Midterm Exam

ISyE8843
Abhyuday Mandal

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1 Problem 1

The model is $y_{ij} = \mu_i + \epsilon_{ij}, \quad i = 1, \ldots, k; j = 1, \ldots, n_i$, where $k = 3$, $n_1 = 9$, $n_2 = 11$ and $n_3 = 14$. Among the nine parameters $\theta = (\mu_1, \mu_2, \mu_3, \psi, \sigma_1, \sigma_2, \sigma_3, \tau^2, \sigma)$, we are interested in comparing μ_1, μ_2 and μ_3. The estimates are obtained by Gibbs Sampling. Figures 1 and 2 summarizes the results. Panels (a) depicts the histograms of the posterior distributions, while the panels (b) depicts the last 500 simulations of the chain. Note that we have a good mixing for all the cases. Estimates are obtained by taking the medians and means of the posterior distributions.

Table 1 : Medians and means of parameters obtained after 10000 simulations with burn-in of 2000

Parameter	Median	Mean
μ_1	23.9989	24.0801
μ_2	20.6510	20.6760
μ_3	21.8425	21.8201
ψ	21.7534	21.8263
σ_1^2	132.8980	158.2477
σ_2^2	98.1583	111.4053
σ_3^2	76.6515	84.9044
τ^2	8.6334	17.6032
σ^2	25.2681	30.2589

We know that the posterior distributions of μ_i's are normal. So we calculate approximate 95% credible sets by $\bar{\mu}_i \pm 1.96 s_i$ where $\bar{\mu}_i$ is the sample mean and s_i is the sample standard deviation of the samples drawn from the corresponding posterior densities. From Table 1, we can say that all the μ_i's are not same. The confidence intervals confirm it.

μ_1	(17.4872 , 30.6730)
μ_2	(15.7696 , 25.5825)
μ_3	(17.4892 , 26.1511)

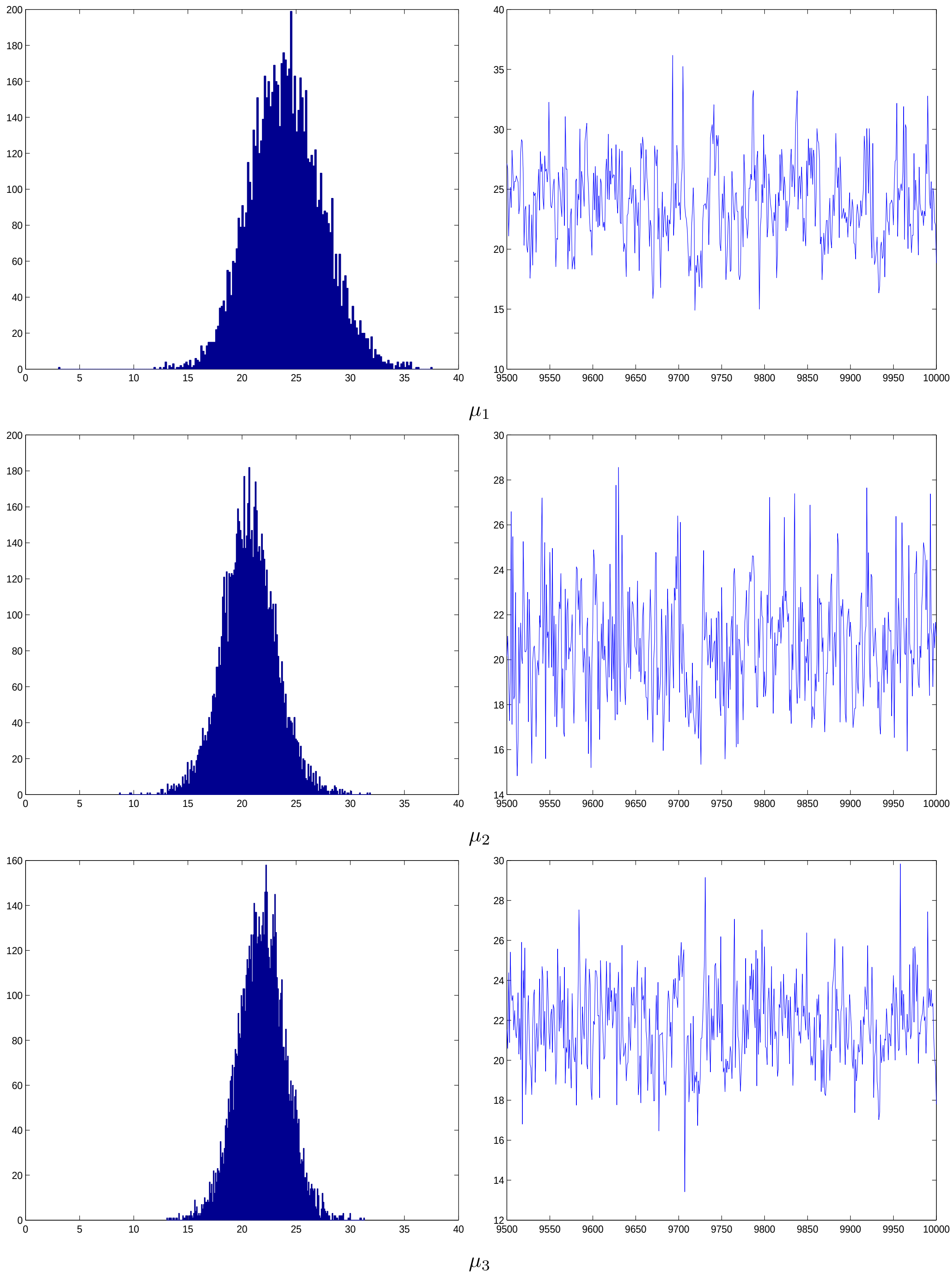
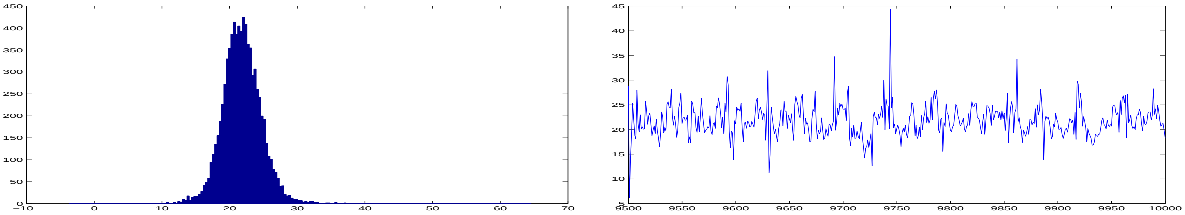
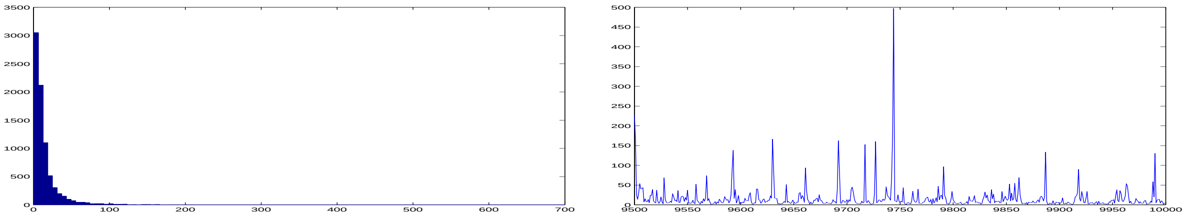


