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ON LIPSCHITZIAN STABILITY OF OPTIMAL SOLUTIONS OF PARAMETRIZED SEMI-INFINITE PROGRAMS

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We study continuity properties of optimal solutions of parametrized semi-infinite programming problems. The involved constraints are formulated in a form of cone constraints and then a slightly modified general result of Shapiro and Bonnans (1992) on Lipschitzian stability of optimal solutions is applied. It is shown that under certain second-order sufficient conditions, optimal solutions of the semi-infinite programs are Lipschitzian stable provided a regularity assumption related to a linearization of the considered programs is satisfied

1. Introduction. Consider the parametric optimization problem:

$$(P_t) \qquad \min_{x \in X} f(x,t) \quad \text{subject to } x \in \Phi(t),$$

where the feasible set $\Phi(t)$ is given in the form

$$(1.1) \qquad \Phi(t) = \{x \in X : h_\tau(x,t) \le 0, \tau \in T\}.$$

Here t is a nonnegative scalar parameter, X is a finite-dimensional vector space, say $X = \mathbb{R}^n$, T is a compact metric space and $h_\tau(x,t) = h(x,t,\tau)$ is a real-valued function. For $t = 0$ we refer to the corresponding optimization problem (P_0) as the *unperturbed* program. Let $\bar{x}(t)$ be an optimal solution of (P_t) converging, as $t \to 0^+$, to an optimal solution x_0 of the unperturbed program (P_0). We show in this paper that under certain second-order sufficient conditions and regularity assumptions, $\bar{x}(t)$ converges to x_0 at a rate of $O(t)$, i.e., $\bar{x}(t)$ is Lipschitz continuous at $t = 0$.

In case the set T is finite, (P_t) becomes a parametrized nonlinear programming problem. The Lipschitzian stability of optimal solutions has been studied in this case in a number of recent publications (cf. Auslender and Cominetti 1990, Bonnans, Ioffe and Shapiro 1992, Gauvin and Janin 1988, Shapiro 1988). Semi-infinite programming problems can be analysed by the so-called reduction method. That is, the feasible set $\Phi(t)$ can be defined by one constraint in the form

$$\Phi(t) = \{x \in X : m(x,t) \le 0\},$$

where $m(x,t) = \max_{\tau \in T} h_\tau(x,t)$. Differentiability properties of the max-function $m(x,t)$ are then derived and applied to investigation of the corresponding semi-infinite program. This approach was employed by several authors in investigation of (second-order) optimality conditions of semi-infinite programming problems (e.g., Hettich and Jongen 1977, Jongen, Wetterling and Zwier 1987, Shapiro 1985). In

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certain situations the function $m(x, t)$ can be represented as maximum of a finite number of smooth functions and hence the corresponding semi-infinite program can be reduced to a nonlinear programming problem with a finite number of constraints (cf. Klatte 1992). This method, however, requires rather restrictive assumptions.

In this paper a different approach will be used. In order to analyse the semi-infinite programs (P_t) we first formulate the feasible set $\Phi(t)$ in a form of cone constraints. That is, suppose that for every $\tau \in T$, the function $h_\tau(x, t)$ is twice differentiable jointly in x and t and that $h_\tau(x, t)$ together with the first- and second-order derivatives are continuous on $X \times \mathbb{R}_+ \times T$. Consider the normed space $Y = C(T)$ of continuous functions y: $T \to \mathbb{R}$, equipped with the sup-norm:

$$\|y\| = \sup\{|y(\tau)|: \tau \in T\},$$

and the cone $K \subset C(T)$ formed by nonpositive-valued continuous functions $y(\tau)$. Consider also the mapping g: $X \times \mathbb{R}_+ \to C(T)$ taking a point (x, t) into the function $y = g(x, t)$, $y(\cdot) = h(x, t, \cdot)$. Then the mapping g is twice continuously differentiable and the feasible set $\Phi(t)$ can be written in the form

$$\Phi(t) = \{x: g(x, t) \in K\}. \tag{1.2}$$

In the next section we describe a slightly modified result due to Shapiro and Bonnans (1992) on Lipschitzian stability of optimal solutions under cone constraints. It will be shown then in §3 how this general result can be applied to the particular case of semi-infinite programming problems.

