Journal of Mathematical Sciences, Vol. 115, No. 4, 2003

SENSITIVITY ANALYSIS OF GENERALIZED EQUATIONS

A. Shapiro UDC 517.977

In this paper, we study sensitivity analysis of generalized equations (variational inequalities) with nonpolyhedral set constraints. We use an approach of local reduction of the corresponding constraint set to a convex cone. This leads us to the introduction of an additional term representing a curvature of the constraint set in a linearization of the generalized equations. We also discuss concepts of nondegeneracy and strict complementarity of set constrained problems.

1. Introduction

In this paper, we study sensitivity analysis of the following parameterized generalized equations:

$$(GE_u) \qquad F(x,u) + D_xG(x,u)^*\lambda = 0, \quad \lambda \in N_K(G(x,u)), \tag{1.1}$$

where X, Y, and U are *finite-dimensional* vector spaces equipped with a scalar product $\langle \cdot, \cdot \rangle$, $F : X \times U \to X$, $G : X \times U \to Y$, $K \subset Y$ is a closed convex set, and $D_xG(x,u)^* : Y \times U \to X$ is the adjoint of the derivative mapping $D_xG(x,u) : X \times U \to Y$. For a given value $u_0 \in U$ of the parameter vector, we consider the corresponding generalized equations (GE_{u_0}) as unperturbed and write $F(x)$, $G(x)$, and (GE) for the corresponding mappings and the generalized equations, respectively.

Let $(\bar{x}(u), \bar{\lambda}(u))$ be a solution of (GE_u) and (x_0, λ_0) be a solution of the unperturbed problem (GE). We investigate the continuity and differentiability properties of $\bar{x}(u)$ and $\bar{\lambda}(u)$ in the vicinity of the points x_0 and λ_0, respectively. We assume that $F(x,u)$ is continuously differentiable and $G(x,u)$ is twice continuously differentiable in a neighborhood of (x_0, u_0).

Let us observe that if the mapping $F(x,u)$ is a derivative of a real-valued function $f : X \times U \to \mathbb{R}$, i.e., $F(x,u) = D_xf(x,u)$, then (GE_u) represent first-order optimality conditions for the following parameterized optimization problem:

$$(P_u) \qquad \operatorname{Min}_{x\in X} f(x,u) \quad \text{subject to} \quad G(x,u) \in K. \tag{1.2}$$

Note also that if $Y = X$ and $G(x,u) := x$ is the identity mapping, then (GE_u) become the variational inequality

$$(VI_u) \qquad -F(x,u) \in N_K(x), \tag{1.3}$$

where $\bar{\lambda}(u) = -F(\bar{x}(u), u)$.

The continuity and differentiability properties of optimal solutions for parameterized optimization problems of the form (1.2) were studied extensively, and the corresponding theory is quite complete. On the other hand, parameterized generalized equations were investigated mainly in cases where the set K is *polyhedral* (see, e.g., [3, 4, 6] and references therein). For a nonpolyhedral set K, an additional term representing a possible curvature of K should appear in the corresponding formulas. For optimization problems, this is discussed in [2], where this additional term appears as the so-called sigma term. Unfortunately, it is not easy to extend the corresponding results from optimization problems to generalized equations (variational inequalities). This is because powerful tools of duality theory cannot be directly applied to generalized equations.

The approach to extending these results to generalized equations suggested in this paper consists of two steps. First, the problem is reduced to a case where the constraint mapping maps (x_0, u_0) into the

---

Translated from Itogi Nauki i Tekhniki, Seriya Sovremennaya Matematika i Ee Prilozheniya. Tematicheskie Obzory. Vol. 92, Optimization and Related Topics–1, 2001.

 1072–3374/03/1154–2554 $ 25.00

null vector and the corresponding constraint set in the image space is a convex cone. For the obtained reduced problem, local analysis is simpler and does not involve the sigma term. Afterwards, formulas derived for the reduced problem are translated back into a formulation for the original problem.

We use the following notation and terminology throughout the paper. Let $C \subset Y$ be a closed convex cone. The largest linear subspace of C is called its *lineality* space and denoted $\text{lin}(C)$. By $\text{span}(C)$, we denote the linear space generated (spanned) by C. We say that a convex cone C is pointed if $\text{lin}(C) = \{0\}$. The polar (negative dual) of the cone C is

$$C^- := \{y^* \in Y : \langle y^*, y \rangle \le 0 \; \forall\, y \in C\}.$$

In particular, if C is a linear subspace of Y, then $C^- = C^\perp$, where $C^\perp$ denotes the orthogonal complement of C in Y. Also, for $y \in Y$, we denote by $y^\perp$ its orthogonal complement, i.e., $y^\perp = \{z \in Y : \langle z, y \rangle = 0\}$. By $T_K(y)$ and $N_K(y)$, we denote the tangent and normal cones to the set K at the point y. Note that by definition $T_K(y)$ and $N_K(y)$ are empty if $y \notin K$. By $\text{int}(K)$ and $\text{ri}(K)$, we denote the interior and the relative interior of K, respectively. By $\text{dist}(y, K) := \inf_{z \in K} \|y - z\|$, we denote the distance from a point $y \in Y$ to a set K.

By A^T, we denote the transpose of matrix A. By $\mathcal{S}^p$, we denote the linear space of $p \times p$ symmetric matrices equipped with the scalar product $\langle A, B \rangle := \text{trace}(AB)$, and by $\mathcal{S}^p_+$ we denote the cone of $p \times p$ positive-semidefinite symmetric matrices. The notation $A \succeq B$ ($A \preceq B$) means that the matrix $A - B$ is positive-semidefinite (negative semidefinite).

2. Nondegeneracy and Strict Complementarity

In this section, we discuss a nondegeneracy concept for sets defined by abstract constraints of the form $G(x) \in K$. Throughout this section, we assume that the mapping $G : X \to Y$ is continuously differentiable and the set $K \subset Y$ is convex and closed. Recall that the feasible set of the optimization problem (P) is defined as $G^{-1}(K) = \{x \in X : G(x) \in K\}$.

Definition 2.1 ([2]). We say that a point $x_0 \in G^{-1}(K)$ is *nondegenerate*, with respect to the mapping G and the set K, if

$$DG(x_0)X + \text{lin}\,(T_K(y_0)) = Y, \tag{2.1}$$

where $y_0 := G(x_0)$.