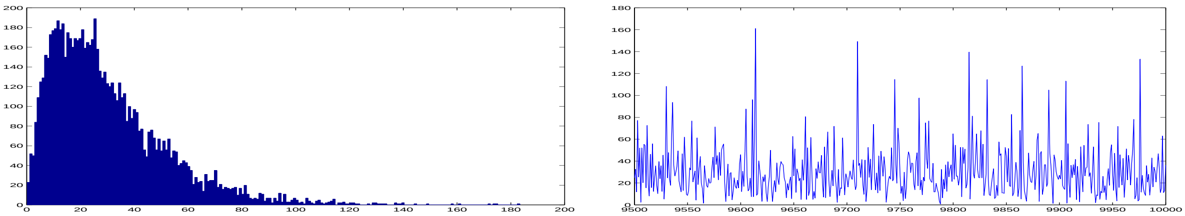
Figure 1 : Simulations of μ_1, μ_2 and μ_3



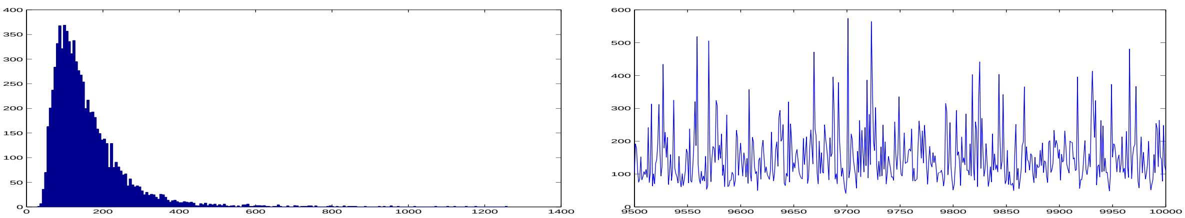
ψ



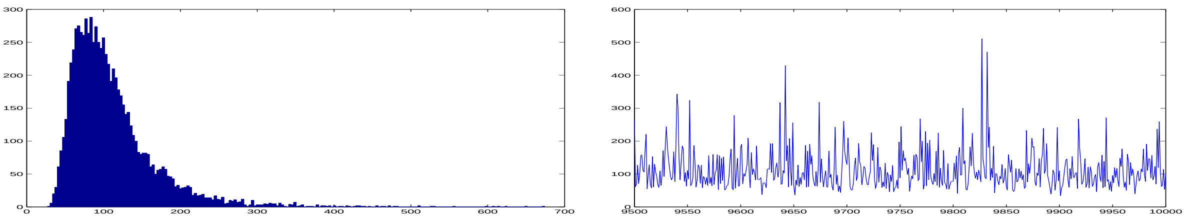
σ_1^2 (truncated above at 700)



