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NONDEGENERACY AND QUANTITATIVE STABILITY OF PARAMETERIZED OPTIMIZATION PROBLEMS WITH MULTIPLE SOLUTIONS*

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Abstract. This paper presents some results on quantitative stability of optimal solutions of parameterized optimization problems having nonisolated optima. The analysis is based on the introduced concept of nondegeneracy, which may be of independent interest. Examples of nonlinear and semidefinite programming are discussed.

Key words. sensitivity analysis, parameterized optimization, quantitative stability, semidefinite programming, nondegeneracy

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1. Introduction.

Consider the following parameterized optimization problem:

$$\min_{x \in X} f(x,u) \quad \text{subject to (s.t.) } x \in \Phi(u), \qquad (P_u) \tag{1.1}$$

depending on the parameter vector $u \in U$. We assume that X is a finite dimensional vector space, say $X = \mathbb{R}^m$, that U is a normed vector space and that the feasible set is defined in the form

$$\Phi(u) = \{x : G(x,u) \in K\}, \tag{1.2}$$

where K is a closed convex subset of a space $Y := \mathbb{R}^n$ and $G : X \times U \to Y$ is a *continuously differentiable* mapping. We investigate continuity properties of the set $\mathcal{S}(u)$ of optimal solutions of (P_u), considered a function of u.

We say that the optimal-solutions set $\mathcal{S}(u)$ is *Lipschitz stable* at a point $u_0 \in U$ if the multifunction $u \to \mathcal{S}(u)$ is upper Lipschitz continuous at u_0; i.e., there exist a constant $\alpha > 0$ and a neighborhood $\mathcal{U}$ of u_0 such that, for all $u \in \mathcal{U}$, the inclusion

$$\mathcal{S}(u) \subset \mathcal{S}(u_0) + \alpha \|u - u_0\| B_X \tag{1.3}$$

holds. (Here B_X denotes the unit ball $B_X := \{x \in X : \|x\| \le 1\}$ in the space X.) In this paper we study Lipschitzian stability of optimal solutions in cases where $\mathcal{S}(u_0)$ possibly is not a singleton. In situations where the feasible set $\Phi(u) \equiv \Phi$ is fixed (independent of u), Lipschitzian stability of optimal solutions holds essentially under a second order growth condition (for the unperturbed problem); see [14]. The general case is considerably more subtle. If $\mathcal{S}(u_0) = \{x_0\}$ is a singleton, then Lipschitzian stability of optimal solutions was studied extensively (see, e.g., [5] and references

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therein). It is known that the second order growth condition does not imply Lipschitzian stability of optimal solutions, even if $\mathcal{S}(u)$ is a singleton and the set K is polyhedral.

Therefore, in order to ensure Lipschitzian stability of optimal solutions, some additional assumptions are needed apart from the second order growth condition. In the next section we discuss a concept of nondegenerate points. We show that if a feasible point of the unperturbed problem is nondegenerate, then locally (P_u) can be reparameterized in such a way that the feasible set of the new problem does not depend on u. That is, in the nondegenerate case the question of Lipschitzian stability of optimal solutions can be reduced to studying parameterized problems with a fixed feasible set.

For the unperturbed problem, i.e., for $u = u_0$, we sometimes delete u_0 and write $G(\cdot)$ for $G(\cdot, u_0)$, etc. The Lagrangian, for the unperturbed problem (P), can be written as

$$L(x, \lambda) := f(x) + \lambda \cdot G(x).$$

With an optimal solution x_0 of (P) is associated a set of Lagrange multipliers λ satisfying the first order optimality conditions:

$$D_x L(x_0, \lambda) = 0,\ x_0 \in \Phi,\ \lambda \in N_K(G(x_0)). \tag{1.4}$$

Here $\Phi := \Phi(u_0)$ is the feasible set of (P), $N_K(y_0)$ denotes the normal cone to K at $y_0 \in K$, and $D_x L(x_0, \lambda)$ stands for the differential of $L(\cdot, \lambda)$ at the point x_0. Under the constraint qualification, due to Robinson [11],

$$0 \in \operatorname{int}\{G(x_0) + DG(x_0)X - K\}, \tag{1.5}$$

the set of Lagrange multipliers, satisfying (1.4), is nonempty and bounded [9, 10]. Note that since the space Y is finite dimensional, (1.5) is equivalent to

$$DG(x_0)X - T_K(G(x_0)) = Y, \tag{1.6}$$

where $T_K(y_0)$ denotes the tangent cone to K at $y_0 \in K$.

