ISyE8843
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Quiz 4

Choose one problem

1. Assume $X|\theta$ is exponential $\mathcal{E}(1/\theta)$ with density $f(x|\theta) = \frac{1}{\theta}e^{-x/\theta},\ x \geq 0$. Let F be the cdf corresponding to f. Assume a prior on θ, $\pi(\theta)$.

Let $m_\pi(x) = \int_\Theta f(x|\theta)\pi(\theta)d\theta$ be the marginal and $M_\pi(x) = \int_0^x m_\pi(t)dt$ be its cumulative distribution function. (a) Show that $\theta = \frac{f(x|\theta)}{1-F(x|\theta).}$

(b) Show that Bayes estimator with respect π is $\hat{\theta} = \frac{m_\pi(x)}{1-M_\pi(x).}$

[Hint. You will need to use a version of Fubini's theorem (Tonelli theorem) and change order of integration. Tonelli theorem allows for change when integrands are nonnegative.]

(c) Suppose you observe $X_i|\theta_i \sim \mathcal{E}(1/\theta_i), i = 1, \ldots, n+1$. Explain how would you estimate θ_{n+1} in the empirical Bayes fashion, using result (b).

2. Assume $[X|\theta] \sim \mathcal{N}(\theta, 1)$ and $[\theta|\mu, \tau^2]\ \sim \mathcal{N}(\mu, \tau^2)$.

(a) Find the marginal for X.

(b) What are the moment matching estimators of μ and τ^2, if the sample $X_i \sim \mathcal{N}(\theta_i, 1),\ i = 1, \ldots, n$, is available.

[Hint. Find the moments of the marginal and be careful, the estimator of the variance need to be non-negative.]

(c) Propose an empirical Bayes estimator of θ based on the considerations in (b).

(d) What modifications in (b) are needed if you use MLE II estimator of μ and τ^2.

[Sol. moment matching. Marginal is $\mathcal{N}(\mu, 1 + \tau^2)$. $\hat{\mu} = \bar{X}$ and estimator of $1 + \tau^2$ is s^2. So $\hat{\tau}^2 = \max\{0, s^2 - 1\}$. For MLE $\hat{\tau}^2 = \max\{0, \frac{n-1}{n}s^2 - 1\}$.

3. If the data $X_i \sim f(x_i|\theta)$ can not be reduced by a sufficient statistics, then so called pseudo-Bayes approach is possible. Let T be an estimator of θ for which distribution $g(t|\theta)$ is known. Instead of finding Bayes rule

$$\delta_\pi(x_1, \ldots, x_n) = \frac{\int_\Theta \theta \prod_{i=1}^n f(x_i|\theta)\pi(\theta)d\theta}{\int_\Theta \prod_{i=1}^n f(x_i|\theta)\pi(\theta)d\theta}$$

one finds pseudo-Bayes rule

$$\delta_\pi^*(t) = \frac{\int_\Theta tg(t|\theta)\pi(\theta)d\theta}{\int_\Theta g(t|\theta)\pi(\theta)d\theta}.$$

Suppose that you have model

$$X_i \sim \mathcal{N}(\theta, 1),$$
$$\theta \sim \mathcal{N}(\mu, \tau^2).$$

For n large, the distribution of the sample median $m = \mathrm{Med}(X_1, X_2, \ldots, X_n)$, is approximately $\mathcal{N}(\theta, \frac{\pi}{2n})$.

Write down pseudo-Bayes estimator of θ using the median estimator of θ and its normal approximation.

4. Another way to justify Stein shrinkage estimator. Mimic the exercise 2. Suppose that you have model

$$X \sim \mathcal{MVN}_p(\boldsymbol{\theta}, I),$$
$$\boldsymbol{\theta} \sim \mathcal{MVN}(0, \tau^2 I).$$

(i) Find marginal distribution.
(ii) Show, in (i) the MLE of τ^2 is

$$\hat{\tau}^2 = \begin{cases} \frac{\sum x_i^2}{p}, & \sum x_i^2 > p \\ 0, & \text{else} \end{cases}$$

(iii) Replacing the MLE in the Bayes estimator

$$\delta_\pi(\boldsymbol{x}) = \frac{\tau^2 \boldsymbol{x}}{1+\tau^2}$$

the *truncated* James-Stein estimator is obtained,

$$\delta_{\text{EB}}(\boldsymbol{x}) = \left(1 - \frac{p}{\sum x_i^2}\right)_+ \boldsymbol{x},$$

where $(x)_+ = \max\{x, 0\}$ is the positive part of x.