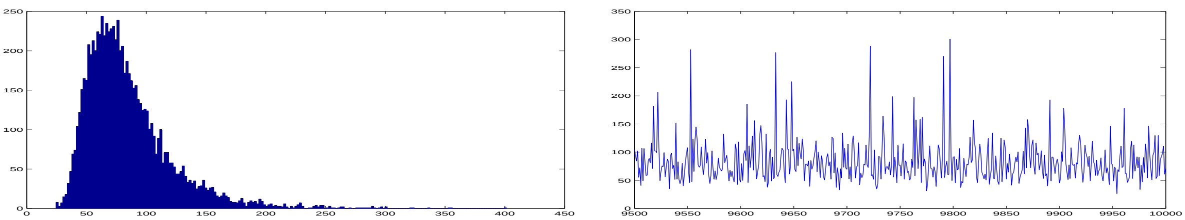
σ_2^2



σ_3^2



τ^2



σ^2

Figure 2 : Simulations of other parameters

Matlab Code

```
%----------------------
% y_ij =  mu_i + epsilon_ij, i=1,...,k;j=1,...,n_i
% epsilon_ij given sigma_i^2 ~ N(0, sigma_i^2)
% mu_i given (psi,tau^2) ~ N(psi,tau^2)
% psi given tau^2 ~ N(psi_0, tau^2/zeta_0)
% sigma_i^2 given sigma^2 ~ IGamma(a_0/2,a_0 sigma^2/2)
% tau^2 ~ IGamma(c_0/2,d_0/2)
% sigma^2 ~ Gamma(f_0/2,g_0/2)
% ------
clear all
close all
rand('state',2); randn('state',2);
%-----------dimensions------------
k = 3;
n1 = 9;
n2 = 11;
n3 = 14;

%-----------data------------------
y1 = [25.8 19.8 28.6 29.4 22.3 33.8 33.8 27.8 29.6]';
y2 = [16.5 23.5 13.5 34.6 16.9 18.8 26.1 18.4 17.2 11.6 20.2]';
y3 = [24.0 29.1 16.0 24.8 27.0 10.9 11.8 23.2 17.7 23.9 24.6 24.0 27.2 23.7]';

%-----------(hyper)parameters------
a0 = 1;
c0 = 1;
d0 = 1;
f0 = 1;
g0 = 0.1;
psi0 = 10;
zeta0 = 0.1;

%------initial value --------------------
mu1 = mean(y1);
mu2 = mean(y2);
mu3 = mean(y3);
sigma1_sq = var(y1);
sigma2_sq = var(y2);
sigma3_sq = var(y3);
sigma_sq = 0.025;
psi = (sum(y1)+sum(y2)+sum(y3))/34;
tau_sq = var([mu1 mu2 mu3]);


% ----------------------------------------
mu1s = [];
mu2s = [];
mu3s = [];
psis = [];
tau_sqs = [];
sigma1_sqs = [];
sigma2_sqs = [];
sigma3_sqs = [];
sigma_sqs = [];
y1bar = mean(y1);
y2bar=mean(y2);
y3bar=mean(y3);


%-----------------------------------------
M = 10000;
h=waitbar(0,'Simulation in progress');
```

```
for r = 1: M



% ------- Distribution of mu_i's given other parameters -------------
mu1 = 1/sqrt(n1/sigma1_sq + 1/tau_sq) * randn + (n1*y1bar  /sigma1_sq+psi/tau_sq)/(n1/sigma1_sq+1/tau_sq );
mu2 = 1/sqrt(n2/sigma2_sq + 1/tau_sq) * randn + (n2*y2bar   /sigma2_sq+psi/tau_sq)/(n2/sigma2_sq+1/tau_sq );
mu3 = 1/sqrt(n3/sigma3_sq + 1/tau_sq) * randn + (n3*y3bar   /sigma3_sq+psi/tau_sq)/(n3/sigma3_sq+1/tau_sq);



% ------- Distribution of psi given other parameters -------------
psi = sqrt(tau_sq/(k+zeta0))*randn +  ( (mu1+mu2+mu3)+psi0*zeta0 ) / (k+zeta0);

% ------- Distribution of tau^2 given other parameters -------------

tau_sq = gamrnd( (c0+k+1)/2, 2/(d0 + (mu1-psi)^2 + (mu2-psi)^2 + (mu3-psi)^2 + zeta0*(psi-psi0)^2) );
tau_sq = 1/tau_sq;

% ------- Distribution of sigma^2_i given other parameters -------------
sigma1_sq = gamrnd((a0+n1)/2,2/(a0*sigma_sq + (n1-1)*(std(y1)^2) + (n2-1)*(std(y2)^2) + (n3-1)*(std(y3)^2) + n1*(mean(y1)-mu1)^2));
sigma2_sq = gamrnd((a0+n2)/2,2/(a0*sigma_sq + (n1-1)*(std(y1)^2) + (n2-1)*(std(y2)^2) + (n3-1)*(std(y3)^2) + n2*(mean(y2)-mu2)^2));
sigma3_sq = gamrnd((a0+n3)/2,2/(a0*sigma_sq + (n1-1)*(std(y1)^2) + (n2-1)*(std(y2)^2) + (n3-1)*(std(y3)^2) + n3*(mean(y3)-mu3)^2));
sigma1_sq = 1/sigma1_sq;
sigma2_sq = 1/sigma2_sq;
sigma3_sq = 1/sigma3_sq;

% ------- Distribution of sigma^2 given other parameters -------------
sigma_sq = gamrnd((f0+k*a0)/2,2/(g0+a0*(1/sigma1_sq+1/sigma2_sq+1/sigma3_sq)));


% ------ Now store the output ----------------