2. Nondegeneracy. In this section we discuss a concept of nondegeneracy for the unperturbed problem

$$\min_{x \in X} f(x) \quad \text{s.t. } G(x) \in K. \qquad (P) \tag{2.1}$$

The material of this section may be of independent interest and will be a basis for a stability analysis of the optimal solutions of (P_u).

Consider a feasible point $x_0 \in \Phi$. Informally speaking we say that x_0 is a *nondegenerate* point of $G(x)$, with respect to the set K, if a local system of coordinates $\xi_1(y), \dots, \xi_k(y)$, $k \le n$, can be constructed in Y such that the set K can be described near $y_0 := G(x_0)$ in that system of coordinates and the composite mapping $x \to (\xi_1(G(x)), \dots, \xi_k(G(x)))$ is nondegenerate at x_0. The precise definition follows.

DEFINITION 2.1. *We say that a feasible point x_0 is a nondegenerate point of $G(x)$, with respect to K, if there exist* (a) *a neighborhood $\mathcal{N} \subset Y$ of $y_0 := G(x_0)$,* (b) *a continuously differentiable mapping $\Xi = (\xi_1, \dots, \xi_k) : \mathcal{N} \to \mathbb{R}^k$, and* (c) *a set $C \subset \mathbb{R}^k$ such that* (i) *$K \cap \mathcal{N} = \Xi^{-1}(C)$ and* (ii) *$D(\Xi \circ G)(x_0) : X \to \mathbb{R}^k$ is onto. We can assume, without loss of generality, that $\Xi(y_0) = 0$.*

Note that the above concept of nondegeneracy involves the mapping G and the set K and also depends on a particular choice of the local system of coordinates $\xi_1(y),\ldots,\xi_k(y)$. The following simple example demonstrates this point.

Example 1. Consider the set $K := \{(y_1,y_2) : y_1 \geq 0\} \subset \mathbb{R}^2$ and the mapping $G(x) := (x,0)$ from $\mathbb{R}$ into $\mathbb{R}^2$. Clearly then $\Phi = \{x : x \geq 0\}$. Consider the standard system of coordinates in $\mathbb{R}^2$, namely $\xi_1(y_1,y_2) := y_1$ and $\xi_2(y_1,y_2) := y_2$. We have then that $\Xi \circ G$ is linear, mapping x into $(x,0)$, and hence, for any $x_0 \in \mathbb{R}$, $D(\Xi \circ G)(x_0) = \Xi \circ G$ is *not* onto. On the other hand the set K is defined by the constraint $y_1 \geq 0$, and hence we can take $\xi_1(y_1,y_2) := y_1$ with $k = 1$. For such choice of coordinates, $\Xi \circ G(x) = x$ and the nondegeneracy follows. Of course in the first representation the coordinate y_2 is redundant and the one-dimensional representation is preferable.

It follows from condition (i) of the above definition that locally the feasible set Φ can be defined by the constraint $(\Xi \circ G)(x) \in C$, i.e., $G^{-1}(K \cap \mathcal{N}) = (\Xi \circ G)^{-1}(C)$. Note that by the chain rule $D(\Xi \circ G) = D\Xi \circ DG$, and hence it follows from (ii) that $D\Xi(y_0) : Y \to \mathbb{R}^k$ is onto; i.e., the gradient vectors $\nabla\xi_1(y_0),\ldots,\nabla\xi_k(y_0)$ are linearly independent, which is a required condition for $\xi_1,\ldots,\xi_k$ to be a local system of coordinates. In fact it is not difficult to see that condition (ii) is equivalent to the following condition:

$$D\Xi(y_0)Y = \mathbb{R}^k \quad \text{and} \quad DG(x_0)X + \operatorname{Ker}[D\Xi(y_0)] = Y. \tag{2.2}$$

