Chapter 2

The Essay

The following is copied from
The Philosophical Transactions of the Royal Society
Volume 53
1763

p370

LII. *An Essay towards solving a problem in the Doctrine of Chances. By the late Rev. Mr.* Bayes, *F.R.S. communicated by Mr.* Price, *in a Letter to* John Canton, *A.M. F.R.S.*

Dear Sir,

Read Dec. 23, 1763

I Now send you an essay which I have found among the papers of our deceased friend Mr. Bayes, and which, in my opinion, has great merit, and well deserves to be preserved. Experimental philosophy, you will find, is nearly interested in the subject of it; and on this account there seems to be particular reason for thinking that a communication of it to the Royal Society cannot be improper.

He had, you know, the honour of being a member of that illustrious Society, and was much esteemed by many in it as a very able mathematician. In an introduction which he has writ to this Essay, he says, that his design at first in thinking on the subject of it was, to find out a method by which we might judge concerning the probability that an event has to happen, in given circumstances, upon supposition that we know nothing concerning it but that, under the same circumstances, *(page 371)* it has happened a certain number of times, and failed a certain other number of times. He adds, that he soon perceived that it would not be very difficult to do this, provided some rule could be found according to which we ought to estimate the chance that the probability for the happening of an event perfectly unknown, should lie between any two named degrees of probability, antecedently to any experiments made about it; and that it appeared to him that the rule must be to suppose the chance the same that it should lie between any two equidifferent degrees; which, if it were allowed, all the rest might be easily calculated in the common method of proceeding in the doctrine of chances. Accordingly, I find among his papers a very ingenious solution of this problem in this way. But he afterwards considered, that the postulate on which he had argued might not perhaps be looked upon by all as reasonable; and therefore he chose to lay down in another form the proposition in which he thought the solution of the problem is contained, and in a scholium to subjoin the reasons why he thought so, rather than to take into his mathematical reasoning any thing that might admit dispute. This, you will observe, is the method which he has pursued in this essay.

Every judicious person will be sensible that the problem now mentioned is by no means merely a curious speculation in the doctrine of chances, but necessary to be solved in order to a sure foundation for all our reasonings concerning past facts, and what is likely to be hereafter. Common sense is indeed sufficient to shew us that, from the observation of what has in former instances been the consequence of a certain cause *(page 372)* or action, one may make a judgement what is likely to be the consequence of it another time, and that the larger number of experiments we have to support a conclusion, so much the more reason we have to take it for granted. But it is certain that we cannot determine, at least not to any nicety, in what degree repeated experiments confirm a conclusion without the particular discussion of the beforementioned problem; which, therefore, is a necessary to be considered by any one who would give a clear account of the strength of analogical or inductive reasoning; concerning, which at present, we seem to know little more than that it does sometimes in fact convince us, and at other times not; and that, as it is the means of [a]cquainting us with many truths, of which otherwise we must have been ignorant; so it is, in all probability, the source of many errors, which perhaps might in some measure be avoided, if the force that this sort of reasoning ought to have with us were more distinctly and clearly understood.

These observations prove that the problem enquired after in this essay is no less important than it is curious. It may be safely added, I fancy, that it is also a problem that has never before been solved. Mr. De Moivre, indeed, the great improver of this part of mathematics, has in his Laws of chance[1], after Bernoulli, and to a greater degree of exactness, given rules to find the probability there is, that if a very great number of trials be made concerning any event, *(page 373)* the proportion of the number of times it will happen, to the number of times it will fail in those trials, should differ less than by small assigned limits from the proportion of the probability of its happening to the probability of its failing in one single trial. But I know of no person

---

[1] See Mr. De Moivre's *Doctrine of Chances*, p. 243, &c. He has omitted the demonstrations of his rules, but these have been since supplied by Mr. Simpson at the conclusion of his treatise on *The Nature and Laws of Chance.*

who has shewn how to deduce the solution of the converse problem to this; namely, 'the number of times an unknown event has happened and failed being given, to find the chance that the probability of its happening should lie somewhere between any two named degrees of probability'. What Mr. De Moivre has done therefore cannot be thought sufficient to make the consideration of this point unnecessary: especially, as the rules he has given are not pretended to be rigorously exact, except on supposition that the number of trials made are infinite; from whence it is not obvious how large the number of trials must be in order to make them exact enough to be depended on in practice.

Mr. De Moivre calls the problem he has thus solved, the hardest that can be proposed on the subject of chance. His solution he has applied to a very important purpose, and thereby shewn that those are much mistaken who have insinuated that the Doctrine of Chances in mathematics is of trivial consequence, and cannot have a place in any serious enquiry[1]. The purpose I mean is, to shew what reason we have for believing that there are in the constitution of things fixt laws according to which events happen, and that, therefore, the frame of the world must be *(page 374)* the effect of the wisdom and power of an intelligent cause; and thus to confirm the argument taken from final causes for the existence of the Deity. It will be easy to see that the converse problem solved in this essay is more directly applicable to this purpose; for it shews us, with distinctness and precision, in every case of any particular order or recurrency of events, what reason there is to think that such recurrency or order is derived from stable causes or regulations in nature, and not from any of the irregularities of chance.

