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On duality theory of convex semi-infinite programming

ALEXANDER SHAPIRO*

School of Industrial and Systems Engineering,
Georgia Institute of Technology, Atlanta, Georgia 30332-0205, USA

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In this article we discuss weak and strong duality properties of convex semi-infinite programming problems. We use a unified framework by writing the corresponding constraints in a form of cone inclusions. The consequent analysis is based on the conjugate duality approach of embedding the problem into a parametric family of problems parameterized by a finite-dimensional vector.

Keywords: Semi-infinite programming; Stochastic programming; Lagrangian duality; Weak and strong duality; Minimax problems

1. Introduction

Consider semi-infinite programming problems (SIPs) of the form

$$\operatorname*{Min}_{x\in\mathbb{R}^n} f(x) \quad \text{subject to} \quad g(x,\omega)\le 0, \quad \omega\in\Omega, \tag{1.1}$$

where Ω is a (possibly infinite) set, $f:\mathbb{R}^n\to\overline{\mathbb{R}}$ is an extended real valued function and $g:\mathbb{R}^n\times\Omega\to\mathbb{R}$. In the above formulation, a feasible point $x\in\mathbb{R}^n$ is supposed to satisfy the constraints $g(x,\omega)\le 0$ for all $\omega\in\Omega$, and no structural assumptions are made about the set Ω. In some situations it is natural to require that these constraints hold for almost every (a.e.) $\omega\in\Omega$. That is, the set Ω is equipped with a sigma algebra $\mathcal{F}$ and a (finite) measure μ on $(\Omega,\mathcal{F})$. Then it is said that a property holds for a.e. $\omega\in\Omega$ if there is a set $A\in\mathcal{F}$ such that $\mu(A)=0$ and the property holds for all $\omega\in\Omega\setminus A$. The formulation "for a.e. $\omega\in\Omega$" is relevant, for example, in stochastic programming (cf [9–11]).

There exists an extensive literature on duality of convex SIPs (see, e.g., [4] and references therein), and in particular on linear SIPs (see [2] and, for a more recent survey, [3]). In this article, we discuss an approach to duality theory of both

*Email: ashapiro@isye.gatech.edu

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formulations in a unified framework by writing the corresponding constraints in a form of cone inclusions. We then embed the dual problem into a parametric family and study the "strong duality" relations between the primal and dual problems from the point of view of conjugate duality (cf [8]). An advantage of the suggested approach is that the strong duality property can be established in a finite-dimensional setting.

2. Weak duality

Both formulations of the semi-infinite programs (i.e., "for every $\omega \in \Omega$" and "for a.e. $\omega \in \Omega$") can be written in the following form.

Define the mapping $G : x \mapsto g(x, \cdot)$, from $\mathbb{R}^n$ into an appropriate linear functional space $\mathcal{Y}$. Choose an appropriate (nonempty convex) cone $K \subset \mathcal{Y}$ and write the corresponding semi-infinite program in the form

$$\operatorname*{Min}_{x \in \mathbb{R}^n} f(x) \quad \text{subject to } G(x) \in K. \tag{2.1}$$

For example, in the case of the formulation "for every $\omega \in \Omega$" we can take $\mathcal{Y} := \mathbb{R}^\Omega$ to be the linear space of real valued functions $\gamma : \Omega \to \mathbb{R}$, and the cone

$$K := \{\gamma \in \mathcal{Y}\colon \gamma(\omega) \leq 0, \ \omega \in \Omega\}. \tag{2.2}$$

In the case of the formulation "for a.e. $\omega \in \Omega$" we assume that, for every $x \in \mathbb{R}^n$, the function $g(x, \cdot)$ belongs to the functional space $\mathcal{Y} := L_p(\Omega, \mathcal{F}, \mu)$, for some $p \in [1, +\infty]$, and take

$$K := \left\{\gamma \in L_p(\Omega, \mathcal{F}, \mu)\colon \gamma(\omega) \leq 0 \text{ for a.e. } \omega \in \Omega\right\}. \tag{2.3}$$