Of course, if the tangent cone $T_K(y_0)$ is pointed, i.e., $\text{lin}\,(T_K(y_0)) = \{0\}$, then condition (2.1) means that the derivative mapping $DG(x_0) : X \to Y$ is onto. Note also that condition (2.1) is stronger than Robinson's constraint qualification, which can be written (since the spaces are finite dimensional) in the form

$$DG(x_0)X + T_K(y_0) = Y \tag{2.2}$$

(see, e.g., [2, Sec. 2.3.4]).

Example 2.1. Assume that $X := \mathbb{R}^m$, $Y := \mathbb{R}^n$, $K := \mathbb{R}^n_+$, and $G(x) = (g_1(x), \dots, g_n(x))$, i.e., the feasible set is defined by the constraints $g_i(x) \ge 0$, $i = 1, \dots, n$. For $y \in \mathbb{R}^n_+$, we denote $I(y) := \{i : y_i = 0,\; i = 1, \dots, n\}$. Then it is straightforward to verify that condition (2.1) is equivalent to the linear independence of the gradient vectors $\nabla g_i(x_0)$, $i \in I(y_0)$, of active at $y_0 := G(x_0)$ inequality constraints. On the other hand, Robinson's constraint qualification (2.2) is equivalent here to the Mangasarian–Fromovitz constraint qualification.

Example 2.2. Assume that $X := \mathbb{R}^m$, $Y := \mathcal{S}^p$, and $K := \mathcal{S}^p_+$, i.e., the feasible set $G^{-1}(K)$ is defined by the semidefinite constraints $G(x) \succeq 0$. Let $y_0 \in \mathcal{S}^p_+$ be a positive-semidefinite matrix of rank r. Denote by $\mathcal{W}_r$ the subset of $\mathcal{S}^p$ formed by matrices of rank r. In this case, the lineality space $\text{lin}\,(T_K(y_0))$ coincides with the tangent space to $\mathcal{W}_r$ at y_0, and hence Eq. (2.1) takes the form

$$DG(x_0)X + T_{\mathcal{W}_r}(y_0) = \mathcal{S}^p. \tag{2.3}$$

For semidefinite programming (eigenvalue optimization), the nondegeneracy condition in the form (2.3) was introduced in [8, Sec. 2] under the name "transversality condition." It is possible to formulate condition (2.3) in various equivalent algebraic forms (see [1,9]).

By taking the orthogonal complement of both sides of (2.1), we can write this equation in the following equivalent form:

$$[DG(x_0)X]^{\perp} \cap \left[\operatorname{lin}(T_K(y_0))\right]^{\perp} = \{0\}. \tag{2.4}$$

Moreover, we have

$$\left[\operatorname{lin}(T_K(y_0))\right]^{\perp} = \operatorname{span}(N_K(y_0)) \tag{2.5}$$

(see, e.g., [2, Proposition 4.73]). If K is a convex cone, then

$$N_K(y_0) = K^- \cap y_0^{\perp}$$

and hence $N_K(y_0)$ forms a face of the polar cone K^- orthogonal to y_0. Therefore, if K is a convex *cone*, then condition (2.1) can be written in the following equivalent form:

$$[DG(x_0)X]^{\perp} \cap \operatorname{span}\left[K^- \cap y_0^{\perp}\right] = \{0\}. \tag{2.6}$$

For a facially exposed convex cone K and an affine mapping $G(x)$, the definition of nondegeneracy in the form (2.6) was suggested in [5].

Now let x_0 be a solution of the generalized equations (GE), i.e., there exists $\lambda \in Y$ such that

$$F(x_0) + DG(x_0)^*\lambda = 0 \quad \text{and} \quad \lambda \in N_K(y_0). \tag{2.7}$$

The following definition of strict complementarity is taken from [2].

Definition 2.2. We say that the *strict complementarity* condition holds at x_0 if there exists $\lambda \in \operatorname{ri}(N_K(y_0))$ satisfying conditions (2.7).

By the above discussion we have that if K is a convex cone, then the strict complementarity condition means that the generalized equations (GE) have a solution (x_0, λ_0) with λ_0 lying in the relative interior of the face of K^- orthogonal to y_0.

Theorem 2.1. *Let x_0 be a solution of the generalized equations (GE). If x_0 is nondegenerate, then λ satisfying (2.7) is unique. Conversely, if λ satisfying (2.7) is unique and the strict complementarity condition holds, then x_0 is nondegenerate.*

Proof. The following proof is a slight modification of the proof of Proposition 4.75 in [2]. Suppose that x_0 is nondegenerate and let λ and λ' satisfy (2.7). Since $\lambda \in N_K(y_0)$ and $N_K(y_0)$ is the polar of the cone $T_K(y_0)$, we see that λ is orthogonal to $\operatorname{lin}(T_K(y_0))$. Similarly λ', and hence $\lambda - \lambda'$, are orthogonal to $\operatorname{lin}(T_K(y_0))$. It also follows from the first equation in (2.7) that $DG(x_0)^*(\lambda - \lambda') = 0$, which is equivalent to orthogonality of $\lambda - \lambda'$ to $DG(x_0)X$. Then, by (2.1), it follows that $\lambda - \lambda'$ is orthogonal to Y and hence $\lambda - \lambda' = 0$.

Conversely, suppose that λ satisfying (2.7) is unique and the strict complementarity condition holds. Further, assume that x_0 is not nondegenerate. We have then by (2.4) and (2.5) that there exists $\mu \in Y$ such that μ is orthogonal to $DG(x_0)X$, and hence $DG(x_0)^*\mu = 0$ and $\mu \in \operatorname{span}(N_K(y_0))$. Since, by strict complementarity, $\lambda \in \operatorname{ri}(N_K(y_0))$, it follows that for some $t > 0$ small enough, $\lambda + t\mu \in N_K(y_0)$, and hence $\lambda + t\mu$ also satisfies (2.7). This contradicts the uniqueness of λ. ☐

The above theorem shows that nondegeneracy is a sufficient and, under the strict complementarity, necessary condition for uniqueness of the multiplier λ.

3. Reduction Approach

The following definition of cone reducibility is taken from [2]. It suffices to assume in this section that the set K is closed, not necessarily convex.

Definition 3.1. We say that the set K is *cone reducible* at a point $y_0 \in K$, if there exist a neighborhood $\mathcal{V} \subset Y$ of y_0, a convex closed pointed cone Q in a finite-dimensional space Z, and a twice continuously differentiable mapping $\Xi : \mathcal{V} \to Z$ such that:

(i) $\Xi(y_0) = 0 \in Z$,
(ii) the derivative mapping $D\Xi(y_0) : Y \to Z$ is onto,
(iii) $K \cap \mathcal{V} = \{y \in \mathcal{V} : \Xi(y) \in Q\}$.