2. Lipschitzian stability of optimal solutions under cone constraints. In this section we consider the parametric optimization problem (P_t) with the feasible set $\Phi(t)$ being given in the form (1.2), X being a Banach space and K being a closed, convex cone in a Banach space Y. By Y^* we denote the dual space of Y and for $\lambda \in Y^*$ we write $\langle \lambda, y \rangle$ for the value $\lambda(y)$. It will be assumed that the corresponding mapping g: $X \times \mathbb{R}_+ \to Y$ is twice continuously differentiable. We present here some general results on Lipschitzian stability of optimal solutions $\bar{x}(t)$ of (P_t). Let x_0 be an optimal solution of the unperturbed program (P_0). We assume that x_0 is a *regular* point of the mapping $g(x) = g(x, 0)$ in the sense of Robinson (1976b). That is,

$$0 \in \operatorname{int}\{g(x_0) + Dg(x_0)X - K\}. \tag{2.1}$$

With programs (P_t) are associated the Lagrangian function

$$L(x, \lambda, t) = f(x, t) + \langle \lambda, g(x, t) \rangle,$$

$\lambda \in Y^*$, and the optimal value function

$$\varphi(t) = \inf\{f(x, t): x \in \Phi(t)\}.$$

Note that $\varphi(0) = f(x_0, 0)$ and hence is finite. Under the regularity assumption (2.1) the set

$$\Lambda_0 = \{\lambda \in K^-: D_x L(x_0, \lambda) = 0, \langle \lambda, g(x_0) \rangle = 0\}, \tag{2.2}$$

of Lagrange multipliers of the program (P_0) at the optimal solution point x_0, is nonempty and bounded (Maurer and Zowe 1979, Robinson 1976a). Here $L(x, \lambda) =$

$L(x, \lambda, 0)$ and

$$K^{-} = \{\lambda \in Y^*: \langle \lambda, v\rangle \leqslant 0 \text{ for all } v \in K\}$$

is the polar (negative dual) cone of the cone K. Note that since the set Λ_0 is convex, closed and bounded, it is a weakly* compact subset of Y^*.

We make use of the following assumption.

ASSUMPTION A. *There exists a Lagrange multiplier $\lambda_0 \in \Lambda_0$ and a vector $v \in X$ such that*

$$\langle \lambda_0, D_x g(x_0, 0)v + D_t g(x_0, 0)\rangle = 0 \tag{2.3}$$

and

$$\operatorname{dist}(g(x_0) + tD_x g(x_0, 0)v + tD_t g(x_0, 0), K) = O(t^2). \tag{2.4}$$

We shall discuss this assumption later. Let us now formulate the following result which is a slight modification of Lemma 1 in Shapiro and Bonnans (1992).

LEMMA 2.1. *Suppose that x_0 is a regular point of $g(x)$ with respect to the cone K and that Assumption* A *holds. Then there exist positive numbers κ and η such that*

$$\varphi(t) - \varphi(0) \leqslant t \max_{\lambda \in \Lambda_0} D_t L(x_0, \lambda, 0) + \kappa t^2 \tag{2.5}$$

for all $t \in [0, \eta)$.

PROOF. Consider vector v specified in Assumption A. We have that

$$g(x_0 + tv, t) = g(x_0) + tD_x g(x_0, 0)v + tD_t g(x_0, 0) + O(t^2). \tag{2.6}$$

Since x_0 is regular, by the Robinson (1976c)-Ursescu (1975) stability theorem it follows from (2.4) and (2.6) that there exists $\tilde{v}(t)$ such that $x_0 + \tilde{v}(t) \in \Phi(t)$ and $\|tv - \tilde{v}(t)\|$ is of order $O(t^2)$. Then

$$\begin{aligned}\varphi(t) &\leqslant f(x_0 + \tilde{v}(t), t) = f(x_0) + D_x f(x_0, 0)\tilde{v}(t) + tD_t f(x_0, 0) + O(t^2)\\ &= f(x_0) + tD_x f(x_0, 0)v + tD_t f(x_0, 0) + O(t^2).\end{aligned}$$