Suppose that the point x_0 is nondegenerate, and consider the set

$$W := \{y \in \mathcal{N} : \xi_i(y) = 0,\ i = 1,\ldots,k\}. \tag{2.3}$$

We have, by the condition (i) and since $\Xi(y_0) = 0$, that $W \subset K$ and, because of the first part of condition (2.2), W is a smooth manifold near y_0, of dimension $n-k$, with the tangent space $T_W(y_0) = \operatorname{Ker}[D\Xi(y_0)]$. Therefore, the second part of condition (2.2) can be written in the form

$$DG(x_0)X + T_W(y_0) = Y. \tag{2.4}$$

The above condition means that G intersects W transversally at the point x_0 (see, e.g., [6] for a discussion of the transversality concept).

Since $T_W(y_0) \subset T_K(y_0)$, it also follows from (2.4) that Robinson's constraint qualification (1.6) holds at x_0; i.e., the nondegeneracy condition is stronger than Robinson's constraint qualification. Therefore, if x_0 is a locally optimal solution of (P), then there exists a Lagrange multiplier λ satisfying (1.4). Moreover, it follows from the nondegeneracy condition that λ is *unique*. Indeed, since $\lambda \in N_K(y_0)$ and $T_W(y_0) \subset T_K(y_0)$, we have that λ is orthogonal to $T_W(y_0)$. If λ' is another Lagrange multiplier, then $\lambda - \lambda'$ is orthogonal to both $DG(x_0)X$ and to $T_W(y_0)$. It follows then from (2.4) that $\lambda - \lambda' = 0$.

Let us note at this point that the above concept of nondegeneracy is somewhat different from the one introduced in [12], although it is aimed at essentially the same goal as the following result shows.

THEOREM 2.2. *Suppose that x_0 is a nondegenerate point of G. Then there exist neighborhoods $\mathcal{X}$ of x_0, $\mathcal{U}$ of u_0, $\mathcal{Z}$ of $0 \in \mathbb{R}^m$, and a continuously differentiable mapping $T : \mathcal{Z} \times \mathcal{U} \to \mathcal{X}$ such that $T(0,u_0) = x_0$ and, for every $u \in \mathcal{U}$, $T(\cdot,u)$ is a diffeomorphism of $\mathcal{Z}$ onto $\mathcal{X}$ (i.e., $T(\cdot,u) : \mathcal{Z} \to \mathcal{X}$ is one-to-one, onto, and its inverse is also continuously differentiable) and*

$$T((C \times \mathbb{R}^{m-k}) \cap \mathcal{Z}, u) = \Phi(u) \cap \mathcal{X}. \tag{2.5}$$

Proof. Consider the mapping $\Psi(x,u) := (\Xi \circ G)(x,u)$ defined on a neighborhood of (x_0,u_0). By condition (ii) of Definition 2.1 we have that $D_x\Psi(x_0,u_0)$ is onto, or equivalently, that the $m \times k$ Jacobian matrix $\nabla_x\Psi(x_0,u_0)$ is of rank k. Without loss of generality we can assume that the upper $k \times k$ submatrix of $\nabla_x\Psi(x_0,u_0)$ is nonsingular. Let $\psi_i(x,u)$, $i = 1,\dots,k$, be coordinate functions of the mapping $\Psi(x,u)$, and consider the following system of equations:

$$z_1 = \psi_1(x,u),\dots,z_k = \psi_k(x,u), z_{k+1} = x_{k+1},\dots,z_m = x_m. \tag{2.6}$$

We have then that the Jacobian, with respect to x, of this system at (x_0,u_0) is nonsingular, and hence, by the implicit function theorem, the above equations can be locally inverted in x for all u near u_0. That is, for every u near u_0, there exists a local diffeomorphism $T(\cdot,u) : \mathcal{Z} \to \mathcal{X}$ such that $z_i = \psi_i(T(z,u),u)$, $i = 1,\dots,k$, and $T(z,u)$ is continuously differentiable. Since locally the set $\Phi(u)$ can be defined by the constraint $(\psi_1(x,u),\dots,\psi_k(x,u)) \in C$, (2.5) follows. □

The above theorem shows that *locally*, near x_0 and for all u in a neighborhood of u_0, problem (P_u) can be reparameterized into the equivalent problem

$$\min_{z\in\mathbb{R}^m} f(T(z,u),u) \ \text{ s.t. } \ (z_1,\dots,z_k) \in C, \tag{2.7}$$

whose feasible set does not depend on u. We will use this in the next section while deriving some stability results for the optimal-solutions set $\mathcal{S}(u)$. Let us consider now some examples.

