ISyE8843
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$$\frac{2}{9} = \boxed{\frac{1}{3}} \cdot \frac{3}{3+15} + \boxed{\frac{1}{5}} \cdot \frac{15}{3+15}$$

Quiz 1

1. Lifetime. A lifetime X of a particular machine is modeled by an exponential distribution with unknown parameter θ.

If the parametrization is $f(x|\theta) = \theta e^{-\theta x}$, $x \geq 0, \theta > 0$, the MLE estimator for θ is $\hat{\theta}_{MLE} = 1/\bar{X}$ on basis of a sample $X_1, \dots, X_n$.

The lifetimes (in years) of $X_1 = 5, X_2 = 6$, and $X_3 = 4$ are observed.

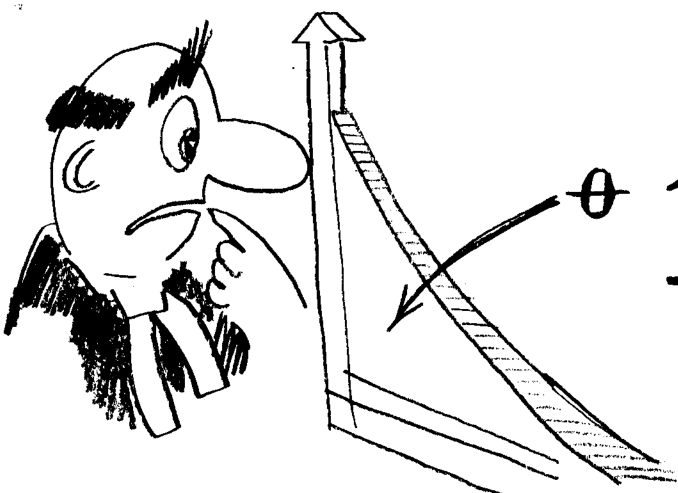
(*a*) Write down the MLE of θ for those observations.

Assume now that an expert believes that θ should have exponential distribution as well and that, on average θ should be $1/3$.

(*b*) Elicit a prior according to expert's beliefs.

(*c*) For the prior in (b), find the posterior. Is the problem conjugate?

(*d*) Find the Bayes estimator $\hat{\theta}_{Bayes}$, and compare it with the MLE estimator from (a). Discuss.



$$f(x|\theta) = \theta e^{-\theta x}, \quad x, \theta \geq 0$$

$$\ell_x(\theta) = \theta^3 e^{-\theta \sum_{i=1}^{3} x_i} = \theta^3 e^{-15\theta}$$

(a) MLE: $\hat{\theta}_{MLE} = \frac{1}{\bar{X}} = \frac{1}{5}$

(b) $\mu = E\theta = 1/3$

$\pi(\theta) = 3e^{-3\theta}, \quad \theta \geq 0$

(c) $\pi(\theta|\underset{\sim}{x}) \propto \ell_x(\theta) \cdot \pi(\theta) \propto \theta^3 e^{-18\theta}, \quad \theta \geq 0$

$\longrightarrow \pi(\theta|\underset{\sim}{x}) \sim \mathcal{G}amma(4, \frac{1}{18})$.

The problem is conjugate since the exponential distribution is a special case of Gamma (shape $\equiv 1$)

(d) $\hat{\theta}_{Bayes} = E^{\theta|X}\theta = 4 \cdot \frac{1}{18} = \frac{2}{9}$

$1/5$ ——▲ $2/9$ —— $1/3$ Bayes est. is compromise between prior mean and MLE