Answer for Quiz 7 (Ni Wang)

Answer: First, let's calculate the Fisher information for θ from the Negative Binomial $\mathcal{NB}(m, \theta)$ distribution.

$$\begin{aligned}
\log f(x|\theta) &= m \log \theta + x \log(1-\theta) + \text{const} \\
-\frac{\partial^2}{\partial \theta} \log f(x|\theta) &= \frac{m}{\theta^2} + \frac{x}{(\theta-1)^2} \\
\mathcal{L}(\theta) &= \frac{m}{\theta^2} + \frac{1}{(\theta-1)^2} \frac{m(1-\theta)}{\theta} \\
&= \frac{m}{\theta^2(1-\theta)}
\end{aligned}$$

To find the minimum message length (MML) estimate of θ, we need to maximize weighted posterior

$$\begin{aligned}
& \frac{\pi(\theta) f(x|\theta)}{|\mathcal{L}(\theta)|^{1/2}} \\
\propto \; & \theta^{\alpha-1}(1-\theta)^{\beta-1}\theta^m(1-\theta)^x|\theta^2(1-\theta)|^{1/2} \\
= \; & \theta^{\alpha+m+1-1}(1-\theta)^{\beta+x+1/2-1}
\end{aligned}$$

This shows θ's weighted posterior is $\mathcal{B}e(\alpha+m+1, \beta+x+1/2)$. Thus the MML estimate for θ will be the mode of this weighted posterior

$$\hat{\theta} = \frac{\alpha+m}{\alpha+m+\beta+x-1/2}$$