Example 2. Suppose that the set K can be defined near y_0 by a finite number of inequality constraints. That is, there exist continuously differentiable functions $\xi_1(y),\dots,\xi_k(y)$ and a neighborhood $\mathcal{N}$ of y_0 such that $\xi_i(y_0) = 0$, $i = 1,\dots,k$, and

$$K \cap \mathcal{N} = \{y \in \mathcal{N} : \xi_1(y) \geq 0,\dots,\xi_k(y) \geq 0\}. \tag{2.8}$$

In that case the set C is given by $\mathbb{R}^k_+$ and, in that system of coordinates, a feasible point x_0 is nondegenerate iff $\nabla\xi_1(y_0),\dots,\nabla\xi_k(y_0)$ are linearly independent and condition (2.4) holds with

$$T_W(y_0) = \{\eta : \eta^T\nabla\xi_i(y) = 0,\ i = 1,\dots,k\}.$$

Example 3. Suppose that the set K is polyhedral, and consider the tangent cone $T_K(y_0)$. Note that, since K is polyhedral, there is a neighborhood $\mathcal{N}$ of y_0 such that K coincides with $y_0 + T_K(y_0)$ in that neighborhood. Let $L := \text{lin}[T_K(y_0)]$ be the lineality space of $T_K(y_0)$, i.e., the largest linear subspace of $T_K(y_0)$. Then the cone $T_K(y_0)$ can be defined by a finite number of linear constraints $T_K(y_0) = \{y : a_i^T y \geq 0,\ i = 1,\dots,p\}$, with $a_i \in M$, where M is the linear space orthogonal to L and such that $L + M = Y$. Let $k := \dim(M)$ and choose a basis $b_1,\dots,b_k$ in M. We have then that $b_1,\dots,b_k$ are linearly independent and every vector a_i, $i = 1,\dots,p$, can be represented as a linear combination of vectors $b_1,\dots,b_k$. It follows that locally the set K can be defined in the following system of coordinates: $\xi_i(y) := b_i^T(y - y_0)$, $i = 1,\dots,k$. In that system of coordinates x_0 is nondegenerate iff

$$DG(x_0)X + \text{lin}[T_K(y_0)] = Y. \tag{2.9}$$

For *polyhedral* sets the above condition of nondegeneracy coincides with the one used in [12]. In particular if $K := \mathbb{R}^n_+$, then x_0 is nondegenerate iff vectors $\nabla g_i(x_0)$, $i \in I(x_0)$, are linearly independent, where $I(x_0) := \{i : g_i(x_0) = 0,\ i = 1,\dots,n\}$.

Finally let us consider the following example of semidefinite programming.

Example 4. Let $Y := S^{p\times p}$ be the space of $p \times p$ symmetric matrices and $K := S_+^{p\times p}$ be the cone of $p \times p$ positive semidefinite symmetric matrices. Let $y_0 \in S_+^{p\times p}$ be a matrix of rank $r < p$. Denote by $\lambda_1(A) \geq \cdots \geq \lambda_p(A)$ the eigenvalues of a $p \times p$ symmetric matrix A and by $e_1(A), \dots, e_p(A)$ an orthonormal set of corresponding eigenvectors. Let $E(A)$ be the $p \times (p-r)$ matrix whose columns are formed from vectors $e_{r+1}(A), \dots, e_p(A)$, and let $E_0 := E(y_0)$. Note that the columns $e_{r+1}(y_0), \dots, e_p(y_0)$ of E_0 generate the null space of y_0. Since $E(A)^T AE(A)$ is a diagonal matrix with diagonal elements $\lambda_{r+1}(A), \dots, \lambda_p(A)$ and the eigenvalues $\lambda_1(y_0), \dots, \lambda_r(y_0)$ are positive, we have that the cone $S_+^{p\times p}$ can be defined in a neighborhood of y_0 by the constraint $E(A)^T AE(A) \succeq 0$. (The notation $B \succeq 0$ means that the matrix B is positive semidefinite.) The latter constraint has value in the set of $(p-r) \times (p-r)$ symmetric matrices. However, this formulation is not suitable for our purpose since the eigenvectors $e_{r+1}(A), \dots, e_p(A)$ are not uniquely defined and $E(A)$ is not a continuous (and hence is not smooth) function of A near y_0 unless 0 is a simple eigenvalue of y_0. In order to overcome this difficulty we proceed as follows.