If K is cone reducible at every point $y_0 \in K$ (possibly to a different cone Q), we simply say that K is cone reducible.

The last condition (iii) of the above definition means that locally, in the vicinity of the point y_0, the set K can be defined by the constraints $\Xi(y) \in Q$. Condition (ii) ensures that the reduction mapping Ξ is nondegenerate at y_0.

The convex set K is cone reducible at every point $y_0 \in \mathrm{ri}(K)$. In this case, the mapping Ξ is given by the projection onto the linear space orthogonal to $\mathrm{ri}(K)$ and the shift of y_0 to the null vector, and $Q = \{0\}$. In applications, the set K, in fact, usually is a cone. Any convex cone K (in a finite-dimensional space) can be represented as a direct sum of its lineality space and a pointed cone. Therefore, any convex cone K is reducible to a pointed cone at the null point $y_0 = 0$. The corresponding mapping Ξ can be taken to be the orthogonal projection onto the linear space orthogonal to the lineality space of K. It turns out that many interesting sets (cones) are cone reducible at all their points. It is easy to see that any polyhedral convex set is cone reducible. The ice-cream cone $\left\{x \in \mathbb{R}^n : x_n \geq \sqrt{x_1^2 + \cdots + x_{n-1}^2}\right\}$ is cone reducible. Indeed, at every (nonzero) boundary point, the tangent cone to the ice-cream cone is a half-space and hence it can be reduced at such a point to the half-line cone $Q := \mathbb{R}_+$.

A nontrivial example of a cone reducible set is the cone $\mathcal{S}_+^p$ of positive-semidefinite matrices. If $y_0 \in \mathcal{S}_+^p$ is a positive-semidefinite matrix of rank r, then it is possible to construct an infinitely differentiable (even analytic) reduction mapping Ξ from a neighborhood of y_0 into the linear space $Z := \mathcal{S}^{p-r}$ with the corresponding cone $Q := \mathcal{S}_+^{p-r}$ (for such a construction, see [2, Example 3.140]).

Proposition 3.1. *Let $Y_1, \ldots, Y_n$ be finite-dimensional vector spaces, $K_i \subset Y_i$ be closed sets, and $y_i \in K_i$, $i = 1, \ldots, n$. Suppose that K_i is reducible at y_i to a cone Q_i, $i = 1, \ldots, n$. Then the set $K := K_1 \times \cdots \times K_n$ is cone reducible at the point $y := (y_1, \ldots, y_n) \in Y$, $Y := Y_1 \times \cdots \times Y_n$, to the cone $Q := Q_1 \times \cdots \times Q_n$.*

Proof. Let $\Xi_i : \mathcal{V}_i \to Z_i$, $i = 1, \ldots, n$, be the corresponding reduction mappings. Then take $\mathcal{V} := \mathcal{V}_1 \times \cdots \times \mathcal{V}_n$ and $\Xi := \Xi_1 \times \cdots \times \Xi_n$. □

Remark 3.1. It may happen that some points y_i, in the above proposition, belong to the interior of the corresponding set K_i. In this case, the corresponding reduced space Z_i has zero dimension (i.e., $Z_i = \{0\}$) and hence can be removed from the direct product.

Proposition 3.2. *Let $S := G^{-1}(K)$, where $G : X \to Y$ is a continuously differentiable mapping, and $x_0 \in S$. Suppose that the convex set K is cone reducible at the point $y_0 := G(x_0)$ and that the point x_0 is nondegenerate with respect to G and K. Then the set S is cone reducible at x_0.*

Proof. Let $\Xi(y)$ be the reduction mapping of the set K to the cone $Q \subset Z$. Since $D\Xi(y_0)$ is onto, we have that

$$T_K(y_0) = \{h \in Y : D\Xi(y_0)h \in Q\}, \tag{3.1}$$

$$\mathrm{lin}\,(T_K(y_0)) = \{h \in Y : D\Xi(y_0)h = 0\}. \tag{3.2}$$

Consider the composite mapping $\mathcal{G}(x) := \Xi(G(x))$. We see that $\mathcal{G}(x_0) = \Xi(y_0) = 0$ and clearly the set S is defined in a neighborhood of x_0 by the constraint $\mathcal{G}(x) \in Q$. Moreover, by the chain rule of differentiation and since $D\Xi(y_0)$ is onto, it follows from (3.2) that $D\mathcal{G}(x_0)X = Z$ iff Eq. (2.1) holds, i.e., iff the point x_0 is nondegenerate. We obtain that if x_0 is nondegenerate, then the set S is cone reducible at x_0 by the mapping $\mathcal{G}$ to the cone Q. □

Now let $K_i \subset Y$, $i = 1, \dots, n$, be closed sets, $K := \bigcap_{i=1}^{n} K_i$, and $y_0 \in K$. Suppose that every set K_i is reducible at y_0 to a cone $Q_i \subset Z_i$ by a mapping Ξ_i. Consider the space $Z := Z_1 \times \cdots \times Z_n$, the cone $Q := Q_1 \times \cdots \times Q_n \subset Z$, and the mapping $\Xi(y) := (\Xi_1(y), \dots, \Xi_n(y))$. Clearly, the set K is defined in a neighborhood of y_0 by the constraint $\Xi(y) \in Q$. Therefore, K is reducible at y_0 to the cone Q by the mapping Ξ if $D\Xi(y_0) : Y \to Z$ is onto. We see that $D\Xi(y_0)h = (A_1 h, \dots, A_n h)$, where $A_i := D\Xi_i(y_0)$, $i = 1, \dots, n$, are the derivative mappings. If we consider the linear mappings A_i, $i = 1, \dots, n$, as the corresponding matrices, then the condition $D\Xi(y_0)Y = Z$ is equivalent to the condition that the matrix $A := [A_1^T, \dots, A_n^T]^T$ has full row rank.

Suppose that the set K is cone reducible at the point $y_0 := G(x_0)$. Then, since $D\Xi(y_0)$ is onto, the set

$$W := \{y \in \mathcal{V} : \Xi(y) = 0\} \tag{3.3}$$

forms a smooth manifold in a neighborhood of y_0 and

$$\operatorname{lin}(T_K(y_0)) = T_W(y_0). \tag{3.4}$$

The nondegeneracy of the point x_0 with respect to $G(x)$ and K is equivalent to the nondegeneracy of x_0 with respect to the composite mapping $\mathcal{G}(x) := \Xi(G(x))$ and the cone Q. Recall that the cone Q is assumed to be pointed. Therefore, the point x_0 is nondegenerate iff $D\mathcal{G}(x_0) : X \to Z$ is onto.

