Problem 1
Consider $S_1, S_2 \subset \mathbb{R} \cup \{+\infty\}$. If you need additional assumptions for a result to hold, then state those assumptions, and motivate why the assumptions are needed. Show that:

1. $\inf\{S_1 + S_2\} = \inf\{S_1\} + \inf\{S_2\}$
2. $\inf\{S_1 \cup S_2\} = \min\{\inf\{S_1\}, \inf\{S_2\}\}$
3. $\inf\{S_1 \cap S_2\} \geq \max\{\inf\{S_1\}, \inf\{S_2\}\}$

Give an example where strict inequality holds.

Problem 2
If you need additional assumptions for a result to hold, then state those assumptions, and motivate why the assumptions are needed.

1. Consider $f, g : \mathcal{X} \mapsto \mathbb{R}$. Show that

   $$\inf_{x \in \mathcal{X}} f(x) + \inf_{x \in \mathcal{X}} g(x) \leq \inf_{x \in \mathcal{X}} \{f(x) + g(x)\}$$

   Give an example where strict inequality holds.

2. Consider $f : \mathcal{X} \mapsto \mathbb{R}$ and $g : \mathcal{Y} \mapsto \mathbb{R}$. Show that

   $$\inf_{x \in \mathcal{X}} f(x) + \inf_{y \in \mathcal{Y}} g(y) = \inf_{x \in \mathcal{X}, y \in \mathcal{Y}} \{f(x) + g(y)\}$$

3. Consider $f : \mathcal{X} \times \mathcal{Y} \mapsto \mathbb{R}$. Show that

   $$\inf_{x \in \mathcal{X}} \{\inf_{y \in \mathcal{Y}} f(x, y)\} = \inf_{x \in \mathcal{X}} \{\inf_{y \in \mathcal{Y}} f(x, y)\} = \inf_{x \in \mathcal{X}, y \in \mathcal{Y}} f(x, y)$$

4. Consider $f : \mathcal{X} \times \mathcal{Y} \mapsto \mathbb{R}$. Show that

   $$\sup_{x \in \mathcal{X}} \{\inf_{y \in \mathcal{Y}} f(x, y)\} \leq \inf_{y \in \mathcal{Y}} \{\sup_{x \in \mathcal{X}} f(x, y)\}$$

   Give an example where strict inequality holds.
Problem 3
Order notation:

(1) Let \( f(n) \) and \( g(n) \) be asymptotically nonnegative functions. Use the definition of the \( \Omega \)-notation in the text (beware: other texts often use \( \Theta \) for the same concept). Show that
\[
\max\{f(n), g(n)\} = \Omega(f(n) + g(n))
\]

(2) Show that for any constants \( a \) and \( b \), with \( b > 0 \), \( (n + a)^b = \Omega(n^b) \).

(3) Is \( 2^{n+1} = O(2^n) \)? Is \( 2^{2n} = O(2^n) \)? Why or why not?

Problem 4
Assume that \( \mathcal{L} \subset \mathbb{R}^n \) is a linear subspace, and that \( B^1 \) and \( B^2 \) are two bases for \( \mathcal{L} \).

(1) Show that \( |B^1| = |B^2| \), where \( |B| \) denotes the number of elements (cardinality) of set \( B \).

(2) Show that the expression of any element of \( \mathcal{L} \) as a linear combination of the elements of \( B^1 \) is unique.

(3) Represent a linear combination by its vector of coefficients. Show that the function that maps the expression of an element of \( \mathcal{L} \) as a linear combination of the elements of \( B^1 \) to the expression of the same element as a linear combination of the elements of \( B^2 \) is an invertible linear function. Show how to construct the matrix representation of the abovementioned function and its inverse.

Problem 5
Equivalence of norms in \( \mathbb{R}^n \):

(1) Let \( \| \cdot \| \) be any norm on \( \mathbb{R}^n \). Show that \( \| \cdot \| \) is a continuous function from \( \mathbb{R}^n \) to \( \mathbb{R} \).

(2) Let \( \| \cdot \| \) and \( \| \cdot \|'' \) be any two norms on \( \mathbb{R}^n \). Show that \( \| \cdot \| \) and \( \| \cdot \|'' \) are equivalent, in the sense that there exists a constant \( c > 0 \) such that \( \|x\|'' \leq c\|x\| \) for all \( x \in \mathbb{R}^n \). (Compare this result with the result (A.35) of the text.)

Problem 6
Consider a symmetric \( n \times n \) matrix \( A \) with eigenvalues \( \lambda_1 \geq \cdots \geq \lambda_n \). Let \( R : \mathbb{R}^n \setminus \{0\} \to \mathbb{R} \) be defined by \( R(x) := x^T Ax / x^T x \).

(1) Show that \( \max_{x \neq 0} R(x) = \lambda_1 \) and \( \min_{x \neq 0} R(x) = \lambda_n \).

(2) Show that the maximum is attained by any eigenvector of \( A \) corresponding to \( \lambda_1 \), and the minimum is attained by any eigenvector of \( A \) corresponding to \( \lambda_n \).

(3) Show that
\[
\lambda_n \|x\|^2 \leq x^T Ax \leq \lambda_1 \|x\|^2
\]
for all \( x \in \mathbb{R}^n \). (Compare this result with the result on p.599 of the text.)
Problem 7

Let $A$ be an $m \times n$ matrix with Euclidean norm

$$
\|A\|_2 := \max_{x \in \mathbb{R}^n \setminus \{0\}} \frac{\|Ax\|_2}{\|x\|_2}
$$

(see (A.38) in the text).

(1) Show that all the eigenvalues of $A^T A$ are nonnegative.

(2) Show that $\|A\|_2^2 = \lambda_1(A^T A)$, where $\lambda_1(A^T A)$ denotes the largest eigenvalue of $A^T A$. (Compare this result with the result (A.39b) of the text.)

(3) Suppose that $A$ is a symmetric $n \times n$ matrix with eigenvalues $\lambda_1(A), \ldots, \lambda_n(A)$. Show that

$$
\|A\|_2 = \max_{i \in \{1, \ldots, n\}} |\lambda_i(A)|
$$

(4) Suppose that $A$ is a symmetric and nonsingular $n \times n$ matrix with eigenvalues $\lambda_1(A), \ldots, \lambda_n(A)$. Show that

$$
\|A^{-1}\|_2 = \frac{1}{\min_{i \in \{1, \ldots, n\}} |\lambda_i(A)|}
$$

(5) Let the condition number $\kappa(A)$ of $A$ be defined by $\kappa(A) := \|A\|_2 \|A^{-1}\|_2$ (see (A.42) in the text). Suppose again that $A$ is a symmetric and nonsingular $n \times n$ matrix with eigenvalues $\lambda_1(A), \ldots, \lambda_n(A)$. Show that

$$
\kappa(A) = \frac{\max_{i \in \{1, \ldots, n\}} |\lambda_i(A)|}{\min_{i \in \{1, \ldots, n\}} |\lambda_i(A)|}
$$