Together with (2.3) this implies

$$\begin{aligned}\varphi(t) - \varphi(0) &\leqslant tD_x f(x_0, 0)v + tD_t f(x_0, 0)\\ &\quad + t\langle \lambda_0, D_x g(x_0, 0)v + D_t g(x_0, 0)\rangle + O(t^2)\\ &= tD_x L(x_0, \lambda_0, 0)v + tD_t L(x_0, \lambda_0, 0) + O(t^2).\end{aligned}$$

By the first-order optimality conditions $D_x L(x_0, \lambda_0, 0) = 0$ and hence (2.5) follows. □

Consider now the following linearization of the program (P_t):

$$(\mathscr{L}) \qquad \begin{array}{ll} \min\limits_{y \in X} & \langle Df(x_0), y\rangle \\ \text{subject to} & D_x g(x_0, 0)y + D_t g(x_0, 0) \in T(g(x_0), K). \end{array}$$

Let us observe that if vector v satisfies the conditions of Assumption A, then v is an optimal solution of the program $(\mathscr{L})$. Indeed, it follows from (2.4) that $D_x g(x_0, 0)v + D_t g(x_0, 0) \in T(g(x_0), K)$. We also have that the polar cone of the tangent cone $T(g(x_0), K)$ is given by

$$T(g(x_0), K)^- = \{\lambda \in K^-: \langle \lambda, g(x_0) \rangle = 0\}.$$

Consequently v and the Lagrange multiplier $\lambda_0 \in \Lambda_0$ satisfying equation (2.3), form a pair satisfying the first-order optimality conditions for the program $(\mathscr{L})$. Since the program $(\mathscr{L})$ is convex, this implies that v is an optimal solution of $(\mathscr{L})$. Conversely, if v solves $(\mathscr{L})$, then since x_0 is regular we have that there exists $\lambda_0 \in \Lambda_0$ such that condition (2.3) holds. Moreover, $D_x g(x_0, 0)v + D_t g(x_0, 0) \in T(g(x_0), K)$ and hence

$$\operatorname{dist}(g(x_0) + tD_x g(x_0, 0)v + tD_t g(x_0, 0), K) = o(t).$$

The last condition is similar to (2.4) but with $O(t^2)$ is replaced by $o(t)$.

The above discussion shows that Assumption A is *equivalent* to the following condition.

ASSUMPTION A*. *The program* $(\mathscr{L})$ *possesses an optimal solution* v *such that condition* (2.4) *holds.*

The dual program of $(\mathscr{L})$ is the problem of maximization of $D_t L(x_0, \lambda, 0)$ subject to $\lambda \in \Lambda_0$. Under the assumption of regularity of x_0, there is no duality gap between the program $(\mathscr{L})$ and its dual (cf. Lempio and Maurer 1980, Shapiro and Bonnans 1992). Note that it follows from the weak* compactness of Λ_0 that the set

$$\Lambda_1 = \operatorname{argmax}\{D_t L(x_0, \lambda, 0) : \lambda \in \Lambda_0\} \tag{2.7}$$

is nonempty. By duality arguments we also have that if λ_0 is a Lagrange multiplier corresponding to the optimal solution v of $(\mathscr{L})$, then $\lambda_0 \in \Lambda_1$.

Let us formulate now some required second-order sufficient conditions. For $\eta \geqslant 0$ consider the cone

$$C_\eta = \{y \in X: Dg(x_0)y \in T(g(x_0), K), \langle Df(x_0), y \rangle \leqslant \eta \|y\|\}, \tag{2.8}$$

and the set Λ_1 defined in (2.7).

ASSUMPTION B (SECOND-ORDER SUFFICIENT CONDITIONS). *There exist* $\alpha > 0$ *and* $\eta > 0$ *such that*

$$\max_{\lambda \in \Lambda_1} \langle y, D^2_{xx} L(x_0, \lambda, 0) y \rangle \geqslant \alpha \|y\|^2, \tag{2.9}$$

for all $y \in C_\eta$.

