Newsvendor Problems

August, 2003
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1 The Basic Model

We consider inventory systems of "perishable" goods. Perishable goods cannot be carried from one period to another. Examples of perishable goods include newspaper, Christmas tree, and many grocery items like milk and banana. Here, a period could be one day, or one week, or any time frame that is appropriate in the application context.

Suppose that a retailer sells a particular perishable product in a particular market, say, The New York Times at Georgia Tech. Assume the retailer's selling price of an item is p. At the beginning to the period, the retailer procures the items at a unit price of $c_v < p$. The salvage value of each leftover item is assumed to be $s_v < c_v$, which can be negative. (When s_v is negative, you pay somebody to get rid of the inventory.) In other words, the per unit holding cost of leftover items is $h = -s_v$.

In The New York Times example, one possible set of values for these parameters are: $p = \$1.00$, $c_v = \$.25$, and $s_v = 0$.

Suppose that the demand during a period is D. If the retailer stocks q items at the beginning of the period, the profit for that period is

$$\text{Profit}(q) = (p - c_v)\min\{D, q\} - (c_v - s_v)(q - D)^+, \tag{1}$$

where, for a real number x, $x^+ = \max(x, 0)$. The first factor $p - c_v$ is the profit margin of each item sold. One cannot sell more than what is demanded, and one cannot sell more than what she has. Thus, $(p - c_v)\min\{D, q\}$ is the major source of profit for the retailer. However, the retailer loses money on each leftover item. The amount of loss for each leftover item is $c_v - s_v$, which is a positive value by our assumption. The number of leftover items is $(q - D)^+$. Thus, the total loss is $(c_v - s_v)(q - D)^+$. Equation (1) says that the profit is equal to money made from sales minus money lost from leftover inventory.

Of course, demand changes from one period to another period. We will model D as a random variable. Thus, the profit is also a random variable. In order to determine the best order-up-to quantity q^*, we need to set appropriate objectives.

2 Expected Profit

One popular choice of performance measure is the expected profit for the period,

$$g(q) = \mathbb{E}[\text{Profit}(q)].$$

A first thought might suggest ordering $q = \mathbb{E}(D)$. A more careful thought suggests that, when the profit margin is high, q^* should be bigger than $\mathbb{E}(D)$.

Example 1. Let us come back to The New York Times example. Assume the demand follows the following distribution:

d	20	25	30	35
$\mathbb{P}\{D = d\}$	.1	.2	.4	.3

Note that obtaining such a demand distribution is often a non-trivial task. It may require a market department many months to get the distribution. Note also that

$$\mathbb{E}(D) = 20(.1) + 25(.2) + 30(.4) + 35(.3) = 29.5.$$

Thus, one possibility is to order 30 copies every morning, which is close to the expected value $\mathbb{E}(D)$. Suppose each morning 30 copies of the newspaper is ordered. Then, the expected profit per day is

$$g(30) = \$.75\mathbb{E}[\min(D, 30)] - \$.25\mathbb{E}[(30 - D)^+].$$

To compute $\mathbb{E}[(30 - D)^+]$, let $Y = (30 - D)^+$ be the number of leftover copies at the end of a day. Then, Y has the following distribution

y	10	5	0
$\mathbb{P}\{Y = y\}$	.1	.2	.7

To see this, note that $Y = 5$ when the demand is 25. Thus, $\mathbb{P}\{Y = 5\} = \mathbb{P}\{D = 25\} = .2$. Similarly, $\mathbb{P}\{Y = 10\} = \mathbb{P}\{D = 20\} = .1$. Finally, $\mathbb{P}\{Y = 0\} = 1 - \mathbb{P}\{Y = 10\} - \mathbb{P}\{Y = 5\} = .7$. Therefore,

$$\mathbb{E}[(30 - D)^+] = \mathbb{E}(Y) = 10(.1) + 5(.2) + 0(.7) = 2.$$

Let $Z = \min(D, 30)$. Then Z has the following distribution

z	20	25	30
$\mathbb{P}\{Z = z\}$	.1	.2	.7

Thus,

$$\mathbb{E}[\min(D, 30)] = 20(.1) + 25(.2) + 30(.7) = 28.$$

Therefore, $g(30) = \$.75(28) - \$.25(2) = \$20.5$.