Next we associate with $\mathcal{Y}$, a dual space $\mathcal{Y}^*$ of linear functionals $\gamma^* : \mathcal{Y} \to \mathbb{R}$. In that way we define the scalar product $\langle \gamma^*, \gamma \rangle := \gamma^*(\gamma)$ for $\gamma^* \in \mathcal{Y}^*$ and $\gamma \in \mathcal{Y}$. We assume that $\mathcal{Y}$ and $\mathcal{Y}^*$ form a pair of locally convex topological vector spaces. For example, with $\mathcal{Y} := \mathbb{R}^\Omega$ we can associate the linear space of functions $\gamma^* : \Omega \to \mathbb{R}$ such that only a finite number of $\gamma^*(\omega)$ are nonzero and define the scalar product

$$\langle \gamma^*, \gamma \rangle := \sum_{\omega \in \Omega} \gamma^*(\omega)\gamma(\omega) \tag{2.4}$$

(the summation in the right-hand side of (2.4) is taken over $\omega \in \Omega$ such that $\gamma^*(\omega) \neq 0$.) In case of $\mathcal{Y} := L_p(\Omega, \mathcal{F}, \mu)$ we use its standard dual $\mathcal{Y}^* := L_q(\Omega, \mathcal{F}, \mu)$, where $q \in [1, +\infty]$ is such that $1/p + 1/q = 1$. Consider the dual

$$K^* := \left\{\gamma^* \in \mathcal{Y}^*\colon \langle \gamma^*, \gamma \rangle \leq 0, \ \gamma \in K\right\} \tag{2.5}$$

of the cone K, and the Lagrangian function $L(x, \lambda) := f(x) + \langle \lambda, G(x) \rangle$. We make the following assumption throughout the article.

Assumption (A1) For any $\gamma \notin K$ there exists $\gamma^* \in K^*$ such that $\langle \gamma^*, \gamma \rangle > 0$.

It follows directly from the definition that the cone

$$K^{**} := \{\gamma \in \mathcal{Y}: \langle \gamma^*, \gamma \rangle \leq 0,\ \gamma^* \in K^*\}$$

contains the cone K. The above assumption (A1) implies that $K = K^{**}$.

Under the assumption (A1), we have that

$$\sup_{\lambda \in K^*} L(x, \lambda) = \begin{cases} f(x), & \text{if } G(x) \in K, \\ +\infty, & \text{if } G(x) \notin K, \end{cases}$$

and hence, problem (2.1) can be written in the min–max form

$$\operatorname*{Min}_{x \in \mathbb{R}^n} \operatorname*{Max}_{\lambda \in K^*} L(x, \lambda). \tag{2.6}$$

Then the (Lagrangian) dual of (2.1) is the problem

$$\operatorname*{Max}_{\lambda \in K^*} \left\{ \psi(\lambda) := \inf_{x \in \mathbb{R}^n} L(x, \lambda) \right\}. \tag{2.7}$$

We refer to problems (2.1) and (2.7) as primal (P) and dual (D) problems, respectively, and denote by val(P) and val(D) their respective optimal values. By Sol(P) we denote the (possibly empty) set of optimal solutions of the primal problem.

In particular, for $\mathcal{Y} := \mathbb{R}^{\Omega}$ and the cone K defined in (2.2) we have that

$$K^* = \{\lambda \in \mathcal{Y}^* : \lambda(\omega) \geq 0,\ \omega \in \Omega\} \tag{2.8}$$

and $L(x, \lambda) = f(x) + \sum_{\omega \in \Omega} \lambda(\omega) g(x, \omega)$ (recall that for $\lambda \in \mathcal{Y}^*$ only a finite number of values $\lambda(\omega)$ do not equal zero, and hence, the corresponding summation over $\omega \in \Omega$ is well defined). For $\mathcal{Y} := L_p(\Omega, \mathcal{F}, \mu)$ and the cone K defined in (2.3) we have that

$$K^* = \{\lambda \in L_q(\Omega, \mathcal{F}, \mu) : \lambda(\omega) \geq 0,\ \text{a.e } \omega \in \Omega\} \tag{2.9}$$

and $L(x, \lambda) = f(x) + \int_{\Omega} \lambda(\omega) g(x, \omega) d\mu(\omega)$. In both cases the assumption (A1) holds.