We have now the following result due to Shapiro and Bonnans (1992). Its proof is based on (2.1), (2.5) and the second-order sufficient conditions (2.9) only.

THEOREM 2.1. *Suppose that* x_0 *is a regular point of* $g(x)$ *with respect to the cone* K, *that Assumptions* A *and* B *hold and let* $\bar{x}(t)$ *be an optimal solution of* (P_t) *converging to* x_0. *Then there is a positive constant* c *such that*

$$\|\bar{x}(t) - x_0\| \leqslant ct$$

for all $t \geqslant 0$ *sufficiently small.*

Note that Assumption A in Theorem 2.1 can be replaced by the equivalent Assumption A*. Note also that if the space X is finite-dimensional, then the cone C_η in Assumption B can be replaced by the critical cone

$$C_0 = \{y \in X: Dg(x_0)y \in T(g(x_0), K), \langle Df(x_0), y\rangle = 0\}. \tag{2.10}$$

Moreover, in this case it is enough to assume that the maximum given in the left-hand side of (2.9) is greater than zero for all nonzero $y \in C_0$.

3. Lipschitzian stability of semi-infinite programs. In this section we specify general results of §2 to the semi-infinite programs (P_t) with the feasible set given in (1.1) and $X = \mathbb{R}^n$. Consider the Banach space $Y = C(T)$, the cone $K \subset C(T)$ of nonpositive valued continuous functions and the corresponding mapping $g(x, t)$. It follows from the assumed differentiability properties of the function $h(x, t, \tau)$ that the mapping $g(x, t)$ is twice continuously differentiable and, for example,

$$[Dg(x)v](\cdot) = v \cdot \nabla h(x, \cdot).$$

(Unless stated otherwise all gradients will be written here with respect to x.) The dual space Y^* of $Y = C(T)$ is the space of finite signed measures on $(T, \mathscr{B})$, where $\mathscr{B}$ is the Borel σ-algebra of T, with the norm given by the total variation of the corresponding measure.

The polar cone $K^- \subset Y^*$ of the cone K is formed by the set of (nonnegative) Borel measures on T. Indeed, let $\mu \in K^-$ and consider the Jordan decomposition of μ. That is, $\mu = \mu^+ - \mu^-$ with μ^+ and μ^- being (nonnegative) Borel measures with disjoint supports T_1 and T_2, respectively. Consider also the total variation measure $\mu^* = \mu^+ + \mu^-$. We argue now by a contradiction. Suppose that the negative part μ^- of the measure μ is not zero, say $\mu^-(T_2) = \alpha > 0$. Then, since every finite Borel measure on a compact metric space is regular, we can find a closed set $A \subset T$ and an open set $B \subset T$ such that $A \subset T_2 \subset B$ and $\mu^*(B \setminus A) < \alpha/2$. Moreover, there is a continuous function $y \in C(T)$ such that $y(\tau) = -1$ for all $\tau \in A$, $y(\tau) = 0$ for all $\tau \in T \setminus B$ and $-1 \leqslant y(\tau) \leqslant 0$ for all $\tau \in T$. We have then that $\mu(A) < -\alpha/2$ and $\mu(B \setminus A) < \alpha/2$ and hence

$$\langle \mu, y\rangle = \int_T y(\tau)\mu(d\tau) > 0.$$

This contradicts the definition of the polar cone K^-.

Denote by $\Delta(x)$ the set

$$\Delta(x) = \{\tau \in T: h_\tau(x, 0) = 0\}$$

of active at $x \in \Phi(0)$ constraints. First-order necessary conditions for the semi-infinite programming problem (P_0) are well known (e.g., Pshenichnyi 1971). If x_0 is an optimal solution of (P_0), then there exist $\tau_i \in \Delta(x_0)$, $i = 1, \ldots, n$, and nonnegative multipliers $\lambda_0, \lambda_1, \ldots, \lambda_n$, not all of them zero, such that

$$\lambda_0 \nabla f(x_0) + \sum_{i=1}^{n} \lambda_i \nabla h(x_0, \tau_i) = 0. \tag{3.1}$$