Denote by $L(A)$ the eigenspace corresponding to the $p-r$ smallest eigenvalues of A, and let $P(A)$ be the orthogonal projection matrix onto $L(A)$. Also let E_0 be a (fixed) $p \times (p-r)$ matrix whose columns are orthonormal and span the space $L(y_0)$. It is well known (see [7]) that $P(A)$ is a continuously differentiable (in fact even analytic) function of A in a sufficiently small neighborhood of y_0. Consequently $F(A) := P(A)E_0$ is also a continuously differentiable function of A in a neighborhood of y_0, and, moreover, $F(y_0) = E_0$. It follows that, for all A sufficiently close to y_0, the rank of $F(A)$ is $p-r$; i.e., its column vectors are linearly independent. Let $U(A)$ be the matrix whose columns are obtained by applying the Gram–Schmidt orthonormalization procedure to the columns of $F(A)$. The matrix $U(A)$ is well defined and continuously differentiable near y_0, and, moreover, satisfies the following conditions: $U(y_0) = E_0$, the column space of $U(A)$ coincides with the column space of $E(A)$, and $U(A)^T U(A) = I_{p-r}$ (cf. [15, p. 557]).

We obtain that in a *neighborhood* of y_0, the cone $S_+^{p\times p}$ can be defined in the form

$$\left\{A \in S^{p\times p} : U(A)^T AU(A) \succeq 0\right\}. \tag{2.10}$$

Consider the mapping $\Xi : A \to U(A)^T AU(A)$ from $\mathcal{N}$ into $S^{(p-r)\times(p-r)}$. The mapping Ξ is continuously differentiable, and, since

$$D\Xi(y_0)A = (DU(y_0)^T A)y_0 U(y_0) + U(y_0)^T AU(y_0) + U(y_0)^T y_0 (DU(y_0)A)$$

and $y_0 U(y_0) = 0$, we have $D\Xi(y_0)A = E_0^T AE_0$. It follows that $D\Xi(y_0)$ is onto. The set $W := \{A \in \mathcal{N} : \Xi(A) = 0\}$ is formed here by symmetric matrices of rank r, it is a smooth manifold and

$$T_W(y_0) = \left\{Z \in S^{p\times p} : E_0^T ZE_0 = 0\right\}. \tag{2.11}$$

We obtain that, in the above system of local coordinates, a feasible point x_0 is nondegenerate iff

$$DG(x_0)X + T_W(y_0) = S^{p\times p}. \tag{2.12}$$

Condition (2.12) means that G intersects *transversally*, at x_0, the smooth manifold of symmetric matrices of rank r and is the standard nondegeneracy condition in semidefinite programming [1], [15, Section 2] (see also [16] for a discussion and equivalent formulations of this condition).

3. Stability analysis. Suppose that, for $u = u_0$, the corresponding set $S_0 := \mathcal{S}(u_0)$, of optimal solutions of the unperturbed problem (P), is nonempty. In this section we investigate Lipschitzian stability of optimal solutions in cases where S_0 possibly is not a singleton.