By the standard min–max theory we have the following (called *weak duality*) relation

$$\text{val}(P) \geq \text{val}(D) \tag{2.10}$$

between the optimal values of the primal and dual problems.

3. Strong duality

In this section we discuss conditions under which the strong duality relation $\mathrm{val}(P) = \mathrm{val}(D)$ holds. For vectors $x, y \in \mathbb{R}^n$ we denote by $\langle y, x\rangle$ their standard scalar product. Let us embed the dual problem into the parametric family

$$(D_y) \qquad \operatorname*{Max}_{\lambda \in \mathcal{Y}^*} \varphi(\lambda, y),$$

where

$$\varphi(\lambda, y) := \begin{cases} \inf_{x \in \mathbb{R}^n} \left\{ L(x, \lambda) - \langle y, x\rangle \right\} & \text{if } \lambda \in K^*, \\ -\infty & \text{if } \lambda \notin K^*. \end{cases} \tag{3.1}$$

Note that the function $\varphi : \mathcal{Y}^* \times \mathbb{R}^n \to \overline{\mathbb{R}}$ is the infimum of linear functions, and hence is concave. It follows that the min-function

$$\vartheta(y) := \inf_{\lambda \in \mathcal{Y}^*} \{-\varphi(\lambda, y)\} = -\sup_{\lambda \in \mathcal{Y}^*} \varphi(\lambda, y) \tag{3.2}$$

is an extended real valued convex function. Clearly $\mathrm{val}(D_y) = -\vartheta(y)$ and, in particular, $\mathrm{val}(\mathrm{D}) = -\vartheta(0)$.

Let us calculate the conjugate of the function $\vartheta(y)$. We have

$$\begin{aligned} \vartheta^*(y^*) &:= \sup_y \langle y^*, y\rangle - \vartheta(y) = \sup_{y, \lambda} \langle y^*, y\rangle + \varphi(\lambda, y) \\ &= \sup_{\lambda \in K^*} \sup_y \inf_{x \in \mathbb{R}^n} \left\{ \langle y^*, y\rangle - \langle x, y\rangle + L(x, \lambda) \right\}. \end{aligned} \tag{3.3}$$

Furthermore,

$$\begin{aligned} \sup_y \inf_{x \in \mathbb{R}^n} \left\{ \langle y^*, y\rangle - \langle x, y\rangle + L(x, \lambda) \right\} &= \sup_y \left\{ \langle y^*, y\rangle - \sup_{x \in \mathbb{R}^n} \{\langle y, x\rangle - L(x, \lambda)\} \right\} \\ &= \sup_y \left\{ \langle y^*, y\rangle - L^*(y, \lambda) \right\} = L^{**}(y^*, \lambda), \end{aligned} \tag{3.4}$$

where $L^*(\cdot, \lambda)$ is the conjugate of the function $L(\cdot, \lambda)$, and $L^{**}(\cdot, \lambda)$ is the conjugate of $L^*(\cdot, \lambda)$. Let us make now the following assumption.

Assumption (A2) For every $\lambda \in K^*$, the function $L(\cdot, \lambda)$ is proper convex and lower semicontinuous.

In both the examples considered, for $\lambda \in K^*$ convexity of $\langle \lambda, G(\cdot)\rangle$ is implied by convexity of $g(\cdot, \omega)$, $\omega \in \Omega$. If, moreover, $f(\cdot)$ is proper convex and lower semicontinuous, then the above assumption (A2) follows.