For the cone $K := \mathcal{S}_+^p$ of positive-semidefinite matrices, the set W defined in (3.3) coincides (in a neighborhood of y_0) with the set $\mathcal{W}_r$ of $p \times p$ symmetric matrices of rank $r = \operatorname{rank}(y_0)$. This implies that for $K := \mathcal{S}_+^p$, Eqs. (2.1) and (2.3) are equivalent.

4. Sensitivity Analysis of Reduced Generalized Equations

In the sequel, we assume that the convex set K is cone reducible at the point $y_0 := G(x_0)$ to a pointed convex cone $Q \subset Z$ by a mapping Ξ. Consider the mapping $\mathcal{G}(x, u) := \Xi(G(x, u))$. Then for all (x, u) in a neighborhood of (x_0, u_0), the generalized equations (GE_u) can be written in the following equivalent form:

$$(\mathcal{G}E_u) \qquad F(x,u) + D_x\mathcal{G}(x,u)^*\mu = 0 \ \text{ and } \ \mu \in N_Q(\mathcal{G}(x,u)), \tag{4.1}$$

i.e., locally $(\bar{x}(u), \bar{\lambda}(u))$ is a solution of (GE_u) iff $(\bar{x}(u), \bar{\mu}(u))$ is a solution of $(\mathcal{G}E_u)$ and

$$\bar{\lambda}(u) = [D\Xi(G(x,u))]^* \bar{\mu}(u). \tag{4.2}$$

Note that by Definition 2.1(ii), we have that for all y in a neighborhood of y_0, the mapping $D\Xi(y)$ is onto, and hence the mapping $[D\Xi(y)]^*$ is one-to-one. Therefore, for all (x, u) sufficiently close to (x_0, u_0), the multiplier $\bar{\mu}(u)$ is defined uniquely by Eq. (4.2).

In particular, for the unperturbed problem, we have that (x_0, μ_0), where $\lambda_0 = [D\Xi(y_0)]^*\mu_0$, is a solution of the generalized equations

$$F(x) + D\mathcal{G}(x)^*\mu = 0 \quad \text{and} \quad \mu \in N_Q(\mathcal{G}(x)). \tag{4.3}$$

It is important to note that for the above reduced equations $\mathcal{G}(x_0) = \Xi(y_0)$, and hence $\mathcal{G}(x_0) = 0 \in Z$. Therefore, the strict complementarity condition holds at x_0 iff there exists $\mu \in Z$ such that (x_0, μ) is a solution of (4.3) and $\mu \in \operatorname{int}(Q^-)$.

Note also that the following conditions are equivalent:

(i) $\mu \in N_Q(\mathcal{G}(x, u))$,

(ii) $\mathcal{G}(x,u) \in N_{Q^-}(\mu)$,
(iii) $\mu \in Q^-$, $\mathcal{G}(x,u) \in Q$ and $\langle \mu, \mathcal{G}(x,u)\rangle = 0$.

Consider the cones

$$\mathcal{K} := \{y \in T_K(y_0) : \langle \lambda_0, y\rangle = 0\}, \tag{4.4}$$

$$\mathcal{Q} := \{z \in Q : \langle \mu_0, z\rangle = 0\}. \tag{4.5}$$

Note that if $\mu_0 \in \operatorname{int}(Q^-)$, i.e., the strict complementarity condition holds at x_0, then

$$\mathcal{K} = \operatorname{lin}(T_K(y_0)) \quad \text{and} \quad \mathcal{Q} = \{0\}.$$

Let us consider a direction $d \in U$ and the following linearization of the generalized equations $(\mathcal{G}E_u)$ at the point (x_0, μ_0) (cf. [2, (5.29), p. 409]):

$$D\left[F + (D_x\mathcal{G})^*\mu_0\right](x_0,u_0)(h,d) + D_x\mathcal{G}(x_0,u_0)^*\eta = 0, \tag{4.6}$$

$$D\mathcal{G}(x_0,u_0)(h,d) \in N_{\mathcal{Q}^-}(\eta). \tag{4.7}$$

In the above expansions,

$$D\left[F + (D_x\mathcal{G})^*\mu_0\right](x_0,u_0)(h,d) = DF(x_0,u_0)(h,d) + \left(D^2_{xx}\mathcal{G}(x_0,u_0)^*\mu_0\right)h + \left(D^2_{xu}\mathcal{G}(x_0,u_0)^*\mu_0\right)d,$$
$$DF(x_0,u_0)(h,d) = D_xF(x_0,u_0)h + D_uF(x_0,u_0)d,$$
$$D\mathcal{G}(x_0,u_0)(h,d) = D_x\mathcal{G}(x_0,u_0)h + D_u\mathcal{G}(x_0,u_0)d.$$

In particular, if $F(x,u) = D_xf(x,u)$, then

$$D\left[F + (D_x\mathcal{G})^*\mu_0\right](x_0,u_0)(h,d) = D^2_{xx}\mathcal{L}(x_0,\mu_0,u_0)h + D^2_{xu}\mathcal{L}(x_0,\mu_0,u_0)d,$$

where $\mathcal{L}(x,\mu,u) := f(x,u) + \langle \mu, \mathcal{G}(x,u)\rangle$ is the Lagrangian of the corresponding optimization problem.

Note that condition (4.7) is equivalent to

$$\eta \in \mathcal{Q}^-, \quad D\mathcal{G}(x_0,u_0)(h,d) \in \mathcal{Q}, \quad \text{and} \quad \langle \eta, D\mathcal{G}(x_0,u_0)(h,d)\rangle = 0, \tag{4.8}$$

and hence to $\eta \in N_{\mathcal{Q}}(D\mathcal{G}(x_0,u_0)(h,d))$. Note also that $\mathcal{Q}^-$ coincides with the topological closure of the sum of Q^- and the linear space generated by μ_0, and hence $\mathcal{Q}^- = T_{Q^-}(\mu_0)$.