mu1s = [mu1s mu1];
mu2s = [mu2s mu2];
mu3s = [mu3s mu3];
psis = [psis psi];
tau_sqs = [tau_sqs tau_sq];
sigma1_sqs = [sigma1_sqs sigma1_sq];
sigma2_sqs = [sigma2_sqs sigma2_sq];
sigma3_sqs = [sigma3_sqs sigma3_sq];
sigma_sqs = [sigma_sqs sigma_sq];

waitbar(r/M)
end
close(h)


burnin = 2000;
bin = 200;

plot((M-500:M), mu1s(M-500:M))
print -depsc 'C:\abhyuday\gtfiles\isye8843\quiz\midterm\p_pic1.eps'
hist(mu1s(burnin:M), bin)
xlim([0, 40])
print -depsc 'C:\abhyuday\gtfiles\isye8843\quiz\midterm\h_pic1.eps'

plot((M-500:M), mu2s(M-500:M))
print -depsc 'C:\abhyuday\gtfiles\isye8843\quiz\midterm\p_pic2.eps'
hist(mu2s(burnin:M), bin)
xlim([0, 40])
print -depsc 'C:\abhyuday\gtfiles\isye8843\quiz\midterm\h_pic2.eps'

plot((M-500:M), mu3s(M-500:M))
```

```
print -depsc 'C:\abhyuday\gtfiles\isye8843\quiz\midterm\p_pic3.eps'
hist(mu3s(burnin:M), bin)
xlim([0, 40])
print -depsc 'C:\abhyuday\gtfiles\isye8843\quiz\midterm\h_pic3.eps'

plot((M-500:M), psis(M-500:M))
print -depsc 'C:\abhyuday\gtfiles\isye8843\quiz\midterm\p_pic4.eps'
hist(psis(burnin:M), bin)
print -depsc 'C:\abhyuday\gtfiles\isye8843\quiz\midterm\h_pic4.eps'


plot((M-500:M), tau_sqs(M-500:M))
print -depsc 'C:\abhyuday\gtfiles\isye8843\quiz\midterm\p_pic5.eps'
hist(tau_sqs(burnin:M), 600)
xlim([0, 700])
print -depsc 'C:\abhyuday\gtfiles\isye8843\quiz\midterm\h_pic5.eps'


plot((M-500:M), sigma_sqs(M-500:M))
print -depsc 'C:\abhyuday\gtfiles\isye8843\quiz\midterm\p_pic6.eps'
hist(sigma_sqs(burnin:M), bin)
print -depsc 'C:\abhyuday\gtfiles\isye8843\quiz\midterm\h_pic6.eps'

plot((M-500:M), sigma1_sqs(M-500:M))
print -depsc 'C:\abhyuday\gtfiles\isye8843\quiz\midterm\p_pic7.eps'
hist(sigma1_sqs(burnin:M), bin)
print -depsc 'C:\abhyuday\gtfiles\isye8843\quiz\midterm\h_pic7.eps'

plot((M-500:M), sigma2_sqs(M-500:M))
print -depsc 'C:\abhyuday\gtfiles\isye8843\quiz\midterm\p_pic8.eps'
hist(sigma2_sqs(burnin:M), bin)
print -depsc 'C:\abhyuday\gtfiles\isye8843\quiz\midterm\h_pic8.eps'

plot((M-500:M), sigma3_sqs(M-500:M))
print -depsc 'C:\abhyuday\gtfiles\isye8843\quiz\midterm\p_pic9.eps'
hist(sigma3_sqs(burnin:M), bin)
print -depsc 'C:\abhyuday\gtfiles\isye8843\quiz\midterm\h_pic9.eps'


[median(mu1s(burnin:M)) mean(mu1s(burnin:M))]
[median(mu2s(burnin:M)) mean(mu2s(burnin:M))]
[median(mu3s(burnin:M)) mean(mu3s(burnin:M))]

[median(psis(burnin:M)) mean(psis(burnin:M))]

[median(sigma1_sqs(burnin:M)) mean(sigma1_sqs(burnin:M))]
[median(sigma2_sqs(burnin:M)) mean(sigma2_sqs(burnin:M))]
[median(sigma3_sqs(burnin:M)) mean(sigma3_sqs(burnin:M))]

[median(tau_sqs(burnin:M)) mean(tau_sqs(burnin:M))]

[median(sigma_sqs(burnin:M)) mean(sigma_sqs(burnin:M))]

% 95% Credible set
[mean(mu1s(burnin:M))-1.96*std(mu1s(burnin:M)) mean(mu1s(burnin:M))+1.96*std(mu1s(burnin:M)) ]
[mean(mu2s(burnin:M))-1.96*std(mu2s(burnin:M)) mean(mu2s(burnin:M))+1.96*std(mu2s(burnin:M)) ]
[mean(mu3s(burnin:M))-1.96*std(mu3s(burnin:M)) mean(mu3s(burnin:M))+1.96*std(mu3s(burnin:M)) ]
```