By the Fenchel–Moreau theorem (e.g., [7]) we have that, under assumption (A2), for all $\lambda \in K^*$ the function $L^{**}(\cdot, \lambda)$ coincides with $L(\cdot, \lambda)$, and hence by (3.3) and (3.4)

$$\vartheta^*(y^*) = \sup_{\lambda \in K^*} L(y^*, \lambda). \tag{3.5}$$

By $\vartheta^{**}(\cdot)$ we denote the conjugate of the function $\vartheta^*(\cdot)$.

PROPOSITION 3.1 *Suppose that the assumptions* (A1) *and* (A2) *hold. Then* $\mathrm{val}(D) = -\vartheta(0)$, $\mathrm{val}(P) = -\vartheta^{**}(0)$ *and* $\mathrm{Sol}(P) = -\partial\vartheta^{**}(0)$.

Proof By (2.6) and (3.5) we have

$$\mathrm{val}(P) = \inf_{y^*} \vartheta^*(y^*) = -\vartheta^{**}(0). \tag{3.6}$$

We also have by convex analysis (e.g., [7]) that

$$\partial\vartheta^{**}(y) = \arg\max_{y^*} \left\{ \langle y^*, y\rangle - \vartheta^*(y^*) \right\}. \tag{3.7}$$

In particular,

$$\partial\vartheta^{**}(0) = -\arg\min_{y^*} \vartheta^*(y^*) = -\arg\min_{y^*} \left\{ \sup_{\lambda\in K^*} L(y^*, \lambda) \right\}. \tag{3.8}$$

That is, $\mathrm{Sol}(P) = -\partial\vartheta^{**}(0)$. ■

We obtain the following results which are quite standard in the conjugate duality theory.

PROPOSITION 3.2 *Suppose that the assumptions* (A1) *and* (A2) *hold and* $\vartheta^{**}(0) < +\infty$. *Then*: (i) $\mathrm{val}(D) = \mathrm{val}(P)$ *iff the function* $\vartheta(y)$ *is lower semicontinuous at* $y = 0$, *and* (ii) $\mathrm{val}(D) = \mathrm{val}(P)$ *and* $\mathrm{Sol}(P)$ *is nonempty iff the function* $\vartheta(y)$ *is subdifferentiable at* $y = 0$.

Proof Since the function $\vartheta(\cdot)$ is convex and $\vartheta^{**}(0) < +\infty$, we have by the Fenchel–Moreau theorem that $\vartheta^{**}(0) = \vartheta(0)$ iff $\vartheta(y)$ is lower semicontinuous at $y = 0$. Since $\mathrm{val}(D) = -\vartheta(0)$ and $\mathrm{val}(P) = -\vartheta^{**}(0)$, property (i) follows.

Now if $\vartheta(y)$ is subdifferentiable at $y = 0$, i.e., $\partial\vartheta(0) \neq \emptyset$, then $\vartheta(0)$ is finite and $\vartheta(y)$ is lower semicontinuous at $y = 0$, and $\partial\vartheta(0) = \partial\vartheta^{**}(0)$. It follows by (i) that $\mathrm{val}(D) = \mathrm{val}(P)$, and since $\mathrm{Sol}(P) = -\partial\vartheta^{**}(0)$, it is nonempty. Conversely, if $\mathrm{val}(D) = \mathrm{val}(P)$ and $\mathrm{Sol}(P)$ is nonempty, then $\vartheta^{**}(0) = \vartheta(0)$ and $\partial\vartheta^{**}(0) \neq \emptyset$. This implies that $\partial\vartheta(0) = \partial\vartheta^{**}(0) \neq \emptyset$, and hence property (ii) follows. ■