Suppose now that there exists a vector $v \in X$ such that

$$v \cdot \nabla h_\tau(x_0) < 0 \quad \text{for all } \tau \in \Delta(x_0). \tag{3.2}$$

In case the set T is finite this is the Mangasarian-Fromovitz constraint qualification. Under condition (3.2) the multiplier λ_0 in (3.1) is nonzero and can be taken to be $\lambda_0 = 1$. It is also not difficult to show that (3.2) is *equivalent* to the condition that the point x_0 is a regular point of the mapping $g(x)$ with respect to the cone K. Indeed, suppose that condition (3.2) holds. Since $\Delta(x_0)$ is a closed subset of the compact set T and the function $a(\tau) = v \cdot \nabla h(x_0, \tau)$ is continuous on T, there is an open neighborhood N of $\Delta(x_0)$ and $\epsilon_1 > 0$ such that $a(\tau) \leqslant -\epsilon_1$ for all $\tau \in N$. Also the function $h(x_0, \cdot\,)$ is negative valued on the compact set $T \setminus N$ and hence $h(x_0, \tau) < -\epsilon_2$ for some $\epsilon_2 > 0$ and all $\tau \in T \setminus N$. Since $a(\tau)$ is bounded on $T \setminus N$ we obtain that there exists $\alpha > 0$ such that for $y = \alpha v$ and all $\tau \in T$,

$$[g(x_0) + Dg(x_0)y](\tau) = h(x_0, \tau) + y \cdot \nabla h(x_0, \tau) < 0.$$

Again since the function $[g(x_0) + Dg(x_0)y](\cdot)$ is continuous and the set T is compact, it follows that there is an $\epsilon > 0$ such that all values of this function are less than $-\epsilon$. Consequently the set $g(x_0) + Dg(x_0)y - K$ contains all continuous functions whose values are not less than $-\epsilon$. This implies that the zero function is an interior point of the set $g(x_0) + Dg(x_0)y - K$ and hence the point x_0 is a regular point of $g(x)$. Conversely, suppose that x_0 is a regular point of $g(x)$. Then there exists a vector v such that $[g(x_0) + Dg(x_0)v](\tau) < 0$ for all $\tau \in T$. Clearly this implies condition (3.2).

Consider now the tangent cone $T(g(x_0), K)$. This tangent cone is given by the topological closure of the radial cone $K + [g(x_0)]$. ($[g(x_0)]$ denotes the linear space generated by vector $g(x_0)$.) Let us show that

$$T(g(x_0), K) = \{y \in C(T) \colon y(\tau) \leqslant 0 \text{ for all } \tau \in \Delta(x_0)\}. \tag{3.3}$$

Indeed, it is clear that if $y \in K + [g(x_0)]$, then $y(\tau) \leqslant 0$ for every $\tau \in \Delta(x_0)$. By the arguments of continuity the same holds for $y \in T(g(x_0), K)$. Conversely, let y be a function belonging to the set given by the right-hand side of (3.3). Then for any $\epsilon > 0$ there is a positive constant c such that for all $\tau \in T$,

$$y(\tau) \leqslant -ch(x_0, \tau) + \epsilon.$$

Consider the function $b(\tau) = y(\tau) + ch(x_0, \tau) - \epsilon$. We have that $b \in K$ and $\|y - (b - cg(x_0))\| = \epsilon$. Since $b - cg(x_0) \in K + [g(x_0)]$ and ϵ is arbitrary, we obtain that formula (3.3) holds.

Formula (3.3) suggests the following linearization of the semi-infinite program (P_t),

$$(\mathscr{L}') \qquad \begin{array}{ll} \min\limits_{y \in X} & y \cdot \nabla f(x_0) \\ \text{subject to} & y \cdot \nabla h_\tau(x_0, 0) + \partial h_\tau(x_0, 0)/\partial t \leqslant 0, \qquad \tau \in \Delta(x_0). \end{array}$$

Program $(\mathscr{L}')$ is a specification of the linearized program $(\mathscr{L})$, introduced in §2, to the present situation. The corresponding Assumption A* can be formulated now as follows.