2 Problem 2

(a)

$$\pi(\theta|y) = \pi(\mu, \sigma|y_1, \ldots, y_n) \propto \frac{1}{\sigma^{n+1}} \exp\left[-\sum_{i=1}^{n} \left\{ \frac{y_i - \mu}{\sigma} + \exp\left(-\frac{y_i - \mu}{\sigma} \right) \right\} - \frac{\mu^2 + (\log\sigma)^2}{200} \right]$$

(b)

$$q(\theta) = q(\mu, \sigma) = N(\mu_0, s_1^2) \times LN(\log\sigma_0, s_2^2) \propto \frac{1}{s_1 s_2 \sigma} \exp\left[-\frac{1}{2} \left\{ \frac{(\mu - \mu_0)^2}{s_1^2} + \frac{(\log\sigma - \log\sigma_0)^2}{s_2^2} \right\} \right]$$

Let

$$f(\mu^*, \sigma^*, \mu_0, \sigma_0) = \frac{1}{s_1 s_2 \sigma^*} \exp\left[-\frac{1}{2} \left\{ \frac{(\mu^* - \mu_0)^2}{s_1^2} + \frac{(\log\sigma^* - \log\sigma_0)^2}{s_2^2} \right\} \right]$$

and

$$\pi(\mu, \sigma) = \frac{1}{\sigma^{n+1}} \exp\left[-\sum_{i=1}^{n} \left\{ \frac{y_i - \mu}{\sigma} + \exp\left(-\frac{y_i - \mu}{\sigma} \right) \right\} - \frac{\mu^2 + (\log\sigma)^2}{200} \right],$$

then

$$\frac{q(\theta_n|x)}{q(x|\theta_n)} = \frac{f(\mu_n, \sigma_n, \mu_{n+1}, \sigma_{n+1})}{f(\mu_{n+1}, \sigma_{n+1}, \mu_n, \sigma_n)} = \frac{\sigma_{n+1}}{\sigma_n}.$$

Let

$$p = \frac{q(\theta_n|x)}{q(x|\theta_n)} . \frac{\pi(\mu_{n+1}, \sigma_{n+1})}{\pi(\mu_n, \sigma_n)} = \frac{\sigma_{n+1}}{\sigma_n} . \frac{\pi(\mu_{n+1}, \sigma_{n+1})}{\pi(\mu_n, \sigma_n)}$$

and U be a random number between 0 and 1. Then $\theta_{n+1} = x$ if $U < p \wedge 1$, or equivalently, $\log U < \log p \wedge 0$ where $\log p = \log\sigma_{n+1} - \log\sigma_n + lp(\mu_n, \sigma_n) - lp(\mu_{n+1}, \sigma_{n+1})$ and

$$lp(\mu, \sigma) = -\log\pi(\mu, \sigma) = (n+1)\log\sigma + \sum_{i=1}^{n} \left\{ \frac{y_i - \mu}{\sigma} + \exp\left(-\frac{y_i - \mu}{\sigma} \right) \right\} + \frac{\mu^2 + (\log\sigma)^2}{200}.$$

