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A Conjecture Related to Chi-Bar-Squared Distributions

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Consider a polyhedral, convex, closed cone C in $\mathbb{R}^n$. We write $\langle x, y \rangle = x_1 y_1 + \cdots + x_n y_n$ and $\|x\| = \langle x, x \rangle^{1/2}$. Let $P: \mathbb{R}^n \to C$ be the projection mapping onto C, which is defined as assigning to each $x \in \mathbb{R}^n$ the closest point in C, that is

$$\|x - P(x)\| = \min_{y \in C} \|x - y\|.$$

Let $X = (X_1, \ldots, X_n)$ be a multivariate random variable having the standard normal distribution, i.e., $X \sim \mathcal{N}(0, \mathrm{I})$. Define numbers $w_i(C)$, $i = 0, 1, \dots, n$, to be the probabilities that $P(X)$ belongs to a face of C of dimension i. If C has no i-dimensional faces, then by definition, $w_i(C)$ is zero.

CONJECTURE. *For every polyhedral, convex, closed cone C, which is not a linear subspace of $\mathbb{R}^n$, one has*

$$\sum_{i=0}^{n} (-1)^i w_i(C) = 0. \tag{1}$$

The numbers $w_i(C)$ are related to the random variable $\bar{\chi}^2$ representing the squared distance to C,

$$\bar{\chi}^2 = \|X - P(X)\|^2.$$

It can be shown that $\bar{\chi}^2$ has the distribution which is a mixture of chi-squared distributions with weights $w_i(C)$ (see [3], [4], [5], [7]). Consequently the problem of finding this distribution (called the chi-bar-squared distribution) is reduced to evaluation of weights $w_i(C)$. Unfortunately it is not easy to calculate numbers $w_i(C)$ for an arbitrary cone C, although in some simple cases the solution is known in a closed form (e.g. [1, pp. 134–148], [3], [7]). Of course, established identities between $w_i(C)$ may facilitate the problem.

There is extensive theoretical and numerical evidence supporting the Conjecture (cf. [1, p. 174], [2], [6], [7]). For instance, if C is the nonnegative orthant $\mathbb{R}^n_+ = \{x : x_i \geq 0\}$, then $w_i(C) = 2^{-n}\binom{n}{i}$ and (1) follows. For $n = 2$ it can be easily verified that $w_1(C) = 1/2$, which implies (1). Already for $n = 3$ the identity (1) in all its generality is not trivial. In $\mathbb{R}^3$ the weights have a simple geometric interpretation ([7], [8]) and this leads to a proof in this case. Namely, the weights $w_0(C)$, $w_1(C)$, $w_2(C)$ $w_3(C)$ are given by $\beta(C^0)/4\pi$, $\alpha(C^0)/4\pi$, $\alpha(C)/4\pi$, $\beta(C)/4\pi$, where $\beta(C)$ denotes the solid angle of C, $\alpha(C)$ denotes the plane angle corresponding to the surface of C, and C^0 denotes the dual (polar) cone of C,

$$C^0 = \{y \in \mathbb{R}^n : \langle y, x \rangle \leq 0, \forall x \in C\}.$$

Consequently (1) becomes

$$\alpha(C^0) + \beta(C) = 2\pi. \tag{2}$$

By passing to a limit, the weights $w_i(C)$ can be associated with any convex (not necessarily polyhedral) cone C [7]. It will be of certain interest if one can extend the geometric interpretation above ($n = 3$) to higher dimensions. Hopefully this will lead to a generalization of the geometrical identity (2).

REFERENCES

1. R. E. Barlow, D. J. Bartholomew, J. M. Bremner, and H. D. Brunk, Statistical Inference under Order Restrictions, Wiley, New York, 1972.
2. V. J. Chacko, Testing Homogeneity Against Ordered Alternatives, Ann. Math. Statist., 34 (1963) 945–956.

3. A. Kudô, A Multivariate Analogue of the One-Sided Test, Biometrika, 50 (1963) 403–418.
4. A. Kudô and J. R. Choi, A Generalized Multivariate Analogue of the One-Sided Test, Mem. Fac. Sci. Kyushu Univ. Ser. A., 29 (1975) 303–328.
5. P. E. Nüesch, On the Problem of Testing Location in Multivariate Populations for Restricted Alternatives, Ann. Math. Statist., 37 (1966) 113–119.
6. T. Robertson and F. T. Wright, Testing for Ordered Alternatives With Increased Precision in one of the Samples, Biometrika, 69 (1982) 579–586.
7. A. Shapiro, Asymptotic Distribution of Test Statistics in the Analysis of Moment Structures Under Inequality Constraints, Biometrika, 72 (1985) 133–144.
8. H. P. Wynn, Integrals for One-Sided Confidence Bounds: a General Result, Biometrika, 62 (1975) 393–396.