Lemma 4.1 ([2, Theorem 5.10, p. 410])**.** *Consider a path $u(t) \in U$, $t \geq 0$, of the form $u(t) = u_0 + td + o(t)$. Let (x_0,μ_0) and $(\bar{x}(t), \bar{\mu}(t))$ be solutions of the generalized equations $(\mathcal{G}E)$ and $(\mathcal{G}E_{u(t)})$, respectively. Then any accumulation point of*

$$\theta(t) := t^{-1}(\bar{x}(t) - x_0, \bar{\mu}(t) - \mu_0), \quad t > 0,$$

is a solution of the generalized linearized equations (4.6)–(4.7)*.*

Proof. Let $(\bar{h}, \bar{\eta})$ be an accumulation point of $\theta(t)$, i.e., there exists a sequence $t_n \downarrow 0$ such that

$$(x_n - x_0, \mu_n - \mu_0) = t_n(\bar{h}, \bar{\eta}) + o(t_n),$$

where $x_n := \bar{x}(t_n)$ and $\mu_n := \bar{\mu}(t_n)$. We also see that $u_n - u_0 = t_nd + o(t_n)$, where $u_n := u(t_n)$. Therefore, by using the first-order expansion of the equation

$$F(x_n,u_n) + D_x\mathcal{G}(x_n,u_n)^*\mu_n = 0$$

at the point (x_0,u_0) and μ_0, we obtain that

$$D\left[F + (D_x\mathcal{G})^*\mu_0\right](x_0,u_0)(x_n - x_0, t_nd) + D_x\mathcal{G}(x_0,u_0)^*(\mu_n - \mu_0) = o(t_n);$$

hence, by passing to the limit, we see that $(\bar{h}, \bar{\eta})$ satisfies Eq. (4.6).

Now by the second condition of (4.1) we have

$$\mu_n \in Q^-, \quad \mathcal{G}(x_n,u_n) \in Q \quad \text{and} \quad \langle \mu_n, \mathcal{G}(x_n,u_n)\rangle = 0. \tag{4.9}$$

Consider a vector $z \in \mathcal{Q}$. Then, since $z \in Q$ and $\langle \mu_0, z\rangle = 0$, we have

$$0 \geq \langle \mu_n, z\rangle = \langle \mu_n - \mu_0, z\rangle = t_n\langle \bar{\eta}, z\rangle + o(t_n);$$

hence, by passing to the limit, we obtain that $\langle \bar\eta, z\rangle \le 0$. It follows that $\bar\eta \in \mathcal{Q}^-$.

Since $\mathcal{G}(x_0,u_0)=0$, by using the second and third conditions of (4.9), we obtain that $t_n^{-1}(\mathcal{G}(x_n,u_n)-\mathcal{G}(x_0,u_0)) \in Q$ and $\langle \mu_n, \mathcal{G}(x_n,u_n)-\mathcal{G}(x_0,u_0)\rangle = 0$, respectively. Since Q is closed and $\mu_n \to \mu_0$, by passing to the limit, we see that $D\mathcal{G}(x_0,u_0)(\bar h,\bar\eta) \in Q$ and $\langle \mu_0, D\mathcal{G}(x_0,u_0)(\bar h,\bar\eta)\rangle = 0$, and hence $D\mathcal{G}(x_0,u_0)(\bar h,\bar\eta) \in \mathcal{Q}$.

Finally, since

$$\langle \mu_0, \mathcal{G}(x_n,u_n)-\mathcal{G}(x_0,u_0)\rangle \le 0, \quad \langle \mu_n, \mathcal{G}(x_n,u_n)-\mathcal{G}(x_0,u_0)\rangle = 0,$$

it follows that

$$\langle \bar\eta, D\mathcal{G}(x_0,u_0)(\bar h,\bar\eta)\rangle \ge 0.$$

Moreover, since

$$\bar\eta \in \mathcal{Q}^- \quad \text{and} \quad D\mathcal{G}(x_0,u_0)(\bar h,\bar\eta) \in \mathcal{Q},$$

we have

$$\langle \bar\eta, D\mathcal{G}(x_0,u_0)(\bar h,\bar\eta)\rangle \le 0,$$

and hence

$$\langle \bar\eta, D\mathcal{G}(x_0,u_0)(\bar h,\bar\eta)\rangle = 0.$$

This completes the proof of conditions (4.8), and hence of (4.7). ☐

A solution $(\bar x(t),\bar\mu(t))$ of the generalized equations $(\mathcal{G}E_{u(t)})$ is said to be upper Lipschitzian at (x_0,μ_0) if

$$\|(\bar x(t)-x_0, \bar\mu(t)-\mu_0)\| = O(t), \quad t>0. \tag{4.10}$$

We have that if for all $t>0$ small enough, the generalized equations $(\mathcal{G}E_{u(t)})$ have a solution $(\bar x(t),\bar\mu(t))$ and this solution is upper Lipschitzian at (x_0,μ_0), then $\theta(t)$ is bounded and hence has an accumulation point. It follows from the above lemma that if, in addition, the linearized generalized equations (4.6)–(4.7) have a *unique* solution, denoted $(\bar h,\bar\eta)$, then

$$(\bar x(t)-x_0, \bar\mu(t)-\mu_0) = t(\bar h,\bar\eta)+o(t), \quad t>0, \tag{4.11}$$

i.e., $(\bar h,\bar\eta)$ is the directional derivative of $(\bar x(t),\bar\mu(t))$ along the path $u(t)$. We discuss next conditions ensuring upper Lipschitzian behavior of solutions of the generalized equations $(\mathcal{G}E_u)$.

In particular, for $d=0$, the linearized generalized equations (4.6)–(4.7) take the form

$$D[F+(D\mathcal{G})^*\mu_0](x_0)h + D\mathcal{G}(x_0)^*\eta = 0, \tag{4.12}$$

$$D\mathcal{G}(x_0)h \in N_{\mathcal{Q}^-}(\eta). \tag{4.13}$$

Note that $(\bar h,\bar\eta)=(0,0)$ is always a solution of (4.12)–(4.13). Moreover, if $(0,0)$ is the unique solution of (4.12)–(4.13), then (x_0,μ_0) is a locally unique solution of $(\mathcal{G}E)$. Indeed, if there exists a sequence (x_n,μ_n) of solutions of $(\mathcal{G}E)$ converging to (x_0,μ_0) and such that $(x_n,\mu_n)\ne(x_0,\mu_0)$, then by taking $u(t)\equiv u_0$ in Lemma 4.1, we obtain that any accumulation point of the sequence $(x_n-x_0,\mu_n-\mu_0)/(\|x_n-x_0\|+\|\mu_n-\mu_0\|)$ is a solution of the linearized generalized equations (4.12)–(4.13). Since this sequence is bounded, it has an accumulation point and, moreover, such an accumulation point is different from $(0,0)$.