In situations where the feasible set $\Phi(u) \equiv \Phi$ is fixed, it is relatively easy to give sufficient conditions for Lipschitzian stability of $\mathcal{S}(u)$. We say that the *second order growth* condition holds for (P) if there exist a constant $c > 0$ and a neighborhood $\mathcal{N}$ of S_0 such that

$$f(x) \geq f_0 + c\,[\operatorname{dist}(x, S_0)]^2 \quad \forall x \in \Phi \cap \mathcal{N}, \tag{3.1}$$

where $f_0 := \inf_{x\in\Phi} f(x)$ (see [2, 3, 4] for a discussion of this condition). Consider the optimization problem

$$\min_{x\in\Phi} \psi(x). \tag{3.2}$$

We can view $\psi(x)$ as a perturbation of the objective function $f(x)$ of the problem (P). It is said that $\bar{x}$ is an ε-optimal solution of (3.2) if $\bar{x} \in \Phi$ and $\psi(\bar{x}) \leq \inf_{x\in\Phi} \psi(x) + \varepsilon$. We have then a simple bound for perturbations of an ε-optimal solution $\bar{x}$ in terms of the constant c and the Lipschitzian constant of the difference function $\psi(x) - f(x)$ [14, Lemma 2.1].

PROPOSITION 3.1. *Suppose that the second order growth condition* (3.1) *holds and that the difference function $\psi(x) - f(x)$ is Lipschitz continuous on the neighborhood $\mathcal{N}$ of S_0, modulus κ, and let $\bar{x} \in \mathcal{N}$ be an ε-optimal solution of* (3.2). *Then*

$$\operatorname{dist}(\bar{x}, S_0) \leq c^{-1}\kappa + c^{-1/2}\varepsilon^{1/2}. \tag{3.3}$$

We can formulate now the main result of this section. Recall that a function (a mapping) $g(\cdot)$ is said to be $C^{1,1}$ if it is continuously differentiable and its gradient $\nabla g(\cdot)$ is locally Lipschitz continuous.

THEOREM 3.2. *Suppose that* (i) *$f(x, u)$ and $G(x, u)$ are $C^{1,1}$,* (ii) *the second order growth condition* (3.1) *(for the unperturbed problem) holds,* (iii) *every point of the set S_0 is a nondegenerate point of G, with respect to K, and with the corresponding coordinate functions $\xi_i(y)$ being $C^{1,1}$ functions, and* (iv) *for all u in a neighborhood of u_0, the optimal-solutions sets $\mathcal{S}(u)$ are uniformly bounded. Then $\mathcal{S}(u)$ is Lipschitz stable at u_0.*

Proof. Consider a sequence $u_n \to u_0$, and let $x_n \in \mathcal{S}(u_n)$. It suffices to show that

$$\operatorname{dist}(x_n, S_0) = O(\|u_n - u_0\|). \tag{3.4}$$

Since $\mathcal{S}(u)$ are uniformly bounded, we can assume, by passing to a subsequence if necessary, that the sequence $\{x_n\}$ converges to a point x_0. By continuity arguments it is not difficult to show (and it is well known) that then $x_0 \in S_0$. Since it is assumed that the point x_0 is nondegenerate, by Theorem 2.2 there exists a continuously differentiable mapping $T(z, u)$ such that, locally near x_0 and for all u in a neighborhood of u_0, problem (P_u) can be reparameterized into the problem (2.7). Note that, for n large enough, the relation $T(z_n, u_n) = x_n$ defines a sequence $z_n \to 0$ of optimal solutions of (2.7) for $u = u_n$. Let Z_0 be the set of optimal solutions of the reparameterized problem (2.7) restricted to a neighborhood of 0. Since $T(\cdot, u)$ is Lipschitz continuous near $z = 0$ with the Lipschitz constant uniformly bounded for all u in a neighborhood of u_0, the estimate (3.4) will follow from