What does $g(30) = \$20.5$ mean? It is *not* the profit per day, which is a random quantity as mentioned before. Rather, \$20.5 is the *average* profit per day over a large number of days, say 100 days. This intuition is justified by the following *strong law of large numbers:*

Theorem 1 (Strong Law of Large Numbers). *Assume that X_1, X_2, ..., X_n, ... are independent, identically distributed random variables with mean μ. Then*

$$\mathbb{P}\left\{\frac{X_1 + \ldots + X_n}{n} \to \mu \text{ as } n \to \infty\right\} = 1.$$

In our case, let X_i be the profit on day i. For each n, the average profit $(X_1 + \ldots + X_n)/n$ over n days is still random. The Strong Law of Large Numbers guarantees that as n gets larger, the average is closer to the expected value.

Expected profit is relevant only when an inventory system is managed repeatedly over many periods (strong law of large numbers!). In this setting, the optimal value q^* should be used for every period, although actual demand in each period fluctuates. If you manage the inventory system for only one period, maximizing expected profit may not make sense. For instance, it may make more sense to choose the most likely value that the demand D will take when D has a discrete distribution.

Coming back to our example, let us show that $q = 30$ is not the optimal order-up-to quantity. To see this, let us compute $g(35)$. Since D is always less than or equal to 35, we have $\mathbb{E}[\min(D, 35)] = \mathbb{E}(D) = 29.5$ and $\mathbb{E}[(35 - D)^+] = \mathbb{E}(35 - D) = 35 - \mathbb{E}(D) = 5.5$. Thus,

$$g(35) = \$.75(29.5) - \$.25(5.5) = \$20.75.$$

If we order $q = 35$ copies, the average profit per day is 25 cents more than the case when $q = 30$.

3 Optimal Order-Up-To Quantity

In this section, we are going to develop formulas for the optimal order-up-to quantity q^*. We use $F(x)$ to denote the cumulative distribution (cdf) of D. We separate the problem into two cases depending on whether the demand D is a continuous random variable or not.

3.1 Continuous Demand

Let $f(x)$ be the pdf of D, and $F(x) = \mathbb{P}(D \le x) = \int_0^x f(y)\,dy$ be the cumulative distribution function (cdf) of D. We assume that $f(x)$ is continuous in $[0,\infty)$ in the following proof. The conclusion in this section still holds when D is a general continuous random variable. We have

$$\mathbb{E}(q-D)^+ = \int_0^\infty (q-x)^+ f(x)\,dx = \int_0^q (q-x)f(x)\,dx,$$
$$\mathbb{E}[\min(D,q)] = \int_0^\infty \min(x,q)f(x)\,dx = \int_0^q xf(x)\,dx + q\int_q^\infty f(x)\,dx,$$

Setting $a = (p-c_v)$ and $b = (s_v - c_v)$, it follows that

$$g(q) = a\left(\int_0^q xf(x)\,dx + q\int_q^\infty f(x)\,dx\right) + b\int_0^q (q-x)f(x)\,dx.$$

To find the maximum of g, we first find a q that satisfies $g'(q)=0$. Using the fundamental theorem of calculus, one has

$$g'(q) = a\int_q^\infty f(x)\,dx + b\int_0^q f(x)\,dx = a(1-F(q)) + bF(q).$$

Setting $g'(q)=0$, we find that q^* must satisfy

$$F(q^*) = \frac{a}{a-b} = \frac{p-c_v}{p-s_v}. \tag{2}$$

To check g has a unique maximum, we take the second derivative of g,

$$g''(q) = (b-a)f(q).$$

Since $b-a = s_v - p < 0$, $g''(q) \le 0$, and thus g is a concave function on $[0,\infty)$. Therefore, the optimal order-up-to quality q^* is given by (2).

