ISyE8843
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Quiz 3

Select one problem!

Weighted Square Error Loss. We have seen that under squared error loss the Bayes rule [minimizer of Bayes risk $r(\pi, \delta)$ or equivalently conditional on x minimizer of posterior expected loss $E^{\theta|x}L(\theta, \delta(x))$] is the posterior expectation,

$$\delta_\pi(x) = E^{\theta|x}\theta = \int_\Theta \theta\pi(\theta|x)d\theta = \frac{\int_\Theta \theta f(x|\theta)\pi(\theta)d\theta}{\int_\Theta f(x|\theta)\pi(\theta)d\theta}.$$

If the loss is *weighted square error*

$$L(\theta, a) = w(\theta)(\theta - a)^2,$$

the Bayes rule is

$$\delta_\pi(x) = \frac{E^{\theta|x}\theta w(\theta)}{E^{\theta|x}w(\theta)} = \frac{\int_\Theta \theta w(\theta)f(x|\theta)\pi(\theta)d\theta}{\int_\Theta w(\theta)f(x|\theta)\pi(\theta)d\theta}.$$

To obtain a particular quality control score, 10 independent zero-mean half-normal attributes with the same scale are squared and summed up. The obtained score $[X|\theta]$ is a scaled χ^2_{10} with the likelihood from $\mathcal{G}(5, 2\theta)$ family,

$$f(x|\theta) = \frac{1}{\Gamma(5)(2\theta)^5}x^{5-1}e^{-x/(2\theta)}, \theta > 0.$$

The parameter θ is of interest.

From the past experience and experts' input you know that θ is skewed to the right, most likely to cluster around 2 with an average close to 4. Use this information to elicit an inverse gamma prior on θ, $[\theta] \sim \mathcal{IG}(\alpha, \beta)$. [1]

Assume that the loss function is weighted square error loss $L(\theta, a) = \left(\frac{a}{\theta} - 1\right)^2$. This reduces penalty if the parameter θ is large in magnitude (scale of attributes are large). If the statistician uses $\delta(x) = 0$, the loss is 1, irrespective of the value for θ. On the other hand, if the parameter θ is close 0 the penalty for an error is severe.

A single measurement of $X = 16$ is available. What is the Bayes estimator of θ under the weighted squared error loss?

Solution.

If the prior is inverse-gamma $\mathcal{IG}(\alpha, \beta)$ then (as in handout 0) the mean is $\frac{1}{\beta(\alpha-1)}$ and the mode is $\frac{1}{\beta(\alpha+1)}$. Solving

$$\begin{aligned} \frac{1}{\beta(\alpha-1)} &= 4 \\ \frac{1}{\beta(\alpha+1)} &= 2 \end{aligned}$$

[1] Recall that for a random variable with $\mathcal{IG}(\alpha, \beta)$ distribution the mean is $\frac{1}{\beta(\alpha-1)}$ and the mode is $\frac{1}{\beta(\alpha+1)}$

we get $\alpha = 3$ and $\beta = 1/8$.

The weight function is $w(\theta) = 1/\theta^2$, from $L(\theta, a) = \left(\frac{a}{\theta} - 1\right)^2 = \frac{1}{\theta^2}\ (\theta - a)^2$.

Since $f(x|\theta)\pi(\theta) \propto \frac{1}{\theta^{(\alpha+5)+1}} \exp\{-\frac{1}{\theta}(\frac{x}{2} + \frac{1}{\beta})\}$, then by completing to appropriate inverse gamma density that integrates to 1,

$$\begin{aligned}\int \theta w(\theta) f(x|\theta)\pi(\theta) d\theta &= \Gamma(\alpha+6) \times (\frac{x}{2} + \frac{1}{\beta})^{-(\alpha+6)} \\ \int w(\theta) f(x|\theta)\pi(\theta) d\theta &= \Gamma(\alpha+7) \times (\frac{x}{2} + \frac{1}{\beta})^{-(\alpha+7)}.\end{aligned}$$

The Bayes rule is

$$\delta_\pi(x) = \frac{\Gamma(\alpha+6) \times (\frac{x}{2} + \frac{1}{\beta})^{-(\alpha+6)}}{\Gamma(\alpha+7) \times (\frac{x}{2} + \frac{1}{\beta})^{-(\alpha+7)}} = \frac{x/2 + 1/\beta}{\alpha + 6}.$$

For $X = 16, \alpha = 3$, and $\beta = 1/8$, the Bayes rule is $\frac{16}{9}$.

Equivalently, show that posterior is $\mathcal{IG}(8, \frac{1}{x/2+8})$ and find the ratio

$$\frac{E^{\theta|x}\frac{1}{\theta}}{E^{\theta|x}\frac{1}{\theta^2}}.$$

Traffic. Marietta Traffic Authority is worried about the repeated accidents at the intersection of Canton and Piedmont Roads. As a city-engineer you would like to find the accident rate, even better, a credible set.

A well known model for modeling the number of road accidents in a particular space/time window is the Poisson distribution. Assume that X represents the number of accidents in a 2 month period at the intersection od Canton and Piedmont Roads. Assume that $[X|\theta] \sim \mathcal{P}oi(\theta)$. Nothing is known a priori about θ, so it is reasonable to assume the noninformative prior

$$\pi(\theta) = \frac{1}{\theta}\mathbf{1}(0 < \theta < \infty).$$

In the six most recent two-month periods the following realizations for X are observed: 3 2 1 1 3 2.

(a) Compare the Bayes estimator and the MLE (For Poisson, recall, $\hat{\theta}_{MLE} = \bar{X}$).

(b) An $(1 - \alpha)100\%$ HPD credible set has no finite form. Describe in principle how would you calculate this set approximately on a computer. Your description may be informal.

(c) The Mathematica code

```
In[1]=        ff = 104976/1925 t^11  Exp[-6 t];
In[2]=        NIntegrate[ff, {t, 0, 1.9447275}]
Out[2]=       0.5
```

gives that the median of the posterior is approximately 1.9447275.

If you test the hypotheses

$$H_0 : \theta \geq 2 \qquad vs \qquad H_1 : \theta < 2,$$

in Bayesian fashion, which one will be favored?

Solution.

(a) The posterior is given by

$$\pi(\theta|x) \propto e^{-n\theta}\theta^{\sum x_i - 1} \propto e^{-6\theta}\theta^{12-1}.$$

The Bayes estimate $\theta_{\text{Bayes}} = 12 \times \frac{1}{6} = 2$. This coincides with the MLE, $\bar{X} = 2$. This is expected since the prior is non-informative.

(b) Numerically solve the equation $\pi(\theta|x) = k$. Select k in the range to obtain two real solutions θ_1 and θ_2 (the posterior density is unimodal). Now, try various k until $\int_{\theta_1}^{\theta^2} \pi(\theta|x)d\theta$ is close to $1 - \alpha$. This integral is a difference of two incomplete gammas and is available in many software packages.

(c) The median is less than 2, meaning that $P(\theta < 2) > 1/2$. That means H_1 is favored.