PROPOSITION 3.3 *Suppose that the assumptions* (A1) *and* (A2) *hold and* $\mathrm{val}(P)$ *is finite. Then* $\mathrm{val}(D) = \mathrm{val}(P)$ *and* $\mathrm{Sol}(P)$ *is nonempty and bounded iff the following condition holds*: (i) *there exists a neighborhood* $\mathcal{N}$ *of* $0 \in \mathbb{R}^n$ *such that for every* $y \in \mathcal{N}$ *there exists* $\lambda \in K^*$ *such that*

$$\inf_{x\in\mathbb{R}^n} \left\{ L(x, \lambda) - \langle y, x\rangle \right\} > -\infty. \tag{3.9}$$

Proof Suppose that condition (i) holds. Then for every $y \in \mathcal{N}$ there exists $\lambda \in \mathcal{Y}^*$ such that $\varphi(\lambda, y) > -\infty$, and hence $\vartheta(y) < +\infty$ for all $y \in \mathcal{N}$. Since $\vartheta(0) \geq \vartheta^{**}(0) = -\mathrm{val}(P)$, and $\mathrm{val}(P)$ is finite, it follows that $\vartheta(0)$ is finite. By convexity of $\vartheta(\cdot)$, we obtain that $\vartheta(y)$ is continuous at $y = 0$. It follows that $\partial\vartheta(0)$ is nonempty and bounded, and hence, by Proposition 3.2, $\mathrm{val}(D) = \mathrm{val}(P)$ and $\mathrm{Sol}(P)$ is nonempty and bounded.

Conversely, suppose that Sol(P) is nonempty and bounded. By Proposition 3.1, it follows that $\partial\vartheta^{**}(0)$ is nonempty and bounded. This implies that $\vartheta^{**}(y)$ is finite valued for all y in a neighborhood of $0 \in \mathbb{R}^n$. This, in turn, implies that $\vartheta(y)$ is finite valued for all y in a neighborhood of $0 \in \mathbb{R}^n$, and hence condition (i) follows. ■

Consider the following linear SIP:

$$\operatorname*{Min}_{x\in\mathbb{R}^n} \langle c, x\rangle \text{ subject to } \langle a(\omega), x\rangle + b(\omega) \leq 0, \quad \omega \in \Omega. \tag{3.10}$$

For the problem we have that, for $\lambda \in K^*$,

$$\varphi(\lambda, y) = \begin{cases} \langle \lambda, b\rangle, & \text{if } y = c + \langle \lambda, a\rangle, \\ -\infty, & \text{otherwise,} \end{cases}$$

and hence

$$\vartheta(y) = \inf\big\{ -\langle \lambda, b\rangle : c + \langle \lambda, a\rangle = y,\ \lambda \in K^*\big\}. \tag{3.11}$$

Therefore for the linear SIP, condition (i) of Proposition 3.3 is equivalent to the condition:

$$0 \in \operatorname{int}\big\{y \in \mathbb{R}^n : y = c + \langle \lambda, a\rangle,\ \lambda \in K^*\big\}. \tag{3.12}$$

Condition (3.12) is well known in the dual theory of linear SIP (cf., [1]).

Proposition 3.4 *Suppose that the assumptions* (A1) *and* (A2) *hold and* Sol(P) *is nonempty and bounded. Then* val(D) = val(P).

Proof Since Sol(P) $= -\partial\vartheta^{**}(0)$, it follows that $\partial\vartheta^{**}(0)$ is nonempty and bounded. This implies that $\vartheta^{**}(y)$, and hence $\vartheta(y)$, is continuous at $y = 0$. ■

In various settings the result of Proposition 3.4 is known, e.g., in the case of "for a.e. $\omega \in \Omega$" setting and $\mathcal{Y} := L_\infty(\Omega, \mathcal{F}, \mu)$ and $\mathcal{Y}^* := L_1(\Omega, \mathcal{F}, \mu)$, [9], and $\mathcal{Y} := L_1(\Omega, \mathcal{F}, \mu)$ and $\mathcal{Y}^* := L_\infty(\Omega, \mathcal{F}, \mu)$, [5], and in the setting of "for all $\omega \in \Omega$" and linear semi-infinite programming, [2]. The above derivations show that the nonemptiness and boundedness of Sol(P) implies the strong duality property for general convex problems under the minimal structural assumption (A1).

