ISyE8843
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Quiz 7

Wallace-Freeman MML Estimator. Recall that the Minimum Message Length (MML) estimate, based on $X_1, \dots, X_n \sim f(x|\theta)$ is defined as

$$\text{argmin}_\theta[-\log \pi(\theta) - \log \prod_{i=1}^{n} f(x_i|\theta) + \frac{1}{2} \log |\mathcal{I}(\theta)|],$$

where $\pi(\theta)$ is the prior and $\mathcal{I}(\theta)$ is the Fisher information matrix. This is equivalent to maximizing weighted posterior

$$\frac{\pi(\theta) \prod_{i=1}^{n} f(x_i|\theta)}{|\mathcal{I}(\theta)|^{1/2}}.$$

Problem. Suppose a single observation $X|\theta$ is coming from the Negative Binomial $\mathcal{NB}(m, \theta)$, with p.m.f.

$$f(x|\theta) = \binom{m+x-1}{x} \theta^m (1-\theta)^x,$$

and that the prior on θ is Beta $\mathcal{B}e(\alpha, \beta)$. For example, the observation X could be interpreted as the number of failures incurred until m successes are recorded.

Find the MML rule.

You will need the following facts:
- The expectation of $X \sim \mathcal{NB}(m, \theta)$, is $EX = \frac{m(1-\theta)}{\theta}$.
- The Fisher information $\left[-E \frac{\partial^2}{\partial \theta} \log f(x|\theta)\right]$ for θ from the Negative Binomial $\mathcal{NB}(m, \theta)$ distribution is

$$\mathcal{I}(\theta) = \frac{m}{\theta^2(1-\theta)}. \quad \textbf{Prove this!}$$

- The mode of Beta $\mathcal{B}e(a, b)$ distribution is $\frac{a-1}{a+b-2}$. Also, $X|\theta \sim \mathcal{NB}(m, \theta)$, and $\mathcal{B}e(\alpha, \beta)$ are conjugate; the posterior is straightforward.