 | 19.34 | 0 | 7.1 | 20 | 10.86 (5.4) | 16.62 | 1887.4 | 1866.3 | 2519.1 | 2318.5 | 2483.5 | 2768.3 |
| (Easy: 15 models) | time | 97.6 | | | 0.8 | 2.7 | 200.6 | | | 288.1 | 272.9 | 451.8 | 360.8 | 391.3 | 684.1 |

Table 7: Effect of providing information in different forms: Cuts, Propagation, Branching

| Problem | | *CPLEX* | | *Cuts* | | *Prop* | | *Prop&Branch* | | *Cuts&Prop* | | *Cuts&Prop&Branch* | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | DS | B&B | B | I | B | I | B | I | B | I | B | I |
| neos-1480121 | # nodes | 88346 | 1711005 | 1410056 | 519044 | 1373113 | 179692 | 572460 | 192307 | 1408791 | 165316 | 483513 | 187946 |
| | time | 394.3 | 2559.4 | 2712.2 | 1138.3 | 2160.7 | 287.3 | 1207 | 374.9 | 2834.9 | 272.6 | 1243.9 | 409.5 |
| mas74 | # nodes | 1695810 | 1068579 | 1068579 | 1068579 | 1068566 | 1068562 | 1062473 | 1130859 | 1068566 | 1068562 | 1062473 | 1130859 |
| | time | 6319.3 | 3488.8 | 3417.6 | 3409.8 | 3638.3 | 3754.1 | 3547.6 | 3862.8 | 3629 | 3716.4 | 3378.1 | 3704.5 |
| neos5 | # nodes | 648299 | 707857 | 707857 | 707857 | 707415 | 677465 | 693074 | 1178505 | 707415 | 677465 | 693074 | 1178505 |
| | time | 2042.4 | 2223.7 | 2124.4 | 2040.9 | 2222.5 | 2229.4 | 2183.4 | 2975.3 | 2142.4 | 2249.8 | 2116.2 | 3012.8 |
| markshare_4_0 | # nodes | 566080 | 521166 | 521166 | 506296 | 521611 | 526805 | 545488 | 500836 | 521611 | 506282 | 545488 | 503334 |
| | time | 148.1 | 199.5 | 199.7 | 194.3 | 218.6 | 228.7 | 226.1 | 206.0 | 213.9 | 217.3 | 215.9 | 199.6 |
| neos23 | # nodes | 51820 | 158582 | 190894 | 174018 | 235803 | 137158 | 229827 | 105414 | 121610 | 163040 | 104033 | 90175 |
| | time | 615.5 | 2959 | 3273.1 | 3518.4 | 4031.5 | 2613.6 | 3436.9 | 1775.9 | 2506.7 | 3402.1 | 1977.3 | 1686.6 |
| neos-1228986 | # nodes | 117852 | 137818 | 137818 | 137818 | 137937 | 101588 | 35550 | 31388 | 137937 | 101588 | 35550 | 31388 |
| | time | 1748.2 | 1458.4 | 1490.9 | 1442.3 | 1509.4 | 1146.7 | 418.8 | 323.3 | 1497.5 | 1140.4 | 422.9 | 332.6 |
| mas76 | # nodes | 159932 | 126188 | 126188 | 126188 | 126181 | 127798 | 130850 | 126488 | 126181 | 127798 | 130850 | 126488 |
| | time | 552.0 | 312.5 | 311 | 319.9 | 307.6 | 348.0 | 337.7 | 308.5 | 315.6 | 355.4 | 347.2 | 314.9 |
| neos-863472 | # nodes | 124665 | 106332 | 116146 | 60422 | 106715 | 105666 | 81350 | 60753 | 111297 | 61668 | 85612 | 57466 |
| | time | 5532.6 | 4026.2 | 4580.4 | 2291.3 | 4029.9 | 4006.0 | 3032.1 | 2134.3 | 4418.3 | 2422.7 | 3310.4 | 2116.1 |
| pk1 | # nodes | 107129 | 85072 | 85072 | 85072 | 83694 | 88063 | 82254 | 87185 | 83694 | 88063 | 82254 | 87185 |
| | time | 277.0 | 257.7 | 260.3 | 258.6 | 256 | 278.9 | 263.6 | 277.3 | 255.9 | 279.6 | 256.8 | 276.2 |
| swath2 | # nodes | 41937 | 73907 | 73907 | 73907 | 74145 | 69949 | 53890 | 54639 | 74145 | 69949 | 53890 | 54639 |
| | time | 2228.8 | 3288.2 | 3255.8 | 3286.6 | 3321.7 | 3174.4 | 2437.1 | 2510.0 | 3298.7 | 3157.3 | 2431.7 | 2541.6 |
| neos-1211578 | # nodes | 40683 | 73016 | 73016 | 73016 | 72999 | 72965 | 60625 | 29235 | 72999 | 72965 | 60625 | 29235 |
| | time | 575.6 | 1026.6 | 1010.4 | 1033.8 | 1041.6 | 1030.8 | 896.9 | 362.4 | 1011.2 | 1045.2 | 899.3 | 368.6 |
| neos3 | # nodes | 21618 | 64223 | 61364 | 48108 | 64318 | 23813 | 55510 | 48633 | 60215 | 36067 | 55418 | 52345 |
| | time | 1477.3 | 4158.3 | 4103.5 | 3294.8 | 4191 | 1497.5 | 3617.8 | 3077.1 | 3964.7 | 2359.3 | 3608.7 | 3432.5 |
| neos14 | # nodes | 70550 | 61106 | 61043 | 44868 | 61054 | 60802 | 61013 | 55164 | 61120 | 44834 | 61044 | 52844 |
| | time | 2154.1 | 2206.2 | 2208.7 | 1728.8 | 2208.5 | 2240.4 | 2247.8 | 1942.3 | 2224.6 | 1765.6 | 2425.1 | 2075.3 |
| neos-803220 | # nodes | 69387 | 41066 | 40954 | 40954 | 41308 | 41308 | 37911 | 37911 | 40868 | 40868 | 37720 | 37720 |
| | time | 1313.9 | 917.8 | 939.2 | 939.2 | 929.5 | 929.5 | 844 | 844.0 | 936.8 | 936.8 | 849.8 | 849.8 |
| pg5_34 | # nodes | 41295 | 37532 | 39151 | 39151 | 39459 | 39459 | 39937 | 39937 | 38931 | 38931 | 39185 | 39185 |
| | time | 1092.8 | 1025.2 | 1200.2 | 1200.2 | 996 | 996.0 | 1005.5 | 1005.5 | 1191.8 | 1191.8 | 992.8 | 992.8 |
| neos-1440447 | # nodes | 47358 | 36721 | 33441 | 24381 | 36721 | 45530 | 20034 | 26319 | 33441 | 26354 | 19849 | 22359 |
| | time | 4038.2 | 1082.4 | 1126.2 | 1045.3 | 1085.9 | 1403.5 | 561 | 704.7 | 1139.3 | 1197.8 | 551.3 | 785.1 |
| neos22 | # nodes | 493 | 31919 | 31469 | 33305 | 33287 | 32540 | 25876 | 27364 | 32043 | 31910 | 31632 | 27596 |
| | time | 30.4 | 1898.8 | 1903.6 | 2013.5 | 2029.1 | 1934.5 | 1617.4 | 1714.8 | 1971.8 | 1932.6 | 1942.3 | 1724.6 |
| neos-1173026 | # nodes | 489 | 29768 | 1731 | 1731 | 29768 | 29768 | 128 | 128 | 7174 | 7174 | 239 | 239 |
| | time | 28.6 | 2279 | 113.5 | 113.5 | 2335.3 | 2335.3 | 3.8 | 3.8 | 604.2 | 604.2 | 7 | 7 |
| neos-538867 | # nodes | 28075 | 24817 | 24817 | 24817 | 24817 | 22828 | 16532 | 14704 | 24817 | 22828 | 16532 | 14704 |
| | time | 1018.6 | 980.3 | 984.5 | 1017.5 | 1001.2 | 920.1 | 530.4 | 381.9 | 1015.1 | 924.8 | 531.4 | 382.5 |
| neos-598183 | # nodes | 571 | 23741 | 18040 | 18040 | 23163 | 23163 | 17445 | 17445 | 17811 | 17811 | 15708 | 15708 |
| | time | 19.4 | 817.2 | 600.7 | 600.7 | 820.6 | 820.6 | 568 | 568.0 | 590.7 | 590.7 | 601.8 | 601.8 |
| stein45 | # nodes | 21882 | 18798 | 18798 | 18798 | 18781 | 18781 | 19235 | 19235 | 18781 | 18781 | 19235 | 19235 |
| | time | 105.8 | 96.2 | 98.4 | 98.4 | 97.3 | 97.3 | 97.1 | 97.1 | 99 | 99.0 | 99.4 | 99.4 |