In the setting "for a.e. $\omega \in \Omega$" and $\mathcal{Y} := L_1(\Omega, \mathcal{F}, \mu)$ and $\mathcal{Y}^* := L_\infty(\Omega, \mathcal{F}, \mu)$, the following extension of Proposition 3.4 is similar to a result presented in [5].

Proposition 3.5 *Suppose that the assumptions* (A1) *and* (A2) *hold and the set* Sol(P) *has the form*

$$\mathrm{Sol}(P) = A + \mathcal{L}, \tag{3.13}$$

where A is a nonempty bounded subset of $\mathbb{R}^n$ and $\mathcal{L}$ is a linear subspace of $\mathbb{R}^n$. Then val(D) = val(P).

Proof Suppose that condition (3.13) holds. Then by Proposition 3.1 we have that $\vartheta^{**}(0)$ is finite and $\partial\vartheta^{**}(0) = -A + \mathcal{L}$. Since

$$\vartheta^{**}(y) \geq \vartheta^{**}(0) + \langle z, y \rangle, \quad \forall\, z \in \partial\vartheta^{**}(0),$$

it follows that $\vartheta^{**}(y) = +\infty$ for any $y \notin \mathcal{L}^{\perp}$, i.e., the domain of ϑ^{**} is contained in the linear space $\mathcal{L}^{\perp}$. Consequently, the domain of ϑ is also contained in $\mathcal{L}^{\perp}$. Since A is bounded, we have that $\vartheta^{**}(\cdot)$, restricted to $\mathcal{L}^{\perp}$, has a nonempty and bounded subdifferential at $0 \in \mathcal{L}^{\perp}$. It follows that $\vartheta^{**}(y)$ is finite for all y restricted to $\mathcal{L}^{\perp}$ in a neighborhood of $0 \in \mathcal{L}^{\perp}$. Consequently, $\vartheta(y) = \vartheta^{**}(y)$ and is finite for all $y \in \mathcal{L}^{\perp}$ in a neighborhood of the null vector. It follows that, restricted to $\mathcal{L}^{\perp}$, $\vartheta(\cdot)$ is continuous at $0 \in \mathcal{L}^{\perp}$. Consequently, $\vartheta(\cdot)$ is lower semicontinuous at $0 \in \mathbb{R}^n$, and hence $\mathrm{val}(D) = \mathrm{val}(P)$. ■

So far we did not discuss the existence of optimal solutions of the dual problem. By the standard min–max theory we have from (2.6) and (2.7) that if $\mathrm{val}(P) = \mathrm{val}(D)$, then $\bar{x}$ and $\bar{\lambda}$ are optimal solutions of the primal and dual problems, respectively iff $(\bar{x}, \bar{\lambda})$ is a saddle point of the Lagrangian $L(x, \lambda)$, i.e., $\bar{x} \in \arg\min_{x \in \mathbb{R}^n} L(x, \bar{\lambda})$ and $\bar{\lambda} \in \arg\max_{\lambda \in K^*} L(\bar{x}, \lambda)$. Because of assumption (A1), the second of the above conditions means that $G(\bar{x}) \in K$ and $\langle \bar{\lambda}, G(\bar{x}) \rangle = 0$. Therefore, if $\mathrm{val}(P) = \mathrm{val}(D)$, then $\bar{x}$ is an optimal solution of the primal problem and $\bar{\lambda}$ is an optimal solution of the dual problem iff

$$\bar{x} \in \arg\min_{x \in \mathbb{R}^n} L(x, \bar{\lambda}), \quad G(\bar{x}) \in K \quad \text{and} \quad \langle \bar{\lambda}, G(\bar{x}) \rangle = 0. \tag{3.14}$$

Let us remark that without additional topological type assumptions, existence of an optimal solution for the dual problem cannot be guaranteed (cf., [6]).