3.2 Discrete Demand

Assume that D is a nonnegative discrete random variable with probability mass function (pmf) given as

$$\mathbb{P}\{D = d_n\} = p_n, \quad n = 0,1,\ldots,$$

where $0 \le d_0 < d_1 < \ldots$. Note that the d_n's need not be integers. As before, let $F(x)$ be the cdf of D, i.e., $F(x) = \sum_{n:d_n\le x} p_n$. We will show that the optimal order-up-to quantity q^* is the smallest q such that

$$F(q) \ge \frac{p-c_v}{p-s_v}. \tag{3}$$

Note that q^* must equal to one of the d_n's. Note also that (3) includes (2) as a special case. Thus, in both the discrete and continuous demand cases, the optimal order-up-to quantity is the smallest q that satisfies (3).

For any nonnegative integer n, we compute $g(d_n)$ and $g(d_{n+1})$ and then figure out how the difference between the two varies as n increases. First, we have

$$g(d_n) = a\sum_{y=0}^{n} d_y p_y + a\sum_{y=n+1}^{\infty} d_n p_y + b\sum_{y=0}^{n}(d_n - d_y)p_y. \tag{4}$$

Replacing n by $n+1$ in the above relation, we have

$$\begin{aligned} g(d_{n+1}) &= a\sum_{y=0}^{n+1} d_y p_y + a\sum_{y=n+2}^{\infty} d_{n+1} p_y + b\sum_{y=0}^{n+1}(d_{n+1} - d_y)p_y \\ &= a\sum_{y=0}^{n} d_y p_y + a d_{n+1} p_{n+1} + a\sum_{y=n+1}^{\infty} d_{n+1} p_y - a d_{n+1} p_{n+1} \\ &\quad + b\sum_{y=0}^{n}(d_{n+1} - d_y)p_y \\ &= a\sum_{y=0}^{n} d_y p_y + a\sum_{y=n+1}^{\infty} d_n p_y + a(d_{n+1} - d_n)\sum_{y=n+1}^{\infty} p_y + b\sum_{y=0}^{n}(d_n - d_y)p_y \\ &\quad + b(d_{n+1} - d_n)\sum_{y=0}^{n} p_y \\ &= g(d_n) + (d_{n+1} - d_n)\Big[a(1 - F(d_n)) + bF(d_n)\Big]. \end{aligned}$$

Clearly, $d_{n+1} - d_n > 0$. Since $a > 0$ and $b < 0$, we have $b - a > 0$. Hence, $a(1 - F(q)) + bF(q)$ is a non-increasing function of n (since F is nondecreasing). So g is maximized at q^* where q^* is the smallest $q \in \{d_n,\ n = 0, 1, \ldots\}$ such that $a(1 - F(q)) + bF(q) \le 0$, which is the same as $F(q) \ge \frac{a}{a-b} = \frac{p-c_v}{p-s_v}$.

4 Other Objectives

Sometimes, it is easier to do the bookkeeping in an inventory problem in terms of cost instead of profit. Let p be the cost for each unit that is short. (Under some service agreement with a customer, if you find yourself short of one unit, you need to go to open market to buy a unit, at market price p, and send it to the customer.) Let h be the holding cost for each unit that is leftover. Clearly, the holding cost is the opposite of the salvage value, i.e., $h = -s_v$. If the order-up-to quantity is q, the expected total cost is

$$l(q) = c_v q + p\mathbb{E}(D - q)^+ + h\mathbb{E}(q - D)^+.$$

Lemma 1. *The expected profit and the expected cost have the following duality relationship.*

$$g(q) = p\mathbb{E}(D) - l(q). \tag{5}$$

Proof. First, notice that $\min\{D,q\} = D - (D-q)^+$ (try deriving this!). Then we rewrite the right-hand-side of (5) as follows:

$$\begin{aligned} p\mathbb{E}(D) - l(q) &= p\mathbb{E}(D) - c_v q - p\mathbb{E}(D-q)^+ - h\mathbb{E}(q-D)^+ \\ &= p\mathbb{E}(\min\{D,q\}) + s_v\mathbb{E}(q-D)^+ - c_v q \\ &= g(q) + c_v(\mathbb{E}(\min\{D,q\}) + \mathbb{E}(q-D)^+ - q) \end{aligned}$$

The term inside the parentheses in the last equality is 0. To see this, note that when $D \le q$, then $\min\{D,q\} + (q-D)^+ = D + (q-D) = q$. On the other hand, when $D \ge q$, then $\min\{D,q\} + (q-D)^+ = q + 0 = q$. So $\mathbb{E}(\min\{D,q\}) + \mathbb{E}(q-D)^+ = q$. This completes the proof. □

Since penalty price p and the expected demand $\mathbb{E}(D)$ are given, minimizing expected total cost is equivalent to maximizing expected profit. Thus, the optimal order-up-to quantity for minimizing expected total cost is also given by the formula in (3).

5 Fixed Cost and Initial Inventory

Suppose that the retailer has an initial inventory level x. It is best to think that these x items are given to the retailer as a "gift". Thus, the retailer does not incur any cost to possess these initial x units of items.

Whenever the retailer procure some items, a fixed cost c_f is incurred. Obviously, when the retailer's initial inventory is high enough, the retailer would not order. So the retailer faces two questions: (i) when to order and (ii) how much to order if she decides to order. We answer the second question first.

In the following we cast the problem in terms of total expected cost. Suppose that $y > 0$ units are ordered. The total expected cost is

$$c_f + c_v y + p\mathbb{E}(D-q)^+ + h\mathbb{E}(q-D)^+ = c_f - c_v x + c_v q + p\mathbb{E}(D-q)^+ + h\mathbb{E}(q-D)^+.$$

where, the order-up-to quantity $q = x + y > x$. The term $c_f + c_v y$ is the fixed cost plus the variable cost. The term $p(D-q)^+$ is the shortage cost, lost from the potential sales. The term $h(q-D)^+$ is the holding cost for leftover items. Since c_f, c_v and x are all fixed, minimizing the total expected total cost is equivalent to finding the optimal order-up-to quantity q to minimize

$$c_v q + p\mathbb{E}(D-q)^+ + h\mathbb{E}(q-D)^+.$$

However, the latter optimization problem has been solved in Section 4. Thus, the optimal order-up-to quantity S, if one decides to order at all, is given by q^* in (3).

Now we need to answer the first question. Suppose that x is the initial inventory level. Clearly, if $x \ge S$, one would not order anything. Whenever an order is to be made, always order enough to make the inventory level exactly at the level S.

Now suppose $x < S$. Let

$$L(q) = p\mathbb{E}(D-q)^+ + h\mathbb{E}(q-D)^+$$

be the expected shortage plus holding cost. Then the expected total cost, if $S - x$ items are ordered, is

$$c_f + c_v(S-x) + L(S).$$

On the other hand, if nothing is ordered, the expected total cost is

$$L(x).$$

Thus, if $L(x) \le c_f + c_v(S - x) + L(S)$, the retailer should not order anything. Otherwise, the retailer orders (up to S). Notice that $L(x) \le c_f + c_v(S - x) + L(S)$ is equivalent to

$$L(x) + c_v x \le c_f + c_v S + L(S). \tag{6}$$

When D is a continuous random variable, let s be the critical point such that

$$L(s) + c_v s = c_f + c_v S + L(S). \tag{7}$$

Then when $x \ge s$, the retailer should not order anything. (One needs to argue that when $x \ge s$, $L(x) + c_v x \le L(s) + c_v s$.)