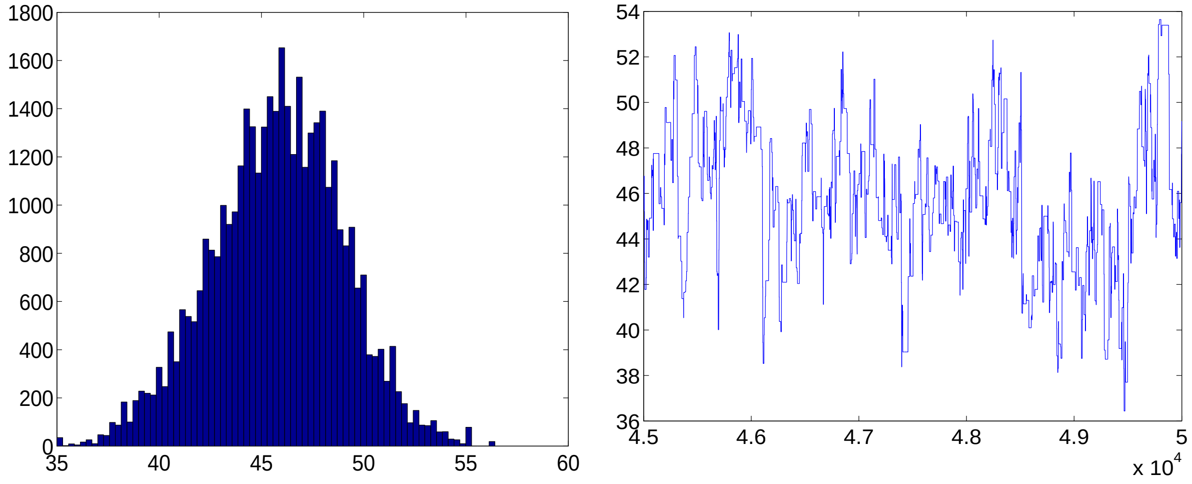
The histograms of the posterior distributions of μ and σ and the last 5000 simulations of the chains are given below in Figure 3. The mean and medians of the posterior distributions are given below.

Parameter	Mean	Median
μ	45.7523	45.8583
σ	21.7073	21.4893

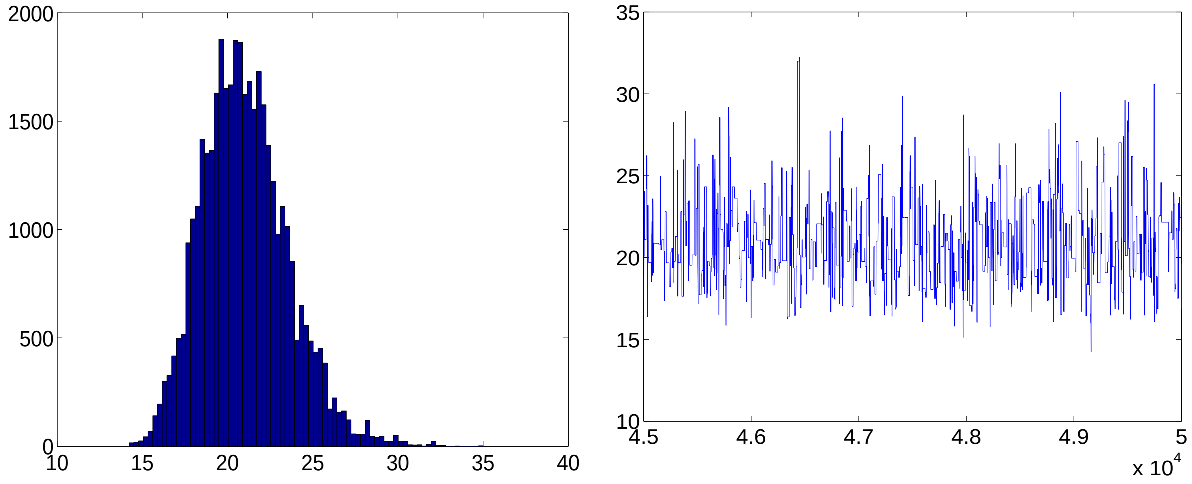
(c)

$$P(y* \geq 410|y) = 1 - F(y^*|\mu^*, \sigma^*) = 1 - \exp\left\{ -\exp\left\{ -\frac{y - \mu^*}{\sigma^*} \right\} \right\} = 4.3729 \times 10^{-8},$$

where $\mu^* = 45.8583$ and $\sigma^* = 21.4893$, as obtained from part (b).