ASSUMPTION A'. *The program* ($\mathscr{L}'$) *possesses an optimal solution* $\bar{y}$ *such that the condition*

$$(3.4)\quad \left[\sup_{\tau\in T}\{h(x_0,0,\tau)+t[\bar{y}\cdot\nabla h(x_0,0,\tau)+\partial h(x_0,0,\tau)/\partial t]\}\right]^{+}=O(t^2)$$

holds. (*Here* $a^+ = \max\{0, a\}$.)

Note that the optimal value of the program ($\mathscr{L}'$) is equal to the optimal value of the corresponding dual program and hence is finite because of the regularity condition (3.2). In particular, if the set $\Delta(x_0)$ is finite, then program ($\mathscr{L}'$) becomes a linear programming problem. In this case regularity assumption (3.2) implies existence of an optimal solution for the program ($\mathscr{L}'$).

Let us consider now condition (3.4). Notice that $\Delta(x_0)$ represents the set of maximizers of the function $h(x_0, 0, \cdot)$ over the set T. Consider the parametric program

$$(3.5)\qquad \max_{\tau\in T} p(\tau, t)$$

where

$$p(\tau,t)=h(x_0,0,\tau)+t[\bar{y}\cdot\nabla h(x_0,0,\tau)+\partial h(x_0,0,\tau)/\partial t].$$

Since

$$\bar{y}\cdot\nabla h(x_0,0,\tau)+\partial h(x_0,0,\tau)/\partial t\leqslant 0$$

for all $\tau \in \Delta(x_0)$, we have then that (3.4) is implied by the following assumption.

ASSUMPTION C. *Functions* $\nabla h(x_0, 0, \cdot)$ *and* $\partial h(x_0, 0, \cdot)/\partial t$ *are Lipschitz continuous in a neighborhood of* $\Delta(x_0)$ *and program* (3.5) *has an optimal solution* $\bar{\tau}(t)$ *such that*

$$(3.6)\qquad \operatorname{dist}(\bar{\tau}(t),\Delta(x_0))=O(t).$$

Note that existence of the optimal solution $\bar{\tau}(t)$ and convergence of the distance $\operatorname{dist}(\bar{\tau}(t), \Delta(x_0))$ to zero, as $t \to 0^+$, follow from compactness of T.

It is relatively easy to give sufficient (second-order) conditions to ensure Lipschitzian stability (3.6) of the optimal solutions of the program (3.5). Suppose now that T is a subset of a normed space. We say that *the second-order growth condition* for the program (3.5) holds if there exist a positive constant α and a neighborhood W of $\Delta(x_0)$ such that

$$(3.7)\qquad -h(x_0,\tau)\geqslant\alpha[\operatorname{dist}(\tau,\Delta(x_0))]^2$$

for all $\tau \in W \cap T$ (see Bonnans and Ioffe (1993) for a discussion of this condition). We have then that if functions $\nabla h(x_0, 0, \cdot)$ and $\partial h(x_0, 0, \cdot)/\partial t$ are Lipschitz continuous in the neighborhood W and the second-order growth condition holds, then (3.6) follows (cf. Shapiro 1992, Lemma 2.1).

Let us finally specify the second-order sufficient conditions of Assumption B. Consider the set M_0 of measures $\mu \in Y^*$ representing Lagrange multipliers for the program (P_0) at the point x_0. That is, M_0 is formed by (nonnegative) measures μ

such that their support is contained in the set $\Delta(x_0)$ and

$$\nabla f(x_0) + \int \nabla h(x_0, \tau)\mu(d\tau) = 0.$$

Note that any extreme point of the set M_0 can be represented in the form

$$\mu = \sum_{i=1}^{n} \lambda_i \delta(\tau_i), \tag{3.8}$$

where $\delta(\tau)$ denotes the measure of mass one at the point τ and the multipliers λ_i and the points $\tau_i \in \Delta(x_0)$ satisfy the equation

$$\nabla f(x_0) + \sum_{i=1}^{n} \lambda_i \nabla h(x_0, \tau_i) = 0. \tag{3.9}$$