Continued on next page

Table 7: Effect of providing information in different forms: Cuts, Propagation, Branching

| Problem | | *CPLEX* | | *Cuts* | | *Prop* | | *Prop&Branch* | | *Cuts&Prop* | | *Cuts&Prop&Branch* | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | DS | B&B | B | I | B | I | B | I | B | I | B | I |
| harp2 | # nodes | 27245 | 14144 | 14144 | 14144 | 7717 | 7717 | 11663 | 11663 | 7717 | 7717 | 10726 | 10726 |
| | time | 1850.9 | 650.9 | 634.2 | 634.2 | 416.8 | 416.8 | 633.5 | 633.5 | 417.4 | 417.4 | 583.6 | 583.6 |
| prod2 | # nodes | 17773 | 13587 | 14183 | 18117 | 13873 | 13327 | 14015 | 13560 | 14068 | 20600 | 14300 | 13526 |
| | time | 396.2 | 326.9 | 338.8 | 454.8 | 329.6 | 321.7 | 335.8 | 319.1 | 340 | 508.1 | 342.9 | 339.4 |
| prod1 | # nodes | 15916 | 13258 | 12647 | 15240 | 13080 | 12532 | 12989 | 12923 | 13060 | 14708 | 12904 | 13175 |
| | time | 165.2 | 149 | 198.7 | 187.9 | 153.1 | 143.0 | 153.1 | 146.7 | 199 | 186.2 | 198.3 | 161.7 |
| neos-631694 | # nodes | 516 | 11988 | 11988 | 3819 | 11988 | 5800 | 16400 | 1261 | 11988 | 10549 | 16400 | 1429 |
| | time | 96.3 | 1639.4 | 1629.1 | 729.9 | 1633.6 | 1232.3 | 1158.9 | 225.3 | 1699.1 | 1686.3 | 1094.2 | 231.1 |
| neos-860244 | # nodes | 3204 | 10482 | 10482 | 3095 | 10502 | 2224 | 2177 | 2265 | 10502 | 2201 | 2177 | 2009 |
| | time | 435.1 | 1975.2 | 1996.4 | 500.0 | 2013 | 337.5 | 327.4 | 364.1 | 1973.1 | 338.7 | 328 | 305.5 |
| neos-538916 | # nodes | 20382 | 10363 | 10363 | 10363 | 10363 | 10231 | 11238 | 7101 | 10363 | 10231 | 11238 | 7101 |
| | time | 920.1 | 529.2 | 523.1 | 524.2 | 528.8 | 531.6 | 540 | 253.9 | 531.4 | 530.1 | 540.6 | 255.6 |
| qiu | # nodes | 11568 | 9121 | 9121 | 13979 | 9114 | 7091 | 10147 | 3447 | 9114 | 13605 | 10147 | 3411 |
| | time | 1662.5 | 1518 | 1413.5 | 1953.5 | 1532.4 | 1238.1 | 1509.2 | 563.9 | 1516.7 | 1868.0 | 1499.5 | 517.2 |
| misc07 | # nodes | 10472 | 8427 | 8171 | 6998 | 9319 | 8865 | 7973 | 8574 | 8307 | 9738 | 8905 | 8250 |
| | time | 408.4 | 305.3 | 318.4 | 265.0 | 344.6 | 331.2 | 277 | 270.3 | 320.6 | 363.7 | 331.5 | 264.4 |
| neos-886822 | # nodes | 8488 | 8408 | 8306 | 8340 | 8340 | 8268 | 8411 | 8686 | 8301 | 8143 | 8232 | 8571 |
| | time | 1096.4 | 1002 | 980.6 | 1010.2 | 1019.8 | 1013.6 | 1004 | 1018.6 | 1011.9 | 1007.7 | 976.7 | 1043.4 |
| neos-480878 | # nodes | 15275 | 8384 | 8384 | 9701 | 8365 | 7397 | 8434 | 13559 | 8365 | 9653 | 8434 | 12150 |
| | time | 1167.0 | 657.1 | 660.8 | 784.2 | 648.8 | 594.6 | 661.6 | 1008.9 | 651.5 | 796.7 | 663 | 942.9 |
| neos-1200887 | # nodes | 10972 | 7117 | 7117 | 7117 | 7117 | 6572 | 4788 | 2154 | 7117 | 6572 | 4788 | 2154 |
| | time | 731.5 | 508.1 | 501.7 | 522.0 | 522.7 | 495.5 | 335.6 | 121.3 | 518.8 | 488.2 | 329.1 | 118 |
| neos-807639 | # nodes | 5928 | 6858 | 6120 | 6120 | 6883 | 6883 | 5844 | 5844 | 6502 | 6502 | 6006 | 6006 |
| | time | 189.0 | 272.6 | 250.4 | 250.4 | 277 | 277.0 | 233.9 | 233.9 | 267.8 | 267.8 | 242.6 | 242.6 |
| neos2 | # nodes | 3163 | 6574 | 6581 | 5928 | 6394 | 5040 | 7033 | 8086 | 6478 | 5725 | 6968 | 8283 |
| | time | 218.1 | 316.4 | 320.2 | 287.9 | 313.6 | 244.1 | 333.7 | 368.4 | 314.9 | 297.5 | 330.4 | 392.5 |
| bienst1 | # nodes | 5953 | 5411 | 5243 | 2484 | 5812 | 3391 | 5177 | 2456 | 5551 | 2512 | 5266 | 2358 |
| | time | 334.6 | 521.6 | 512.5 | 266.9 | 555.3 | 321.1 | 536.2 | 244.4 | 560.4 | 245.7 | 546.2 | 257.9 |
| neos-504674 | # nodes | 4078 | 5384 | 4816 | 3440 | 4799 | 4712 | 4868 | 4724 | 5331 | 3342 | 4830 | 4949 |
| | time | 546.4 | 450.9 | 416 | 307.5 | 416.6 | 394.0 | 403.3 | 379.6 | 481.2 | 306.2 | 412.8 | 412.6 |
| rout | # nodes | 11115 | 4865 | 3928 | 3849 | 15792 | 6181 | 5829 | 5586 | 4718 | 3072 | 6185 | 6337 |
| | time | 1474.3 | 322 | 237.2 | 257.8 | 1602.9 | 433.8 | 384.7 | 294.7 | 316.5 | 173.0 | 384.5 | 366.2 |
| neos-1396125 | # nodes | 3474 | 4660 | 3724 | 4387 | 4627 | 4376 | 3175 | 842 | 3402 | 4445 | 3904 | 2144 |
| | time | 930.4 | 1414.7 | 1098.4 | 1348.3 | 1359.1 | 1269.0 | 877.6 | 168.6 | 983.8 | 1388.8 | 1140.7 | 513.2 |
| neos21 | # nodes | 4370 | 4649 | 4649 | 4649 | 4770 | 4770 | 3225 | 3274 | 4770 | 4770 | 3225 | 3274 |
| | time | 803.5 | 850 | 839.3 | 814.2 | 842.9 | 854.8 | 573.1 | 586.1 | 882.8 | 869.0 | 560.8 | 570.8 |
| neos-892255 | # nodes | 4340 | 3642 | 3642 | 3335 | 3642 | 3095 | 4419 | 3632 | 3642 | 3549 | 4419 | 3989 |
| | time | 1835.4 | 1646.6 | 1630.7 | 1538.9 | 1693.3 | 1446.4 | 1650.2 | 1314.9 | 1621.2 | 1618.8 | 1616.3 | 1416.5 |
| neos-1109824 | # nodes | 2686 | 3439 | 2126 | 2048 | 3439 | 3969 | 2327 | 2449 | 2126 | 2147 | 2280 | 2242 |
| | time | 582.2 | 452.8 | 256.3 | 230.6 | 457.2 | 527.7 | 225.1 | 243.0 | 259.5 | 245.2 | 221.6 | 212.7 |
| lrn | # nodes | 1778 | 2850 | 861 | 861 | 2431 | 2431 | 1580 | 1580 | 1199 | 1199 | 1331 | 1331 |
| | time | 642.5 | 642.2 | 294.9 | 294.9 | 577.1 | 577.1 | 431.4 | 431.4 | 363 | 363.0 | 375.2 | 375.2 |
| neos-570431 | # nodes | 3579 | 2433 | 1816 | 2552 | 2411 | 2411 | 925 | 2142 | 1759 | 2868 | 887 | 1137 |