$$\operatorname{dist}(z_n, Z_0) = O(\|u_n - u_0\|). \tag{3.5}$$

It remains to show that the Lipschitzian stability result (3.5), for the reparameterized problem, holds. First let us observe that since the mapping $T(\cdot, u_0)$ and its inverse are continuously differentiable, and hence are Lipschitz continuous near x_0, we have that the second order growth condition for the problem (2.7), possibly restricted to a neighborhood of $z = 0$ and for $u = u_0$, follows from the second order growth condition (3.1). Therefore, by (3.3), we only have to verify that the Lipschitz constant of the function $h(\cdot, u) := f(T(\cdot, u), u) - f(T(\cdot, u_0), u_0)$, in a neighborhood $\mathcal{Z}$ of 0, is of order $O(\|u - u_0\|)$. Note that by the mean value theorem such a Lipschitz constant is given by $\sup_{z \in \mathcal{Z}} \|\nabla_z h(z, u)\|$. Therefore, by the chain rule of differentiation and since f is a $C^{1,1}$ function, we have that the Lipschitz constant of $h(\cdot, u)$, on $\mathcal{Z}$, is of order $O(\|u - u_0\|)$ if

$$\sup_{z \in \mathcal{Z}} \|\nabla_z T(z, u) - \nabla_z T(z, u_0)\| = O(\|u - u_0\|). \tag{3.6}$$

Since the coordinate mapping Ξ is assumed to be $C^{1,1}$, the composite mapping $\Xi \circ G$ is also $C^{1,1}$. The implicit function theorem implies then that the mapping $T(z, u)$ is also $C^{1,1}$. This completes the proof. □

In the case of nonlinear programming, i.e., when the set K is polyhedral, the above stability result was derived in [13] (under somewhat restrictive assumptions) and in [8] by different techniques. This result is new in the framework of semidefinite programming. The stability result can be also extended to ε-optimal-solutions set of (P_u) if $\varepsilon = O(\|u - u_0\|^2)$.

REFERENCES

[1] F. Alizadeh, J.P. Haeberly, and M.L. Overton, *Complementarity and nondegeneracy in semidefinite programming*, Math. Programming, 77 (1997), pp. 111–128.

[2] J.F. Bonnans, *Extended quadratic tangent optimization problems*, in Mathematical Programming with Data Perturbations, A.V. Fiacco, ed., Marcel Dekker, New York, Basel, 1998, pp. 31–45.

[3] J.F. Bonnans and A.D. Ioffe, *Second-order sufficiency and quadratic growth for nonisolated minima*, Math. Oper. Res., 20 (1995), pp. 801–817.

[4] J.F. Bonnans and A.D. Ioffe, *Quadratic growth and stability in convex programming problems with multiple solutions*, J. Convex Anal., 2 (1995), pp. 41–57.

[5] J.F. Bonnans and A. Shapiro, *Optimization problems with perturbations: A guided tour*, SIAM Rev., 40 (1998), pp. 228–264.

[6] M. Golubitsky and V. Guillemin, *Stable Mappings and their Singularities*, Springer-Verlag, New York, 1973.

[7] T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, New York, 1984.

[8] D. Klatte, *On quantitative stability for non-isolated minima*, Control Cybernet., 23 (1994), pp. 183–200.

[9] S. Kurcyusz, *On the existence and nonexistence of Lagrange multipliers in Banach spaces*, J. Optim. Theory Appl., 20 (1976), pp. 81–110.

[10] S.M. Robinson, *First order conditions for general nonlinear optimization*, SIAM J. Appl. Math., 30 (1976), pp. 597–607.

[11] S.M. Robinson, *Stability theory for systems of inequalities, part* II: *Differentiable nonlinear systems*, SIAM J. Numer. Anal., 13 (1976), pp. 497–513.

[12] S.M. Robinson, *Local structure of feasible sets in nonlinear programming.* II. *Nondegeneracy*, Math. Programming Stud., 22 (1984), pp. 217–230.

[13] A. Shapiro, *Perturbation theory of nonlinear programs when the set of optimal solutions is not a singleton*, Appl. Math. Optim., 18 (1988), pp. 215–229.

[14] A. Shapiro, *Perturbation analysis of optimization problems in Banach spaces*, Numer. Funct. Anal. Optim., 13 (1992), pp. 97–116.

[15] A. Shapiro and M.K.H. Fan, *On eigenvalue optimization*, SIAM J. Optim., 5 (1995), pp. 552–569.

[16] A. Shapiro, *First and second order analysis of nonlinear semidefinite programs. Semidefinite programming*, Math. Programming Ser. B, 77 (1997), pp. 301–320.