Finally, let us consider the following min–max problem

$$\operatorname*{Min}_{x \in X} \left\{ \phi(x) := \sup_{\omega \in \Omega} h(x, \omega) \right\}. \tag{3.15}$$

We make the following assumptions:

Assumption (B1) The set X is a nonempty closed convex subset of $\mathbb{R}^n$.
Assumption (B2) The set Ω is a nonempty convex subset of a linear space Σ.
Assumption (B3) The function $h : \mathbb{R}^n \times \Sigma \to \mathbb{R}$ is such that $h(\cdot, \omega)$ is convex for every $\omega \in \Omega$.

With the problem (3.15) is associated the following SIP

$$\operatorname*{Min}_{x \in X,\ z \in \mathbb{R}} z \quad \text{subject to} \ \ h(x, \omega) - z \leq 0, \ \ \omega \in \Omega. \tag{3.16}$$

The optimal values of problems (3.15) and (3.16) are equal to each other, and $\bar{x}$ is an optimal solution of (3.15) iff $(\bar{x}, \bar{z})$, with $\bar{z}$ being the optimal value of (3.15), is an optimal solution of (3.16). The dual of problem (3.16) can be written in the form

$$\operatorname*{Max}_{\substack{\lambda\in K^* \\ \sum_{\omega\in\Omega}\lambda(\omega)=1}} \left\{ \inf_{x\in X} \sum_{\omega\in\Omega} \lambda(\omega) h(x,\omega) \right\}, \tag{3.17}$$

where $\lambda : \Omega \to \mathbb{R}$ is such that only a finite number of $\lambda(\omega)$ are nonzero and K^* is defined in (2.8). The assumption (A1) holds here and assumption (A2) is implied by the assumptions (B1) and (B3). Therefore, we can apply the developed theory to problems (3.16) and (3.17) in a straightforward way. In particular, we obtain that the common optimal value of (3.15) and (3.16) is equal to the optimal value of (3.17) if the set $\mathrm{Sol}(P)$ of optimal solutions of (3.15) can be represented in the form (3.13), with A being a nonempty bounded subset of $\mathbb{R}^n$ and $\mathcal{L}$ being a linear subspace of $\mathbb{R}^n$.

Further, let us make the following assumption.

Assumption (B4) For every $x \in X$, the function $h(x,\cdot) : \Sigma \to \mathbb{R}$ is concave.

Under assumption (B4), we have that for any $\lambda \in K^*$ such that $\sum_{\omega\in\Omega}\lambda(\omega) = 1$, the inequality

$$\sum_{\omega\in\Omega} \lambda(\omega) h(x,\omega) \geq h(x,\bar{\omega})$$

holds with $\bar{\omega} := \sum_{\omega\in\Omega}\lambda(\omega)\omega$. Note that by convexity of Ω, we have that $\bar{\omega} \in \Omega$. Therefore, under the assumptions (B3) and (B4), the optimal value of (3.17) is equal to the optimal value of the max–min problem

$$\operatorname*{Max}_{\omega\in\Omega} \left\{ \eta(\omega) := \inf_{x\in X} h(x,\omega) \right\}. \tag{3.18}$$

The above discussion together with Proposition 3.5 implies the following result.

Proposition 3.6 *Suppose that the assumptions* (B1)–(B4) *hold and the set of optimal solutions of* (3.15) *can be represented in the form $A + \mathcal{L}$, with A being a nonempty bounded subset of $\mathbb{R}^n$ and $\mathcal{L}$ being a linear subspace of $\mathbb{R}^n$. Then the optimal values of* (3.15) *and* (3.18) *are equal to each other.*

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