Continued on next page

Table 7: Effect of providing information in different forms: Cuts, Propagation, Branching

| Problem | | *CPLEX* | | *Cuts* | | *Prop* | | *Prop&Branch* | | *Cuts&Prop* | | *Cuts&Prop&Branch* | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | DS | B&B | B | I | B | I | B | I | B | I | B | I |
| | time | 598.9 | 443.3 | 351.2 | 505.1 | 437.1 | 459.5 | 180.7 | 348.8 | 345.9 | 582.0 | 178.9 | 204.4 |
| neos-512201 | # nodes | 4482 | 2186 | 1626 | 1461 | 1997 | 2060 | 1588 | 1907 | 1696 | 1759 | 1602 | 2203 |
| | time | 596.0 | 170.7 | 124 | 134.4 | 163 | 162.9 | 113.4 | 138.1 | 105.7 | 169.7 | 117.2 | 170.4 |
| neos-806323 | # nodes | 2615 | 2041 | 2052 | 2052 | 1998 | 1998 | 2509 | 2509 | 1869 | 1869 | 2471 | 2471 |
| | time | 186.3 | 152.7 | 153.9 | 153.9 | 148.8 | 148.8 | 190.1 | 190.1 | 137.2 | 137.2 | 191.3 | 191.3 |
| neos-803219 | # nodes | 2591 | 1987 | 1935 | 1935 | 1941 | 1941 | 1898 | 1898 | 1938 | 1938 | 1893 | 1893 |
| | time | 84.6 | 68.3 | 69.5 | 69.5 | 66.6 | 66.6 | 62.9 | 62.9 | 69.5 | 69.5 | 65.1 | 65.1 |
| neos-504815 | # nodes | 1381 | 1575 | 1559 | 1333 | 1461 | 1616 | 1382 | 1482 | 1577 | 1349 | 1389 | 1412 |
| | time | 97.0 | 75.8 | 82.9 | 75.1 | 71.3 | 98.1 | 65.8 | 60.2 | 86.2 | 90.3 | 67.5 | 60.1 |
| neos-820879 | # nodes | 801 | 1348 | 1348 | 993 | 1144 | 911 | 1262 | 1280 | 1144 | 647 | 1262 | 1489 |
| | time | 91.3 | 147.3 | 147.5 | 108.7 | 124.8 | 94.3 | 135.8 | 121.5 | 126.2 | 74.8 | 138.6 | 152.3 |
| neos-807705 | # nodes | 1633 | 1231 | 1160 | 1160 | 1208 | 1208 | 1370 | 1370 | 1147 | 1147 | 1320 | 1320 |
| | time | 125.6 | 100.6 | 97.3 | 97.3 | 99.4 | 99.4 | 107.2 | 107.2 | 99 | 99.0 | 107.3 | 107.3 |
| bc1 | # nodes | 1176 | 1210 | 874 | 677 | 1056 | 766 | 757 | 579 | 726 | 554 | 608 | 479 |
| | time | 981.7 | 993.7 | 705.5 | 593.7 | 942.6 | 691.3 | 658.1 | 486.6 | 599.7 | 524.4 | 554 | 412.7 |
| seymour1 | # nodes | 1291 | 1171 | 1171 | 1171 | 1148 | 1148 | 1405 | 1405 | 1148 | 1148 | 1405 | 1405 |
| | time | 1399.5 | 1359.3 | 1351.2 | 1351.2 | 1339.1 | 1339.1 | 1587.5 | 1587.5 | 1390.3 | 1390.3 | 1584.5 | 1584.5 |
| Arith.Mean | # nodes | 81421.9 | 104274.6 | 98267.5 | 78190.6 | 99286 | 71424.2 | 78554.3 | 76915.4 | 96766.1 | 69657.1 | 74498.6 | 76399.6 |
| (All: 51 models) | time | 1026.7 | 1116.2 | 1059.8 | 926.2 | 1158.1 | 949.7 | 877.8 | 740.1 | 1059.8 | 920.9 | 861.4 | 752.6 |
| Geo.Mean | # nodes | 11277.7 | 15729.7 | 13826.9 | 12353.0 | 15774.6 | 13243.6 | 11782.9 | 10050.7 | 13971.1 | 12277.0 | 11688.6 | 10095.2 |
| (All: 51 models) | time | 524.3 | 677.3 | 607.8 | 554.8 | 694.4 | 588.9 | 488.3 | 402.8 | 626.7 | 560.6 | 494.2 | 417.2 |
| Arith.Mean | # nodes | 287286.2 | 376527 | 356392.8 | 278861.0 | 356427 | 249255.9 | 281874.2 | 277031.2 | 350429.3 | 244892.1 | 265678.8 | 275569.9 |
| (Hard: 13 models) | time | 1851.2 | 2166.5 | 2226.8 | 1842.9 | 2241.3 | 1756.6 | 1834.8 | 1548.5 | 2177.9 | 1721.8 | 1741 | 1574.7 |
| Geo.Mean | # nodes | 125247.4 | 187595.9 | 188082.9 | 157273.9 | 190045 | 140600.5 | 149230.1 | 121213.8 | 180646.8 | 136568.1 | 139124.4 | 119371.1 |
| (Hard: 13 models) | time | 1083.2 | 1486.4 | 1510.9 | 1295.2 | 1523.4 | 1162.6 | 1229.7 | 949.6 | 1495.5 | 1153.6 | 1185.5 | 965.6 |
| Arith.Mean | # nodes | 16108.0 | 16689.9 | 15046.4 | 14350.5 | 16563.9 | 15801.2 | 13402.4 | 12580.3 | 15070.9 | 14628.5 | 13540 | 12210.6 |
| (Medium: 23 models) | time | 776.8 | 866.1 | 767.9 | 682.9 | 869.6 | 753.4 | 594.4 | 498.8 | 797.9 | 729.8 | 608.5 | 500.5 |
| Geo.Mean | # nodes | 7686.5 | 13503.9 | 11602.5 | 10202.5 | 13212 | 11311.5 | 9169.7 | 7460.1 | 12146.8 | 10987.0 | 9465.2 | 7512.4 |
| (Medium: 23 models) | time | 383.3 | 654.8 | 574.9 | 513.1 | 651.2 | 564.5 | 408.4 | 332.1 | 620.9 | 560.3 | 428.3 | 342.3 |
| Arith.Mean | # nodes | 3154.1 | 2619.1 | 2164.7 | 2164.2 | 3271 | 2592.1 | 2243.4 | 2129.0 | 2190.7 | 2164.1 | 2278.7 | 2208.4 |
| (Easy: 15 models) | time | 695.3 | 589.3 | 496 | 504.9 | 661.7 | 551.2 | 482.9 | 409.4 | 492.4 | 519.7 | 486.9 | 426.9 |
| Geo.Mean | # nodes | 2519.1 | 2318.5 | 1883.1 | 1825.5 | 2393.8 | 2176.1 | 1916.3 | 1833.7 | 1883 | 1803.3 | 1887.4 | 1866.3 |
| (Easy: 15 models) | time | 451.8 | 360.8 | 300.4 | 299.7 | 387.5 | 348.3 | 288.1 | 257.3 | 298.8 | 299.9 | 288.1 | 272.9 |