μ_1



μ_2

Figure 3 : Simulations of μ and σ

Matlab Code

```
rand('seed',1); randn('seed',1)
n = 48;

data=[
1951 154
1952 49.6
1953 46.7
1954 58.3
1955 70.5
1956 90
1957 70.1
1958 105.7
1959 37.4
1960 40.8
1961 34.7
1962 58.9
1963 72.2
1964 30
1965 71.6
1966 100
1967 33.7
1968 49.9
1969 56.1
1970 142.3
1971 28.6
1972 54.8
1973 74.1
1974 60
1975 50.9
1976 38.6
1977 53.4
1978 132.5
1979 50.7
1980 40.8
1981 84.3
1982 38.8
1983 27.4
1984 67
1985 118.7
1986 23.2
1987 55
1988 67.9
1989 87.3
1990 89
1991 98.7
1992 47.1
1993 71.6
1994 83.6
1995 44.3
1996 41.2
1997 35.9
1998 44.3];
y=data(:,2);

cov=[2.5 0; 0 1];
mu = [];
sigma = [];
mu_old = 0;
sigma_old = 1;
M = 50000;

h=waitbar(0,'Simulation in progress');
```

```
for r = 1: M

   u = rand;
   lsigma_old = log(sigma_old);
   theta = mvnrnd([mu_old lsigma_old],cov);
   mu_new = theta(1);
   sigma_new = exp(theta(2));
   logp = log(sigma_old) - log(sigma_new) + lpai(y,mu_old,sigma_old) - lpai(y,mu_new,sigma_new);
   if log(u) < min(logp,0)
       sigma_old = sigma_new;
       mu_old = mu_new;
   end
   mu = [mu mu_old];
   sigma = [sigma sigma_old];
waitbar(r/M)
end
close(h)


burnin = 10000;
bin = 75;

plot((M-5000:M), mu(M-5000:M))
print -depsc 'C:\abhyuday\gtfiles\isye8843\quiz\midterm\problem2\p_pic1.eps'
hist(mu(burnin:M), bin)
print -depsc 'C:\abhyuday\gtfiles\isye8843\quiz\midterm\problem2\h_pic1.eps'


plot((M-5000:M), sigma(M-5000:M))
print -depsc 'C:\abhyuday\gtfiles\isye8843\quiz\midterm\problem2\p_pic2.eps'
hist(sigma(burnin:M), bin)
print -depsc 'C:\abhyuday\gtfiles\isye8843\quiz\midterm\problem2\h_pic2.eps'

s = median(sigma(burnin:M))
m = median(mu(burnin:M))

F=exp(-exp((m-410)/s));
prob=1-F


function lpai = lpai(y,mu,sigma)
n=48;
lpai = sum((y-mu)/sigma + exp((mu-y)/sigma )) + (mu^2 + log(sigma)^2)/200 + (n+1)*log(sigma);
```

3 Problem 3

The histograms are given below.