If $(0,0)$ is the unique solution of (4.12)–(4.13), then, in particular, $\bar\eta=0$ is the unique solution of the equations

$$D\mathcal{G}(x_0)^*\eta = 0 \quad \text{and} \quad 0 \in N_{\mathcal{Q}^-}(\eta). \tag{4.14}$$

Since

$$\{\eta : D\mathcal{G}(x_0)^*\eta = 0\} = \{\eta : \langle \eta, D\mathcal{G}(x_0)h\rangle = 0 \ \forall\, h\in X\} = [D\mathcal{G}(x_0)X]^\perp$$

and $0\in N_{\mathcal{Q}^-}(\eta)$ means that $\eta\in\mathcal{Q}^-$, we obtain that (4.14) is equivalent to

$$[D\mathcal{G}(x_0)X]^\perp \cap \mathcal{Q}^- = \{0\}, \tag{4.15}$$

which, in turn, is equivalent (since the space Z is finite dimensional) to the condition

$$D\mathcal{G}(x_0)X + \mathcal{Q} = Z \tag{4.16}$$

(see, e.g., [2, Proposition 2.97]).

Lemma 4.2. *Let (x_0, μ_0) be a solution of the generalized equations $(\mathcal{G}E)$. Suppose that the linearized generalized equations (4.12)–(4.13)) have a unique solution $(0, 0)$. Then there exist $\kappa > 0$ and neighborhoods $\mathcal{N}_{x_0}$ and $\mathcal{N}_{u_0}$ of x_0 and u_0, respectively, such that if $u \in \mathcal{N}_{u_0}$, $(\bar{x}(u), \bar{\mu}(u))$ is a solution of $(\mathcal{G}E_u)$, and $\bar{x}(u) \in \mathcal{N}_{x_0}$, then*

$$\|\bar{x}(u) - x_0\| + \|\bar{\mu}(u) - \mu_0\| \leq \kappa \|u - u_0\|. \tag{4.17}$$

Proof. Let $u_n \in U$ be a sequence converging to u_0 and (x_n, μ_n) be a solution of $(\mathcal{G}E_{u_n})$ such that $x_n \to x_0$. It suffices to show that then

$$\|x_n - x_0\| + \|\mu_n - \mu_0\| = O(\|u_n - u_0\|). \tag{4.18}$$

As was mentioned earlier, the assumption that equations (4.12)–(4.13) have a unique solution implies condition (4.16). Condition (4.16), in turn, implies that

$$\|\mu_n - \mu_0\| = O(\|x_n - x_0\|)$$

(see [2, Proposition 4.47]). Therefore, it follows that $\mu_n \to \mu_0$.

Assume that (4.18) is false. This means that $\|u_n - u_0\| = o(t_n)$, where $t_n := \|x_n - x_0\| + \|\mu_n - \mu_0\|$. By using the result of Lemma 4.1 with $d = 0$, we obtain then that any accumulation point of the sequence $t_n^{-1}(x_n - x_0, \mu_n - \mu_0)$ is a solution of Eqs. (4.12)–(4.13), and hence is $(0, 0)$. However, by standard compactness arguments, this sequence has a nonzero accumulation point; a contradiction. □

The above lemma shows that if Eqs. (4.12)–(4.13) have a unique solution, then the set of solutions of the generalized equations $(\mathcal{G}E_u)$ is locally upper Lipschitzian at u_0 near (x_0, μ_0). The converse of that is also true if the parameterization $(\mathcal{G}E_u)$ is rich enough. Let us consider the parameterization

$$F(x, u) := F(x) + u_1,\ \mathcal{G}(x, u) = \mathcal{G}(x) + u_2, \quad u = (u_1, u_2) \in X \times Z, \tag{4.19}$$

of $(\mathcal{G}E)$ at $u_0 = (0, 0)$.

Lemma 4.3. *Let (x_0, μ_0) be a solution of the generalized equations $(\mathcal{G}E)$ and $(\bar{h}, \bar{\eta})$ be a solution of the generalized equations (4.12)–(4.13) such that*

$$\mu_0 + \bar{t}\bar{\eta} \in Q^- \quad \textit{for some} \quad \bar{t} > 0. \tag{4.20}$$

Consider parameterization (4.19). Then there exists a path $u(t) \in U := X \times Z$, $t \geq 0$, such that for all $t > 0$ in a neighborhood of zero,

$$(\bar{x}(t), \bar{\mu}(t)) := (x_0 + t\bar{h}, \mu_0 + t\bar{\eta})$$

is a solution of the generalized equations $(\mathcal{G}E_{u(t)})$ and $\|u(t)\| = o(t)$.

Proof. Define

$$\begin{aligned} u_1(t) &:= F(x_0) + D\mathcal{G}(x_0)^*\mu_0 - [F(x_0 + t\bar{h}) + D\mathcal{G}(x_0 + t\bar{h})^*(\mu_0 + t\bar{\eta})], \\ u_2(t) &:= D\mathcal{G}(x_0)(t\bar{h}) - \mathcal{G}(x_0 + t\bar{h}). \end{aligned}$$

By using the Taylor expansion and since $\mathcal{G}(x_0) = 0$ and by (4.12), we obtain that $\|u_1(t)\| = o(t)$ and $\|u_2(t)\| = o(t)$. Also we have

$$F(\bar{x}(t)) + u_1(t) + D\mathcal{G}(\bar{x}(t))^*\bar{\mu}(t) = F(x_0) + D\mathcal{G}(x_0)^*\mu_0 = 0,$$
$$D[\mathcal{G}(\bar{x}(t)) + u_2(t)] = D\mathcal{G}(\bar{x}(t)).$$

Moreover, we have

$$\mathcal{G}(\bar{x}(t)) + u_2(t) = D\mathcal{G}(x_0)(t\bar{h}) \in \mathcal{Q} \subset Q,$$
$$\langle \bar{\mu}(t), D\mathcal{G}(x_0)(t\bar{h})\rangle = 0.$$

By assumption (4.20) we see that $\mu_0 + t\bar{\eta} \in Q^-$ for all $t > 0$ small enough. Therefore, we obtain that $\bar{\mu}(t) \in N_Q(\mathcal{G}(\bar{x}(t)) + u_2(t))$ for $t > 0$ small enough. Consequently, $(\bar{x}(t), \bar{\mu}(t))$ is a solution of $(\mathcal{G}E_{u(t)})$ for all $t > 0$ small enough. □

Note that since $\bar{\eta} \in \mathcal{Q}^- = T_{Q^-}(\mu_0)$, it follows that

$$\operatorname{dist}(\mu_0 + t\bar{\eta}, Q^-) = o(t).$$

Assumption (4.20) is stronger of course. If the cone Q is polyhedral, then its polar cone Q^- is also polyhedral. In this case, Q^- coincides with $\mu_0 + T_{Q^-}(\mu_0)$ in a neighborhood of μ_0, and hence assumption (4.20) follows. In general, for a nonpolyhedral cone Q, there is a gap between the sufficient and necessary conditions given in Lemmas 4.2 and 4.3, respectively. This requires a further investigation.