Figure 5: Comparison of # of nodes in the restart phase for solvers with basic information

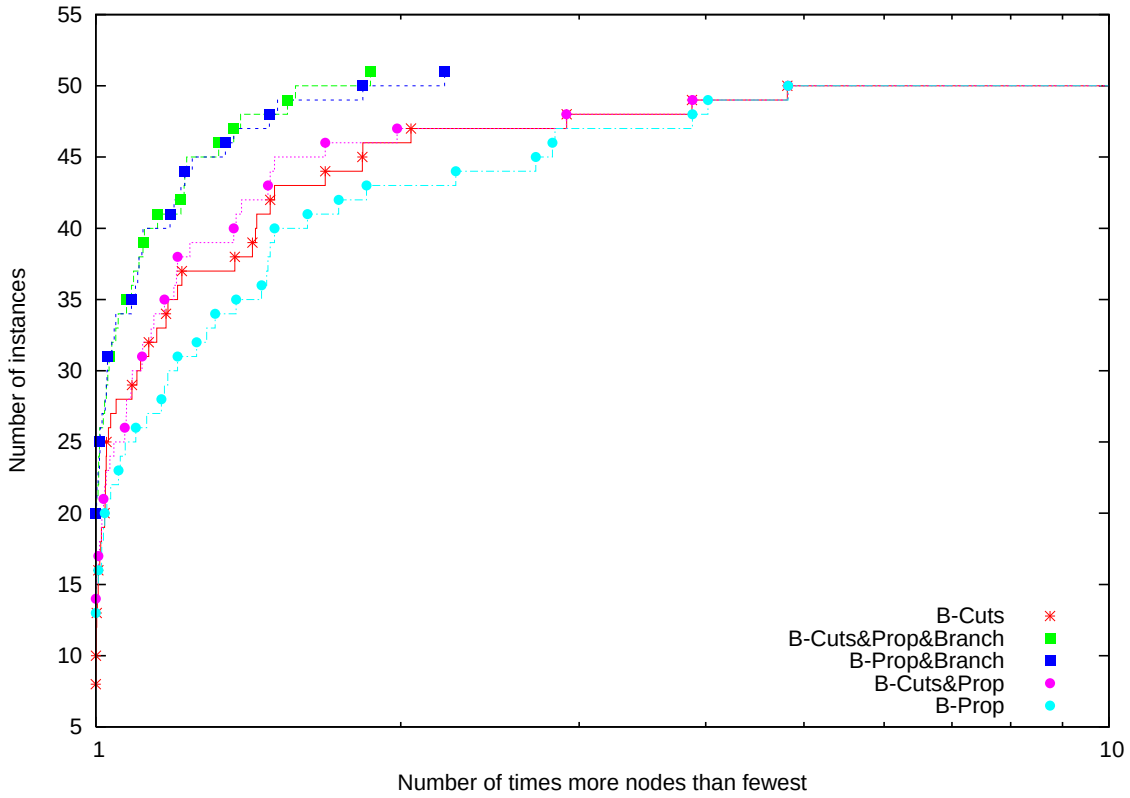



Table 9: Results for collecting and using more fathoming information

| | Clauses collected | | Clauses improved | | Collection phase | | Improvement phase | | Restart phase | |
|---|---|---|---|---|---|---|---|---|---|---|
| | # | size | # | size | # nodes | time | # nodes | time | # nodes | time |
| markshare_4_0 | *CPLEX-BB* | | | | | | | | 521166 | 199.52 |
| | *CPLEX-DS* | | | | | | | | 566080 | 148.06 |
| | 10 | 13.20 | 8 | 10.00 | 49 | 0.03 | 1 | 0.12 | 555338 | 218.63 |
| | 100 | 17.78 | 75 | 11.68 | 186 | 0.07 | 405 | 1.80 | 490470 | 198.28 |
| | 200 | 18.12 | 135 | 12.47 | 336 | 0.16 | 935 | 3.53 | 503334 | 199.56 |
| | 500 | 18.83 | 343 | 12.72 | 716 | 0.32 | 2534 | 8.95 | 495809 | 200.09 |
| | 1000 | 19.52 | 660 | 13.28 | 1369 | 0.60 | 5020 | 17.89 | 501697 | 219.34 |
| mas74 | *CPLEX-BB* | | | | | | | | 1068579 | 3488.83 |
| | *CPLEX-DS* | | | | | | | | 1695810 | 6319.29 |
| | 10 | 27.60 | 8 | 14.50 | 102 | 0.45 | 317 | 1.02 | 989581 | 3010.77 |
| | 100 | 25.69 | 52 | 18.51 | 615 | 2.67 | 2087 | 7.37 | 1138897 | 3542.71 |
| | 200 | 26.41 | 118 | 17.69 | 1190 | 6.27 | 3836 | 15.77 | 1130859 | 3704.51 |
| | 500 | 27.11 | 294 | 18.08 | 2737 | 22.81 | 11778 | 41.66 | 1205746 | 4079.91 |
| | 1000 | 27.13 | 612 | 17.67 | 5245 | 94.72 | 28725 | 89.03 | 1194174 | 4352.65 |
| mas76 | *CPLEX-BB* | | | | | | | | 126188 | 312.48 |
| | *CPLEX-DS* | | | | | | | | 159932 | 551.98 |
| | 10 | 22.90 | 8 | 11.90 | 95 | 0.44 | 119 | 0.60 | 130336 | 317.31 |
| | 100 | 23.00 | 52 | 16.02 | 647 | 2.95 | 1081 | 4.76 | 131033 | 319.96 |
| | 200 | 22.00 | 109 | 14.96 | 1195 | 5.64 | 1662 | 9.27 | 126488 | 314.93 |
| | 500 | 21.61 | 292 | 13.97 | 2621 | 17.68 | 3749 | 22.70 | 157512 | 411.20 |
| | 1000 | 21.79 | 573 | 14.17 | 4612 | 52.97 | 7001 | 45.46 | 155997 | 448.78 |
| neos-1211578 | *CPLEX-BB* | | | | | | | | 73016 | 1026.63 |
| | *CPLEX-DS* | | | | | | | | 40683 | 575.56 |
| | 10 | 10.00 | 6 | 7.60 | 39 | 2.61 | 5 | 2.01 | 47855 | 1818.27 |