(Recall that n is the dimensionality of the space X.) Indeed, consider a measure $\mu \in M_0$ such that it cannot be represented in the form (3.8). Then $\mu = \mu_1 + \cdots + \mu_{n+1}$, where μ_i, $i = 1, \ldots, n+1$, are positive measures with disjoint supports. Consider vectors $b_i = \int \nabla h(x_0, \tau)\mu_i(d\tau)$, $i = 1, \ldots, n+1$. By dimensionality arguments these vectors are linearly dependent and hence there exist numbers α_i, $i = 1, \ldots, n+1$, not all of them zeros, such that $|\alpha_i| < 1$ and $\sum_{i=1}^{n+1} \alpha_i b_i = 0$. Consider now the two following measures $\mu' = \sum_{i=1}^{n+1}(1 - \alpha_i)\mu_i$ and $\mu'' = \sum_{i=1}^{n+1}(1 + \alpha_i)\mu_i$. Clearly $\mu', \mu'' \in M_0$ and $\mu = (\mu' + \mu'')/2$. Therefore the measure μ cannot be an extreme point of M_0.

Consider the Lagrangian

$$L(x, \mu, t) = f(x, t) + \langle \mu, g(x, t) \rangle = f(x, t) + \int h(x, t, \tau)\mu(d\tau)$$

and the set

$$M_1 = \operatorname{argmax}\{D_t L(x_0, \mu, 0) : \mu \in M_0\}.$$

The set M_1^* of extreme points of M_1 is formed by a set of discrete measures μ representable in the form (3.8) with λ_i and $\tau_i \in \Delta(x_0)$ satisfying the equation (3.9) and such that $\sum_{i=1}^{n} \lambda_i \partial h(x_0, 0, \tau_i)/\partial t$ attains the corresponding maximum. The required second-order sufficient conditions can be formulated now as follows.

ASSUMPTION B′. *For all nonzero* $y \in C_0$,

$$\max_{\mu \in M_1^*} y \cdot \nabla_{xx}^2 L(x_0, \mu, 0) y > 0, \tag{3.10}$$

where C_0 *is the critical cone*

$$C_0 = \{y \in X : y \cdot \nabla h_\tau(x_0, 0) \leqslant 0, \tau \in \Delta(x_0), y \cdot \nabla f(x_0) = 0\}$$

and

$$\nabla_{xx}^2 L(x_0, \mu, 0) = \nabla_{xx}^2 f(x_0) + \sum_{i=1}^{n} \lambda_i \nabla_{xx}^2 h(x_0, 0, \tau_i)$$

for $\mu = \sum_{i=1}^{n} \lambda_i \delta(\tau_i)$.

Note that here Assumptions B and B′ coincide due to finite dimensionality of the space X. We can formulate now the main result of this paper.

THEOREM 3.1. *Suppose that the regularity condition* (3.2) *and Assumptions A′ and B′ hold and let $\bar{x}(t)$ be an optimal solution of (P_t) converging to x_0. Then there is a positive constant c such that*

$$\|\bar{x}(t) - x_0\| \leqslant ct$$

for all $t \geqslant 0$ sufficiently small.

Assumption A′ in Theorem 3.1 holds if the program $(\mathscr{L}')$ has an optimal solution and Assumption C is satisfied. As we mentioned earlier in case the set $\Delta(x_0)$ is finite, existence of an optimal solution of the program $(\mathscr{L}')$ follows under the regularity condition (3.2).

Let us finally mention the following application of the Lipschitzian stability result of Theorem 3.1. Consider the optimal value $\varphi(t)$ of the semi-infinite program (P_t). It follows from the Lipschitzian stability of the optimal solution $\bar{x}(t)$ that the optimal value function $\varphi(t)$ is differentiable at $t = 0$ (in the positive direction) and

$$\varphi'(t) = \max_{\mu \in M_0} D_t L(x_0, \mu, 0) \tag{3.11}$$

(cf. Lempio and Maurer 1980). Note that the above maximum can be calculated with respect to discrete measures μ in the form (3.8) with λ_i and $\tau_i \in \Delta(x_0)$ satisfying (3.9).

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