If $\mu_0 \in \operatorname{int}(Q^-)$, i.e., the strict complementarity condition holds, then $\mathcal{Q} = \{0\}$ and hence $\mathcal{Q}^- = Z$. In this case, conditions (4.7) and (4.13) become the equations $D\mathcal{G}(x_0, u_0)(h, d) = 0$ and $D\mathcal{G}(x_0)h = 0$, respectively. In this case, we also see that for all μ sufficiently close to μ_0 such that $\mu \in \operatorname{int}(Q^-)$, the second (inclusion) condition in (4.1) can be replaced by the equation $\mathcal{G}(x, u) = 0$, and hence locally the generalized equations $(\mathcal{G}E_u)$ can be written as the equations

$$F(x, u) + D_x\mathcal{G}(x, u)^*\mu = 0, \quad \mathcal{G}(x, u) = 0, \tag{4.21}$$

i.e., if the strict complementarity condition holds, then (4.6)–(4.7) become linear equations representing linearization of Eqs. (4.21). It follows then by the implicit function theorem that if for $d = 0$ the corresponding linearized equations (4.12)–(4.13) have the unique zero solution, then for all u in a neighborhood of u_0, Eqs. (4.21) have a unique solution $(\bar{x}(u), \bar{\mu}(u))$ in a neighborhood of (x_0, μ_0), which is Lipschitz continuous and continuously differentiable. In other words, under the strict complementarity condition, the local behavior of generalized equations (4.1) is completely described by the classical implicit function theorem. As the following remark shows, in the absence of the strict complementarity condition the situation is more delicate.

Remark 4.1. The assumption that the linearized generalized equations (4.12)–(4.13) have a unique solution $(0, 0)$ does not imply that Eqs. (4.6)–(4.7) have a solution for any d. For example, consider equations

$$0 \in Ah + d + N_Q(h),$$

where $Q := \mathbb{R}^2_+$ and $Ah := (-h_1 - h_2, -h_1 - h_2)$. These equations represent the complementarity problem

$$\langle Ah + d, h\rangle = 0, \quad Ah + d \geq 0, \quad h \geq 0,$$

and for $d = 0$ have a unique solution $\bar{h} = 0$. On the other hand, for $d = (-1, -1)$, these equations do not have a solution.

5. Sensitivity Analysis of Original Generalized Equations

In this section, we discuss how the results obtained in the previous section for the reduced generalized equations $(\mathcal{G}E_u)$ can be formulated in terms of the original generalized equations (GE_u). For a set $S \subset Y$, we denote by

$$\sigma(\lambda, S) := \sup_{y \in S} \langle \lambda, y\rangle$$

its support function. For a point $y \in K$, the second-order tangent set to K at y in a direction $d \in Y$ is defined by

$$T^2_K(y, d) := \left\{ w \in Y : \operatorname{dist}\left(y + td + \frac{1}{2}t^2 w, K\right) = o(t^2),\ t \geq 0 \right\}.$$

Let us show how the linearized generalized equations (4.6)–(4.7) can be formulated in terms of linearization of the original generalized equations (GE_u). By the chain rule of differentiation, we have

$$D\mathcal{G}(x,u) = D\Xi(\mathcal{G}(x,u))DG(x,u),$$

and for y near y_0,

$$N_{\mathcal{K}}(y) = [D\Xi(y)]^* N_{\mathcal{Q}}(\Xi(y)). \tag{5.1}$$

Therefore, condition (4.7) is equivalent to

$$DG(x_0,u_0)(h,d) \in N_{\mathcal{K}^-}(\zeta), \tag{5.2}$$

where $\zeta = [D\Xi(y_0)]^*\eta$. Now

$$D[(D_x\mathcal{G})^*\mu_0](x_0,u_0)(h,d) = D[(D_xG)^*\lambda_0](x_0,u_0)(h,d) + \Delta(h,d), \tag{5.3}$$

where

$$\Delta(h,d) := (D_xG^*)[D^2\Xi(y_0)^*\mu_0](D_xG)h + (D_xG^*)[D^2\Xi(y_0)^*\mu_0](D_uG)d$$

and all derivatives of $G(x,u)$ are taken at (x_0,u_0).

The additional term $\Delta(h,d)$ has the following interpretation. Consider the function

$$\delta(h,d) := \langle \mu_0, D^2\Xi(y_0)(DG(x_0,u_0)(h,d), DG(x_0,u_0)(h,d))\rangle,$$

which is a quadratic function of (h,d). Then

$$\Delta(h,d) = \frac{1}{2}D_h\delta(h,d).$$

By the chain rule for second-order tangent sets (see, e.g., [2, Proposition 3.33]), we obtain

$$T_K^2(y_0,d) = D\Xi(y_0)^{-1}\left[T_Q^2(\Xi(y_0), D\Xi(y_0)d) - D^2\Xi(y_0)(d,d)\right].$$

Moreover, since $\Xi(y_0) = 0$ and Q is a convex cone, we see that if $D\Xi(y_0)d \in Q$, then

$$T_Q^2(\Xi(y_0), D\Xi(y_0)d) = \mathrm{cl}\left\{Q + \mathrm{span}(D\Xi(y_0)d)\right\}.$$

Consequently, if (h,d) satisfies (5.2), then

$$\delta(h,d) = -\sigma\left(\lambda_0, T_K^2(y_0, DG(x_0,u_0)(h,d)\right).$$

The function in the right-hand side of the above equation is called the sigma term and appears in sensitivity analysis of optimization problems.

Therefore, for a given d, the linearized generalized equations (4.6)–(4.7) can be written in the form

$$D[F + (D_xG)^*\lambda_0](x_0,u_0)(h,d) + D_xG(x_0,u_0)^*\zeta + \Delta(h,d) = 0. \tag{5.4}$$

$$DG(x_0,u_0)(h,d) \in N_{\mathcal{K}^-}(\zeta), \tag{5.5}$$

where the cone $\mathcal{K}$ is defined in (4.4) and

$$\Delta(h,d) = -\frac{1}{2}D_h\sigma\left(\lambda_0, T_K^2(y_0, DG(x_0,u_0)(h,d)\right). \tag{5.6}$$

Note that, since it is assumed that K is cone reducible at y_0, we have by the above discussion that the sigma term in the right hand side of (5.6) is a quadratic function of (h,d), and hence $\Delta(h,d)$ is a linear mapping.

By Lemmas 4.2 and 4.1, we have the following results.