Continued on next page

Table 9: Results for collecting and using more fathoming information

| | Clauses collected | | Clauses improved | | Collection phase | | Improvement phase | | Restart phase | |
|---|---|---|---|---|---|---|---|---|---|---|
| | # | size | # | size | # nodes | time | # nodes | time | # nodes | time |
| | 100 | 12.98 | 53 | 9.55 | 230 | 10.59 | 183 | 24.36 | 34838 | 1250.06 |
| | 200 | 13.69 | 112 | 9.66 | 419 | 6.55 | 463 | 18.93 | 29235 | 368.56 |
| | 500 | 14.30 | 266 | 10.17 | 951 | 38.91 | 1007 | 120.88 | 40298 | 1404.29 |
| | 1000 | 14.76 | 539 | 10.32 | 1976 | 79.16 | 1673 | 241.19 | 35356 | 1215.76 |
| neos-1228986 | *CPLEX-BB* | | | | | | | | 137818 | 1458.37 |
| | *CPLEX-DS* | | | | | | | | 117852 | 1748.15 |
| | 10 | 10.90 | 7 | 6.70 | 50 | 1.41 | 6 | 1.54 | 52175 | 582.33 |
| | 100 | 14.99 | 56 | 9.99 | 555 | 13.08 | 225 | 12.05 | 29737 | 306.52 |
| | 200 | 16.25 | 106 | 11.16 | 1031 | 23.66 | 469 | 22.18 | 31388 | 332.64 |
| | 500 | 17.12 | 256 | 11.99 | 2519 | 56.48 | 959 | 54.00 | 18826 | 314.22 |
| | 1000 | 17.49 | 542 | 11.96 | 4242 | 96.10 | 2031 | 113.41 | 15581 | 204.04 |
| neos-1480121 | *CPLEX-BB* | | | | | | | | 1711005 | 2559.37 |
| | *CPLEX-DS* | | | | | | | | 88346 | 394.29 |
| | 10 | 33.60 | 4 | 23.20 | 92 | 1.05 | 0 | 0.16 | 413084 | 1057.73 |
| | 100 | 29.32 | 47 | 19.05 | 408 | 3.18 | 0 | 1.86 | 292622 | 904.54 |
| | 200 | 38.17 | 50 | 32.63 | 798 | 5.67 | 0 | 3.85 | 187946 | 409.54 |
| | 500 | 45.62 | 72 | 41.54 | 1685 | 10.11 | 0 | 9.81 | 244621 | 837.22 |
| | 1000 | 44.35 | 164 | 39.10 | 3072 | 18.81 | 0 | 19.59 | 251375 | 1006.89 |
| neos14 | *CPLEX-BB* | | | | | | | | 61106 | 2206.18 |
| | *CPLEX-DS* | | | | | | | | 70550 | 2154.09 |
| | 10 | 17.20 | 1 | 15.90 | 119 | 6.03 | 0 | 6.03 | 52066 | 1894.66 |
| | 100 | 18.45 | 1 | 18.32 | 765 | 37.03 | 0 | 42.66 | 53045 | 1998.21 |
| | 200 | 18.73 | 2 | 18.61 | 1468 | 73.57 | 11 | 85.79 | 52844 | 2075.28 |
| | 500 | 18.70 | 4 | 18.60 | 3158 | 160.21 | 16 | 211.93 | 55624 | 2243.42 |
| | 1000 | 18.45 | 16 | 18.28 | 5572 | 314.07 | 83 | 431.67 | 57515 | 2445.37 |
| neos23 | *CPLEX-BB* | | | | | | | | 158582 | 2959.02 |
| | *CPLEX-DS* | | | | | | | | 51820 | 615.54 |
| | 10 | 14.40 | 4 | 11.00 | 107 | 3.11 | 5 | 1.76 | 52851 | 948.91 |
| | 100 | 19.34 | 25 | 16.10 | 586 | 16.49 | 58 | 14.89 | 93090 | 1390.19 |
| | 200 | 21.13 | 46 | 17.31 | 1030 | 27.43 | 105 | 29.15 | 90175 | 1686.58 |
| | 500 | 21.27 | 95 | 18.20 | 2187 | 61.40 | 221 | 70.22 | 44058 | 885.46 |
| | 1000 | 22.20 | 180 | 19.10 | 3632 | 99.60 | 453 | 138.38 | 58041 | 1430.29 |
| neos3 | *CPLEX-BB* | | | | | | | | 64223 | 4158.27 |
| | *CPLEX-DS* | | | | | | | | 21618 | 1477.31 |
| | 10 | 97.50 | 7 | 30.50 | 337 | 24.10 | 35 | 22.49 | 64497 | 4383.01 |
| | 100 | 40.38 | 84 | 10.64 | 433 | 30.04 | 155 | 117.74 | 65631 | 4471.54 |
| | 200 | 39.98 | 180 | 7.10 | 538 | 34.91 | 155 | 195.42 | 52345 | 3432.49 |
| | 500 | 55.72 | 405 | 15.92 | 1396 | 85.09 | 244 | 481.69 | 55079 | 3705.66 |
| | 1000 | 69.98 | 834 | 18.08 | 2468 | 148.69 | 491 | 936.04 | 59075 | 3974.79 |
| neos5 | *CPLEX-BB* | | | | | | | | 707857 | 2223.66 |
| | *CPLEX-DS* | | | | | | | | 648299 | 2042.44 |
| | 10 | 25.40 | 9 | 8.20 | 63 | 0.50 | 6 | 0.46 | 522599 | 1706.32 |
| | 100 | 25.55 | 76 | 12.10 | 616 | 4.38 | 147 | 5.26 | 1057156 | 2772.46 |
| | 200 | 24.98 | 144 | 12.47 | 1212 | 8.32 | 234 | 10.08 | 1178505 | 3012.84 |
| | 500 | 25.26 | 344 | 13.08 | 2803 | 25.62 | 573 | 24.26 | 1092201 | 3392.59 |
| | 1000 | 26.91 | 676 | 13.89 | 4608 | 62.23 | 1524 | 49.28 | 876418 | 4550.92 |
| neos-863472 | *CPLEX-BB* | | | | | | | | 106332 | 4026.16 |
| | *CPLEX-DS* | | | | | | | | 124665 | 5532.57 |
| | 10 | 14.30 | 6 | 7.10 | 47 | 1.67 | 8 | 4.69 | 101335 | 3592.25 |
| | 100 | 21.68 | 67 | 11.04 | 498 | 14.47 | 122 | 40.23 | 59535 | 2190.73 |
| | 200 | 26.50 | 124 | 13.75 | 1008 | 33.22 | 381 | 80.08 | 57466 | 2116.06 |
| | 500 | 30.84 | 308 | 15.59 | 2155 | 76.92 | 789 | 188.74 | 56136 | 2115.58 |
| | 1000 | 31.79 | 595 | 16.50 | 4082 | 162.16 | 2411 | 384.88 | 48185 | 2084.20 |
| pk1 | *CPLEX-BB* | | | | | | | | 85072 | 257.67 |
| | *CPLEX-DS* | | | | | | | | 107129 | 276.97 |
| | 10 | 16.20 | 7 | 9.90 | 82 | 0.40 | 14 | 0.57 | 75300 | 230.45 |
| | 100 | 27.47 | 58 | 17.65 | 641 | 2.79 | 1854 | 7.90 | 81433 | 263.40 |
| | 200 | 29.86 | 113 | 19.39 | 1447 | 7.48 | 4255 | 17.51 | 87185 | 276.23 |
| | 500 | 32.64 | 287 | 20.73 | 2908 | 22.31 | 19525 | 53.72 | 87214 | 271.47 |
| | 1000 | 32.09 | 579 | 20.36 | 4843 | 63.26 | 31542 | 100.74 | 85137 | 294.40 |
| swath2 | *CPLEX-BB* | | | | | | | | 73907 | 3288.24 |
| | *CPLEX-DS* | | | | | | | | 41937 | 2228.85 |
| | 10 | 36.80 | 5 | 23.60 | 91 | 7.26 | 2 | 28.46 | 57699 | 2608.63 |

Continued on next page

Table 9: Results for collecting and using more fathoming information

| | Clauses collected | | Clauses improved | | Collection phase | | Improvement phase | | Restart phase | |
|---|---|---|---|---|---|---|---|---|---|---|
| | # | size | # | size | # nodes | time | # nodes | time | # nodes | time |
| | 100 | 46.93 | 39 | 28.48 | 410 | 30.75 | 2 | 244.54 | 73333 | 3301.31 |
| | 200 | 46.53 | 76 | 28.43 | 656 | 43.80 | 2 | 465.20 | 54639 | 2541.62 |
| | 500 | 46.26 | 201 | 27.92 | 1186 | 68.37 | 2 | 1174.08 | 55826 | 2561.50 |
| | 1000 | 47.36 | 352 | 31.00 | 2320 | 125.68 | 2 | 2331.50 | 55009 | 2500.33 |

Figure 6: Comparison of solution times in the restart phase for solvers with basic information