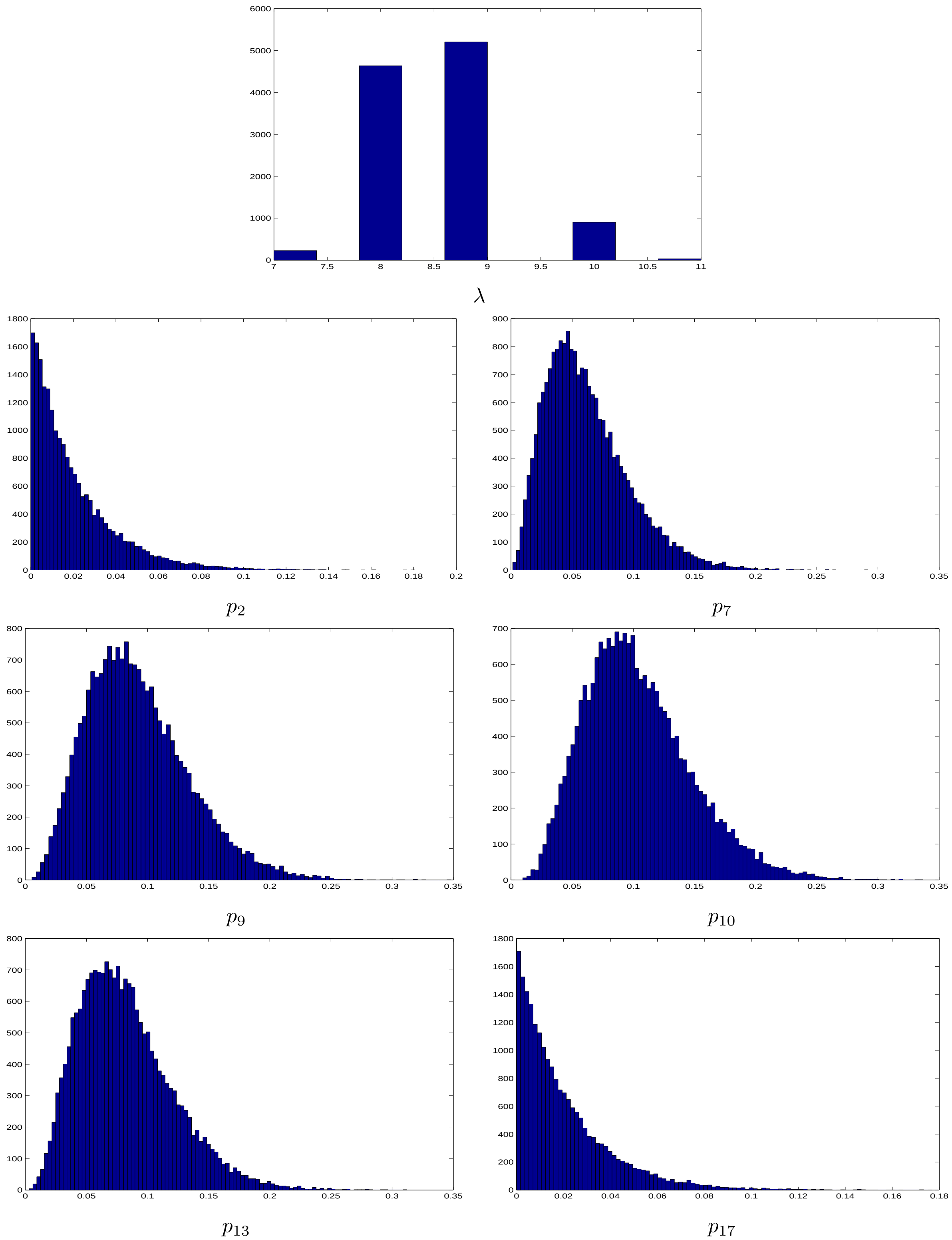


Figure 4 : Histograms

The posterior statistics are given below.

	mean	sd	MCerror	val2.5pc	median	val97.5pc	start	sample
λ	8.63	0.6691	0.005293	8.0	9.0	10.0	1001	21000
p_1	0.02029	0.01981	1.356×10^{-4}	5.643×10^{-4}	0.01429	0.07335	1001	21000
p_2	0.02043	0.01993	1.249×10^{-4}	5.415×10^{-4}	0.0144	0.07451	1001	21000
p_3	0.02021	0.01981	1.202×10^{-4}	5.329×10^{-4}	0.01424	0.07341	1001	21000
p_4	0.04088	0.02805	1.909×10^{-4}	0.004943	0.03476	0.1113	1001	21000
p_5	0.04117	0.02806	1.92×10^{-4}	0.005228	0.03516	0.1109	1001	21000
p_6	0.06111	0.03412	2.384×10^{-4}	0.01322	0.05494	0.1424	1001	21000
p_7	0.06182	0.0341	2.308×10^{-4}	0.01318	0.05573	0.1436	1001	21000
p_8	0.0902	0.04173	3.104×10^{-4}	0.02628	0.08431	0.1863	1001	21000
p_9	0.09181	0.04172	3.091×10^{-4}	0.02668	0.08622	0.1874	1001	21000
p_{10}	0.1036	0.04336	2.653×10^{-4}	0.03511	0.09783	0.2026	1001	21000
p_{11}	0.122	0.04652	3.469×10^{-4}	0.04692	0.1169	0.2266	1001	21000
p_{12}	0.102	0.04303	2.918×10^{-4}	0.03512	0.09649	0.2006	1001	21000
p_{13}	0.0818	0.03864	2.621×10^{-4}	0.02343	0.0764	0.1713	1001	21000
p_{14}	0.0609	0.03376	2.32×10^{-4}	0.01285	0.05474	0.1417	1001	21000
p_{15}	0.04101	0.02804	2.037×10^{-4}	0.005344	0.03487	0.111	1001	21000
p_{16}	0.02023	0.01993	1.486×10^{-4}	5.049×10^{-4}	0.01414	0.07467	1001	21000
p_{17}	0.02057	0.0199	1.403×10^{-4}	5.308×10^{-4}	0.01462	0.07348	1001	21000