Theorem 5.1. *Let (x_0,λ_0) be a solution of the generalized equations (GE). Suppose that the set K is cone reducible at the point $y_0 := G(x_0,u_0)$. Then the following holds.*

(i) *Assume that for* $d = 0$*, the linearized generalized equations* (5.4)–(5.5) *have a unique solution* $(0, 0)$*. Then there exist* $\kappa > 0$ *and neighborhoods* $\mathcal{N}_{x_0}$ *and* $\mathcal{N}_{u_0}$ *of* x_0 *and* u_0*, respectively, such that if* $u \in \mathcal{N}_{u_0}$*,* $(\bar{x}(u), \bar{\lambda}(u))$ *is a solution of* (GE_u)*, and* $\bar{x}(u) \in \mathcal{N}_{x_0}$*, then*

$$\|\bar{x}(u) - x_0\| + \|\bar{\lambda}(u) - \lambda_0\| \leq \kappa \|u - u_0\|. \tag{5.7}$$

(ii) *For some* $d \in U$*, consider a path* $u(t) \in U$*,* $t \geq 0$*, of the form* $u(t) := u_0 + td + o(t)$*. Let* $(\bar{x}(t), \bar{\lambda}(t))$ *be a solution of the generalized equations* $(GE_{u(t)})$*. Then any accumulation point* $(\bar{h}, \bar{\zeta})$ *of*

$$t^{-1}(\bar{x}(t) - x_0, \bar{\lambda}(t) - \lambda_0), \quad t > 0,$$

is a solution of the generalized linearized equations (5.4)–(5.5).

It is also possible to reformulate the result of Lemma 4.3 in terms of the original generalized equations.

In particular, for the variational inequality (1.3), the linearization (5.4)–(5.5) at a solution point x_0 takes the form

$$-DF(x_0, u_0)(h, d) + \Delta(h) \in N_{\mathcal{K}^-}(h), \tag{5.8}$$

where

$$\mathcal{K} = \{x \in T_K(x_0) : \langle F(x_0, u_0), x \rangle = 0\}, \tag{5.9}$$

$$\Delta(h) = -\frac{1}{2} D_h \sigma\left(-F(x_0, u_0), T_K^2(x_0, h)\right). \tag{5.10}$$

If $\lambda_0 \in \mathrm{ri}(N_K(y_0))$, i.e., the strict complementarity condition holds, then $\mathcal{K} = \mathrm{lin}(T_K(y_0))$, and hence for any $\zeta \in \mathcal{K}^-$, we have

$$\mathcal{K}^- = N_{\mathcal{K}^-}(\zeta) = [\mathrm{lin}(T_K(y_0))]^\perp = \mathrm{span}\left(N_K(y_0)\right).$$

Therefore, in this case, condition (5.5) becomes

$$DG(x_0, u_0)(h, d) \in \mathrm{span}\left(N_K(y_0)\right) \quad \text{and} \quad \zeta \in \mathrm{span}\left(N_K(y_0)\right). \tag{5.11}$$

In other words, under the strict complementarity condition, the linearized generalized equations (5.4)–(5.5) can be written as a system of linear equations. As was discussed in the previous section, in this case the local behavior of the generalized equations (GE_u) is completely described by the implicit function theorem.

Finally, let us note that in the case of the semidefinite constraints of Example 2.2, the sigma term can be written explicitly as

$$h^T \mathcal{H}_{xx}(x_0, \Lambda_0, u_0)h + 2h^T \mathcal{H}_{xu}(x_0, \Lambda_0, u_0)d + d^T \mathcal{H}_{uu}(x_0, \Lambda_0)d,$$

where ij-elements of the above matrices are

$$\mathcal{H}_{xx}(x_0, \Lambda_0, u_0)_{ij} = -2\,\mathrm{tr}\left\{\Lambda_0 \left(\frac{\partial G(x_0, u_0)}{\partial x_i}\right) [G(x_0, u_0)]^\dagger \left(\frac{\partial G(x_0, u_0)}{\partial x_j}\right)\right\},$$
$$\mathcal{H}_{xu}(x_0, \Lambda_0, u_0)_{ij} = -2\,\mathrm{tr}\left\{\Lambda_0 \left(\frac{\partial G(x_0, u_0)}{\partial x_i}\right) [G(x_0, u_0)]^\dagger \left(\frac{\partial G(x_0, u_0)}{\partial u_j}\right)\right\},$$
$$\mathcal{H}_{uu}(x_0, \Lambda_0, u_0)_{ij} = -2\,\mathrm{tr}\left\{\Lambda_0 \left(\frac{\partial G(x_0, u_0)}{\partial u_i}\right) [G(x_0, u_0)]^\dagger \left(\frac{\partial G(x_0, u_0)}{\partial u_j}\right)\right\},$$

Λ_0 is the multiplier matrix of the corresponding generalized equations, and $A^\dagger$ denotes the Moore–Penrose pseudoinverse of a matrix A [9].

Acknowledgment. This work was supported by the National Science Foundation, grant DMS-0073770.

REFERENCES

1. F. Alizadeh, J.-P. A. Haeberly, and M. L. Overton, "Complementarity and nondegeneracy in semidefinite programming," *Math. Program., Ser. B,* **77**, 111–128 (1997).
2. J. F. Bonnans and A. Shapiro, *Perturbation Analysis of Optimization Problems*, Springer-Verlag, New York (2000).
3. A. L. Dontchev and R. T. Rockafellar, "Characterizations of Lipschitzian stability in nonlinear programming," In: *Mathematical Programming with Data Perturbations*, Lect. Notes Pure Appl. Math., Vol. 195, Dekker, New York (1998), pp. 65–82.
4. Z. Q. Luo, J. S. Pang, and D. Ralph, *Mathematical Programs with Equilibrium Constraints*, Cambridge University Press, Cambridge (1996).
5. G. Pataki, " The geometry of semidefinite programming," In: *Semidefinite Programming and Applications Handbook* (R. Saigal, L. Vandenberghe, and H. Wolkowicz, Eds.), Kluwer Academic Publishers, Boston (2000), pp. 29–66.
6. S. M. Robinson, "Sensitivity analysis of variational inequalities by normal-map techniques," In: *Variational Inequalities and Network Equilibrium Problems*, Plenum Press, New York (1995), pp. 257–269.
7. R. T. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton, New Jersey (1970).
8. A. Shapiro and M. K. H. Fan, "On eigenvalue optimization," *SIAM J. Optim.,* **5**, 552–569 (1995).
9. A. Shapiro, "First and second order analysis of nonlinear semidefinite programs," *Math. Program., Ser. B,* **77**, 301–320 (1997).