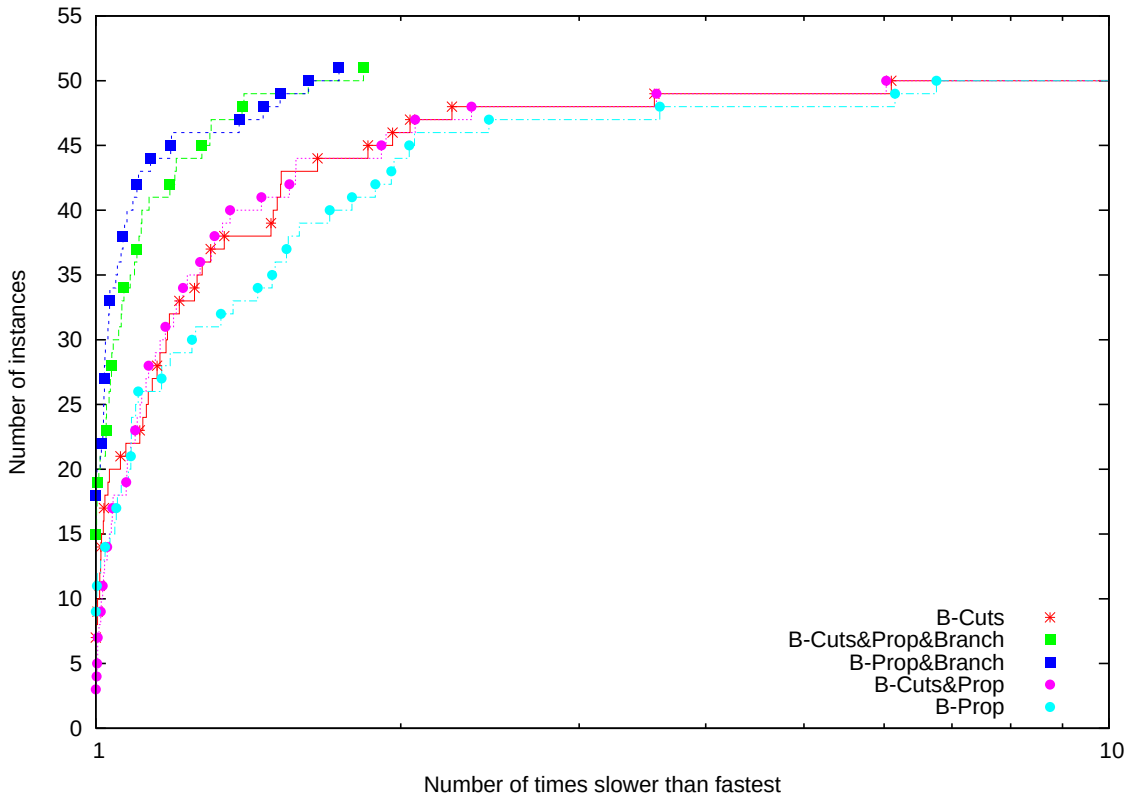


Table 8: Number of times the CPLEX branching is overwritten

| | # decisions in | | # decisions in | | | | | |
|---|---|---|---|---|---|---|---|---|
| | *B-Prop* | *I-Prop* | *B-Prop&Branch* | | | *I-Prop&Branch* | | |
| Problem | Prop | Prop | Prop | Branch | (%) | Prop | Branch | (%) |
| neos-1480121 | 151 | 5 | 4 | 2188 | 0.38 | 13 | 978 | 0.51 |
| mas74 | 24 | 1009 | 49 | 578 | 0.05 | 2213 | 52578 | 4.65 |
| neos5 | 14 | 28126 | 30 | 1129 | 0.16 | 83956 | 512698 | 43.50 |
| markshare_4_0 | 37 | 218 | 87 | 476 | 0.09 | 285 | 8182 | 1.63 |
| neos23 | 1488 | 6582 | 702 | 1168 | 0.51 | 4632 | 5244 | 4.97 |
| neos-1228986 | 7 | 4066 | 15 | 2010 | 5.65 | 996 | 12462 | 39.70 |
| mas76 | 23 | 545 | 41 | 605 | 0.46 | 947 | 15831 | 12.52 |
| neos-863472 | 112 | 6475 | 116 | 1034 | 1.27 | 1560 | 6323 | 10.41 |
| pk1 | 21 | 94 | 50 | 1067 | 1.30 | 345 | 18547 | 21.27 |
| swath2 | 82 | 27 | 1 | 334 | 0.62 | 11 | 329 | 0.60 |
| neos-1211578 | 4 | 1935 | 19 | 1142 | 1.88 | 981 | 7825 | 26.77 |
| neos3 | 275 | 47 | 112 | 173 | 0.31 | 262 | 1030 | 2.12 |
| neos14 | 9 | 645 | 24 | 1018 | 1.67 | 19 | 2416 | 4.38 |
| neos-803220 | 105 | 105 | 44 | 387 | 1.02 | 44 | 387 | 1.02 |
| pg5_34 | 164 | 164 | 189 | 607 | 1.52 | 189 | 607 | 1.52 |
| neos-1440447 | 0 | 563 | 16 | 573 | 2.86 | 436 | 6037 | 22.94 |
| neos22 | 50 | 50 | 0 | 1093 | 4.22 | 3 | 1083 | 3.96 |
| neos-1173026 | 32 | 32 | 3 | 58 | 45.31 | 3 | 58 | 45.31 |
| neos-538867 | 0 | 569 | 2 | 4339 | 26.25 | 366 | 6495 | 44.17 |
| neos-598183 | 474 | 474 | 57 | 192 | 1.10 | 57 | 192 | 1.10 |
| stein45 | 10 | 10 | 137 | 953 | 4.95 | 137 | 953 | 4.95 |
| harp2 | 50 | 50 | 71 | 265 | 2.27 | 71 | 265 | 2.27 |
| prod2 | 76 | 127 | 71 | 202 | 1.44 | 106 | 232 | 1.71 |
| prod1 | 124 | 251 | 78 | 124 | 0.95 | 101 | 209 | 1.62 |
| neos-631694 | 0 | 1 | 0 | 2 | 0.01 | 1 | 2 | 0.16 |
| neos-860244 | 25 | 47 | 50 | 211 | 9.69 | 39 | 220 | 9.71 |
| neos-538916 | 0 | 144 | 18 | 981 | 8.73 | 227 | 3127 | 44.04 |
| qiu | 23 | 170 | 57 | 567 | 5.59 | 59 | 990 | 28.72 |
| misc07 | 12 | 286 | 29 | 718 | 9.01 | 380 | 2947 | 34.37 |
| neos-886822 | 10 | 90 | 56 | 1151 | 13.68 | 96 | 1529 | 17.60 |
| neos-480878 | 69 | 607 | 54 | 455 | 5.39 | 287 | 2397 | 17.68 |
| neos-1200887 | 1 | 479 | 61 | 369 | 7.71 | 164 | 963 | 44.71 |
| neos-807639 | 340 | 340 | 56 | 224 | 3.83 | 56 | 224 | 3.83 |
| neos2 | 112 | 117 | 110 | 219 | 3.11 | 110 | 543 | 6.72 |
| bienst1 | 20 | 537 | 77 | 232 | 4.48 | 400 | 851 | 34.65 |
| neos-504674 | 26 | 337 | 46 | 406 | 8.34 | 309 | 1517 | 32.11 |
| rout | 178 | 292 | 103 | 250 | 4.29 | 529 | 1434 | 25.67 |
| neos-1396125 | 14 | 242 | 18 | 205 | 6.46 | 35 | 299 | 35.51 |
| neos21 | 60 | 60 | 179 | 182 | 5.64 | 176 | 215 | 6.57 |
| neos-892255 | 0 | 91 | 119 | 1631 | 36.91 | 142 | 1696 | 46.70 |
| neos-1109824 | 0 | 30 | 50 | 390 | 16.76 | 87 | 433 | 17.68 |
| lrn | 69 | 69 | 77 | 76 | 4.81 | 77 | 76 | 4.81 |
| neos-570431 | 53 | 55 | 156 | 111 | 12.00 | 257 | 693 | 32.35 |
| neos-512201 | 30 | 273 | 51 | 244 | 15.37 | 253 | 822 | 43.10 |
| neos-806323 | 21 | 21 | 50 | 303 | 12.08 | 50 | 303 | 12.08 |
| neos-803219 | 63 | 63 | 77 | 186 | 9.80 | 77 | 186 | 9.80 |
| neos-504815 | 51 | 164 | 64 | 199 | 14.40 | 149 | 489 | 33.00 |
| neos-820879 | 16 | 3 | 93 | 170 | 13.47 | 77 | 284 | 22.19 |
| neos-807705 | 33 | 33 | 55 | 250 | 18.25 | 55 | 250 | 18.25 |
| bc1 | 106 | 45 | 97 | 137 | 18.10 | 55 | 119 | 20.55 |
| seymour1 | 17 | 17 | 77 | 318 | 22.63 | 77 | 318 | 22.63 |
| Arith.Mean(All) | 91.78 | 1113.37 | 74.47 | 625.53 | 0.80 | 1999.16 | 13409.57 | 17.43 |
| Geo.Mean(All) | 26.94 | 151.36 | 41.67 | 375.12 | 3.18 | 140.90 | 966.61 | 9.62 |
| Arith.Mean(Hard) | 172.85 | 3828.77 | 96.15 | 994.00 | 0.35 | 7401.54 | 49572.54 | 17.89 |
| Geo.Mean(Hard) | 42.09 | 561.49 | 35.68 | 816.67 | 0.55 | 494.70 | 7304.12 | 6.03 |
| Arith.Mean(Medium) | 74.91 | 241.30 | 55.74 | 622.96 | 4.65 | 158.30 | 1383.83 | 11.00 |
| Geo.Mean(Medium) | 21.99 | 134.48 | 31.06 | 332.80 | 3.63 | 83.51 | 586.21 | 7.86 |
| Arith.Mean(Easy) | 47.40 | 97.20 | 84.40 | 310.13 | 13.82 | 139.73 | 507.80 | 23.85 |
| Geo.Mean(Easy) | 24.88 | 57.81 | 74.38 | 229.42 | 11.97 | 105.22 | 359.92 | 19.63 |

Table 10: Comparison of different branching rules

| Instance | neos-1480121 | | markshare_4_0 | | mas74 | | mas76 | | neos-1211578 | | neos-1228986 | | neos14 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | # nodes | time | # nodes | time | # nodes | time | # nodes | time | # nodes | time | # nodes | time | # nodes | time |
| *Cplex-BB* | 1711005 | 2559.4 | 521166 | 199.5 | 1068579 | 3488.8 | 126188 | 312.5 | 73016 | 1026.6 | 137818 | 1458.4 | 61106 | 2206.2 |
| *Cplex-DS* | 88346 | 394.3 | 566080 | 148.1 | 1695810 | 6319.3 | 159932 | 552.0 | 40683 | 575.6 | 117852 | 1748.2 | 70550 | 2154.1 |
| 0 0 0 | 17534 | 36.5 | 549777 | 236.3 | 2565355 | 8331.7 | 237325 | 528.3 | 51965 | 574.9 | 138314 | 1320.8 | 56038 | 1615.7 |
| 0 0 1 | 758611 | 1337.8 | 519487 | 216.4 | 2206070 | 7242.9 | 225329 | 523.3 | 45861 | 512.4 | 86486 | 801.3 | 73817 | 2939.5 |
| 0 0 2 | 758611 | 1340.4 | 519487 | 221.9 | 2206070 | 7249.3 | 225329 | 511.7 | 45861 | 520.3 | 86486 | 806.7 | 73817 | 2943.8 |
| 0 0 3 | 758611 | 1351.9 | 519487 | 215.3 | 2206070 | 7066.6 | 225329 | 507.8 | 45861 | 526.5 | 86486 | 833.2 | 73817 | 3046.6 |
| 0 1 0 | 248686 | 740.9 | 515253 | 215.8 | 1234792 | 3984.9 | 136498 | 319.0 | 27953 | 365.3 | 33854 | 344.3 | 52879 | 1933.3 |
| 0 1 1 | 181807 | 377.6 | 506762 | 209.1 | 1109522 | 3625.8 | 131002 | 315.0 | 26619 | 335.4 | 30615 | 327.9 | 54423 | 2163.3 |
| 0 1 2 | 796786 | 1204.9 | 523997 | 219.6 | 1117438 | 3693.0 | 129931 | 308.9 | 29639 | 369.9 | 33766 | 348.7 | 57964 | 2090.7 |
| 0 1 3 | 439069 | 645.3 | 522452 | 211.3 | 1142533 | 3802.2 | 134446 | 316.1 | 33130 | 389.4 | 34871 | 363.4 | 57150 | 2297.7 |
| 1 0 0 | 43965 | 79.8 | 551287 | 242.1 | 1293548 | 4285.5 | 156522 | 362.8 | 31955 | 396.1 | 81830 | 775.8 | 55628 | 1775.0 |
| 1 0 1 | 468725 | 707.4 | 516363 | 218.7 | 1161677 | 3870.4 | 144213 | 345.7 | 25718 | 303.0 | 45084 | 423.4 | 58414 | 2175.2 |
| 1 0 2 | 478332 | 798.8 | 527078 | 228.0 | 1304613 | 4292.5 | 141494 | 349.5 | 29492 | 331.2 | 44709 | 427.7 | 59058 | 2249.6 |
| 1 0 3 | 459352 | 718.8 | 518931 | 212.5 | 1182208 | 3886.7 | 140405 | 336.6 | 28428 | 336.8 | 46716 | 437.7 | 58407 | 2247.8 |
| 1 1 0 | 189542 | 543.1 | 525383 | 223.8 | 1206303 | 4111.9 | 137729 | 329.9 | 28055 | 364.8 | 38105 | 395.8 | 52150 | 1865.7 |
| 1 1 1 | 188823 | 403.2 | 504477 | 208.4 | 1118096 | 3708.4 | 130100 | 319.6 | 28678 | 359.7 | 27275 | 289.5 | 53406 | 2079.8 |
| 1 1 2 | 357131 | 519.1 | 515125 | 222.6 | 1149457 | 3795.1 | 132473 | 313.0 | 27293 | 344.3 | 38737 | 398.6 | 57549 | 2091.8 |
| 1 1 3 | 327387 | 503.7 | 522256 | 217.2 | 1132087 | 3621.1 | 131181 | 312.1 | 31643 | 370.3 | 35435 | 362.9 | 56516 | 2240.3 |
| 2 0 0 | 154375 | 416.1 | 522047 | 225.1 | 1230660 | 3760.9 | 151139 | 365.8 | 23615 | 298.3 | 65629 | 597.2 | 54012 | 1794.1 |
| 2 0 1 | 221596 | 331.7 | 548933 | 234.7 | 1141976 | 3658.3 | 155819 | 372.4 | 26007 | 324.4 | 46362 | 443.7 | 58130 | 2164.1 |
| 2 0 2 | 216935 | 319.8 | 499564 | 210.3 | 1241280 | 4045.8 | 136762 | 327.1 | 27499 | 382.6 | 41390 | 402.6 | 58420 | 2189.5 |
| 2 0 3 | 225994 | 347.0 | 514733 | 214.7 | 1220537 | 4095.1 | 139539 | 335.4 | 28383 | 339.8 | 47208 | 440.5 | 58401 | 2310.4 |
| 2 1 0 | 206275 | 586.1 | 529317 | 223.4 | 1216750 | 4492.0 | 146341 | 351.3 | 21177 | 274.1 | 48599 | 458.1 | 51416 | 1785.5 |
| 2 1 1 | 308781 | 971.9 | 532446 | 224.9 | 1151198 | 3714.1 | 133330 | 331.5 | 26195 | 332.9 | 38281 | 375.1 | 52720 | 2018.1 |
| 2 1 2 | 165427 | 216.4 | 510626 | 212.5 | 1193989 | 3861.2 | 130925 | 323.2 | 26172 | 327.1 | 34553 | 334.2 | 56440 | 2062.4 |
| 2 1 3 | 264460 | 391.0 | 524261 | 212.2 | 1183494 | 4093.2 | 130229 | 305.2 | 31308 | 354.6 | 43752 | 418.7 | 56201 | 2156.3 |
| 3 0 0 | 51596 | 82.9 | 492079 | 212.5 | 1262014 | 4190.0 | 150202 | 378.5 | 28183 | 360.7 | 74634 | 715.0 | 55486 | 1756.7 |
| 3 0 1 | 489340 | 740.6 | 501219 | 206.5 | 1213767 | 3978.6 | 137179 | 335.0 | 29754 | 364.2 | 50798 | 504.7 | 59344 | 2151.0 |
| 3 0 2 | 551640 | 873.1 | 510224 | 217.0 | 1263149 | 4161.6 | 130904 | 323.2 | 29349 | 333.7 | 51609 | 509.2 | 59334 | 2164.9 |
| 3 0 3 | 551640 | 915.7 | 509511 | 209.5 | 1245883 | 4134.6 | 129198 | 314.3 | 30859 | 354.3 | 55410 | 543.3 | 59341 | 2264.8 |
| 3 1 0 | 23063 | 41.2 | 525676 | 221.5 | 1184422 | 4049.5 | 136781 | 318.3 | 26333 | 337.4 | 45103 | 459.6 | 52454 | 1813.0 |
| 3 1 1 | 187946 | 409.5 | 503334 | 199.6 | 1130859 | 3704.5 | 126488 | 314.9 | 29235 | 368.6 | 31388 | 332.6 | 52844 | 2075.3 |
| 3 1 2 | 333776 | 516.6 | 517149 | 222.3 | 1120143 | 3724.5 | 123465 | 303.0 | 28277 | 354.1 | 38661 | 385.0 | 56985 | 2082.0 |
| 3 1 3 | 407426 | 686.8 | 515805 | 213.6 | 1151893 | 3842.0 | 126103 | 304.7 | 28881 | 341.0 | 36829 | 361.4 | 55574 | 2105.3 |

Table 11: Comparison of different branching rules (continued)

| Instance | neos23 | | neos3 | | neos5 | | neos-863472 | | pk1 | | swath2 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | # nodes | time | # nodes | time | # nodes | time | # nodes | time | # nodes | time | # nodes | time |
| *Cplex-BB* | 158582 | 2959.0 | 64223 | 4158.3 | 707857 | 2223.7 | 106332 | 4026.2 | 85072 | 257.7 | 73907 | 3288.2 |
| *Cplex-DS* | 51820 | 615.5 | 21618 | 1477.3 | 648299 | 2042.4 | 124665 | 5532.6 | 107129 | 277.0 | 41937 | 2228.9 |
| 0 0 0 | 100393 | 2075.7 | 68867 | 5057.0 | 2094950 | 5315.1 | 78064 | 2988.3 | 106348 | 316.2 | 41062 | 2004.9 |
| 0 0 1 | 144538 | 3337.8 | - | - | 1801991 | 4290.8 | 79310 | 3032.0 | 113699 | 351.2 | 40924 | 1979.7 |
| 0 0 2 | 144538 | 3334.3 | - | - | 1801991 | 4321.7 | 79310 | 3052.0 | 113699 | 349.3 | 40924 | 1976.3 |
| 0 0 3 | 144538 | 3617.1 | - | - | 1801991 | 4231.6 | 79310 | 3095.8 | 113699 | 353.5 | 39095 | 3895.6 |
| 0 1 0 | 66250 | 1172.5 | 52228 | 3631.3 | 1657584 | 4154.6 | 61907 | 2308.1 | 86314 | 272.6 | 46230 | 2227.5 |
| 0 1 1 | 69354 | 1205.2 | 48261 | 3323.6 | 1434448 | 3758.5 | 60533 | 2246.7 | 90266 | 286.5 | 70944 | 3222.7 |
| 0 1 2 | 90957 | 1824.3 | 59768 | 4084.6 | 1459104 | 3773.8 | 59181 | 2198.9 | 93582 | 297.6 | 32066 | 1614.0 |
| 0 1 3 | 61177 | 1336.0 | 49094 | 3461.9 | 1427416 | 3616.9 | 59242 | 2177.1 | 90437 | 289.3 | 27807 | 2809.7 |
| 1 0 0 | 114353 | 2107.4 | 55795 | 3854.7 | 1462784 | 3669.4 | 64580 | 2393.3 | 92472 | 289.2 | 40802 | 2045.0 |
| 1 0 1 | 118877 | 2640.9 | 76399 | 5200.1 | 1147562 | 2914.9 | 60246 | 2235.7 | 90910 | 283.8 | 37341 | 1875.2 |
| 1 0 2 | 105646 | 2353.9 | 48916 | 3352.2 | 1271969 | 3208.7 | 65818 | 2379.9 | 91321 | 283.2 | 32018 | 1585.5 |
| 1 0 3 | 113044 | 2560.1 | 58980 | 4142.5 | 1198360 | 2915.3 | 63530 | 2326.3 | 94049 | 297.0 | 37052 | 3803.1 |
| 1 1 0 | 67880 | 1270.2 | 48352 | 3326.2 | 1543945 | 4009.9 | 60492 | 2301.3 | 86294 | 271.5 | 34636 | 1662.9 |
| 1 1 1 | 65717 | 1134.1 | 49699 | 3421.6 | 1300260 | 3257.8 | 58635 | 2184.7 | 87743 | 280.7 | 55967 | 2608.3 |
| 1 1 2 | 70084 | 1575.6 | 48029 | 3230.7 | 1391487 | 3604.2 | 59927 | 2266.8 | 88310 | 276.0 | 28592 | 1437.7 |
| 1 1 3 | 81330 | 1899.5 | 48316 | 3484.7 | 1393404 | 3525.0 | 62879 | 2348.9 | 90097 | 287.6 | 33624 | 3213.7 |
| 2 0 0 | 103872 | 1996.4 | 59603 | 4222.0 | 1257659 | 3160.0 | 60080 | 2208.4 | 95945 | 300.5 | 20127 | 1024.2 |
| 2 0 1 | 96540 | 2024.6 | 72436 | 5005.7 | 1119178 | 2775.1 | 59102 | 2178.6 | 92984 | 291.1 | 37936 | 1900.7 |
| 2 0 2 | 119607 | 2550.0 | 73675 | 4934.7 | 1247392 | 3098.4 | 67868 | 2534.6 | 90955 | 283.5 | 35093 | 1725.3 |
| 2 0 3 | 108035 | 2724.5 | 43111 | 3061.7 | 1227191 | 3053.9 | 59621 | 2140.8 | 94299 | 295.4 | 34269 | 3335.6 |
| 2 1 0 | 113932 | 2120.8 | 50609 | 3511.5 | 1345051 | 3388.2 | 59797 | 2175.2 | 90719 | 284.6 | 39466 | 1909.6 |
| 2 1 1 | 89430 | 1980.7 | 64203 | 4359.3 | 1208661 | 3058.6 | 58637 | 2148.4 | 90906 | 288.2 | 70604 | 3248.6 |
| 2 1 2 | 129408 | 2835.4 | 76478 | 5091.3 | 1241326 | 3231.1 | 60324 | 2264.4 | 90038 | 284.1 | 35080 | 1716.8 |
| 2 1 3 | 98389 | 2287.5 | 44188 | 3158.1 | 1348723 | 3460.1 | 60573 | 2228.5 | 90082 | 285.0 | 39022 | 3894.0 |
| 3 0 0 | 73051 | 1445.1 | 55893 | 3877.1 | 1422365 | 3613.7 | 60850 | 2230.3 | 93439 | 286.8 | 60376 | 2828.4 |
| 3 0 1 | 67214 | 1403.7 | 66867 | 4546.3 | 1203288 | 2994.0 | 59899 | 2162.8 | 90157 | 281.2 | 24075 | 1245.6 |
| 3 0 2 | 85956 | 1709.3 | 66551 | 4435.5 | 1276960 | 3250.8 | 59523 | 2151.0 | 94446 | 293.9 | 23824 | 1220.4 |
| 3 0 3 | 115810 | 2637.2 | 67469 | 4661.4 | 1256291 | 3316.2 | 63011 | 2535.9 | 90315 | 287.4 | 21519 | 2329.0 |
| 3 1 0 | 88218 | 1424.1 | 46198 | 3306.4 | 1360340 | 3492.3 | 65938 | 2554.0 | 87351 | 273.7 | 47897 | 2327.1 |
| 3 1 1 | 90175 | 1686.6 | 52345 | 3432.5 | 1178505 | 3012.8 | 57466 | 2116.1 | 87185 | 276.2 | 54639 | 2541.6 |
| 3 1 2 | 101024 | 1975.8 | 54658 | 3707.6 | 1237898 | 3171.4 | 64995 | 2458.9 | 86295 | 271.3 | 28411 | 1439.9 |
| 3 1 3 | 92562 | 2085.5 | 51184 | 3584.2 | 1211536 | 3223.6 | 65163 | 2601.5 | 89369 | 284.2 | 33643 | 3266.4 |