The two last rules in this essay are given without the deductions of them. I have chosen to do this because these deductions, taking up a good deal of room, would swell the essay too much; and also because these rules, though of considerable use, do not answer the purpose for which they are given as perfectly as could be wished. They are however ready to be produced, if a communication of them should be thought proper. I have in some places writ short notes, and to the whole I have added an application of the rules in the essay to some particular cases, in order to convey a clearer idea of the nature of the problem and to shew how far a solution of it has been carried.

I am sensible that your time is so much taken up that I cannot reasonably expect that you should minutely examine every part of what I now send you. Some of the calculations, particularly in the appendix, no one can make without a good deal of labour. I have taken so much care about them, that I

[1] See his Doctrine of Chances, p. 252, &c.

believe there can be no material error in any of them; but should there be any such errors, I am the only person who ought to be considered as answerable for them.

(page 375)

Mr. Bayes has thought fit to begin his work with a brief demonstration of the general laws of chance. His reason for doing this, as he says in his introduction, was not merely that his reader might not have the trouble of searching elsewhere for the principles on which he has argued, but because he did not know whither to refer him for a clear demonstration of them. He has also made an apology for the peculiar definition he has given of the word chance or probability. His design herein was to cut off all dispute about the meaning of the word, which in common language is used in different senses by persons of different opinions, and according as it is applied to past or future facts. But whatever different senses it may have, all (he observes) will allow that an expectation depending on the truth on any past fact, or the happening of any future event, ought to be estimated so much the more valuable as the fact is more likely to be true, or the event more likely to happen. Instead therefore, of the proper sense of the word probability, he has given that which all will allow to be its proper measure in every case where the word is used. But it is time to conclude this letter. Experimental philosophy is indebted to you for several discoveries and improvements; and, therefore, I cannot help thinking that there is a peculiar propriety in directing to you the following essay and appendix. That your enquiries may be rewarded with many further successes, and that you may enjoy every *[sic]* valuable blessing, is the sincere wish of, Sir,

Your very humble servant,
Richard Price

Newington-Green,
Nov. 10 1763.

[376]

PROBLEM.

> *Given* the number of times in which an unknown event has happened and failed: *Required* the chance that the probability of its happening in a single trial lies somewhere between any two degrees of probability that can be named.

SECTION I.

DEFINITION 1. Several events are *inconsistent*, when if one of them happens, none of the rest can.

2. Two events are *contrary* when one, or other of them must; and both together cannot happen.

3. An event is said to *fail*, when it cannot happen; or, which comes to the same thing, when its contrary has happened.

4. An event is said to be determined when it has either happened or failed.

5. The *probability of any event* is the ratio between the value at which an expectation depending on the happening of the event ought to be computed, and the value of the thing expected upon it's happening.

6. By *chance* I mean the same as probability.

7. Events are independent when the happening of any one of them does neither increase nor abate the probability of the rest.

PROP. 1.

When several events are inconsistent the probability of the happening of one or other of them is the sum of the probabilities of each of them.

Suppose there be three such events, and which ever of them happens I am to receive N, and that the probability of the 1st, 2d, and 3d are respectively $\frac{a}{\text{N}}, \frac{b}{\text{N}}, \frac{c}{\text{N}}$. Then (by the definition of probability) the value of my expectation from the 1st will be a, from the 2d b, and from the 3d c. Wherefore the value of my expectations from all three will be $a + b + c$. But the sum of my expectations from all three is in this case an expectation of receiving N upon the happening of one or other of them. Wherefore (by definition 5) the probability of one or other of them is $\frac{a+b+c}{\mathrm{N}}$ or $\frac{a}{\mathrm{N}}+\frac{b}{\mathrm{N}}+\frac{c}{\mathrm{N}}$. The sum of the probabilities of each of them.

Corollary. If it be certain that one or other of the three events must happen, then $a + b + c = \mathrm{N}$. For in this case all the expectations together amounting to a certain expectation of receiving N, their values together must be equal to N. And from hence it is plain that the probability of an event added to the probability of its failure (or of its contrary) is the ratio of equality. For these are two inconsistent events, one of which necessarily happens. Wherefore if the probability of an event is $\frac{\mathrm{P}}{\mathrm{N}}$ that of it's failure will be $\frac{\mathrm{N}-\mathrm{P}}{\mathrm{N}}$.

PROP. 2.

If a person has an expectation depending on the happening of an event, the probability of the event is to the probability of its failure as his loss if it fails to his gain if it happens.

Suppose a person has an expectation of receiving N, depending on an event the probability of which is $\frac{\mathrm{P}}{\mathrm{N}}$. *(page 378)* Then (by definition 5) the value of his expectation is P, and therefore if the event fail, he loses that which in value is P; and if it happens he receives N, but his expectation ceases. His gain therefore is N – P. Likewise since the probability of the event is $\frac{\mathrm{P}}{\mathrm{N}}$, that of its failure (by corollary prop. 1) is $\frac{\mathrm{N}-\mathrm{P}}{\mathrm{N}}$. But $\frac{\mathrm{P}}{\mathrm{N}}$ is to $\frac{\mathrm{N}-\mathrm{P}}{\mathrm{N}}$ as P is to N – P, i.e. the probability of the event is to the probability of it's failure, as his loss if it fails to his gain if it happens.

PROP. 3.

The probability that two subsequent events will both happen is a ratio compounded of the probability of the 1st, and the probability of the 2d on supposition that the 1st happens.

Suppose that, if both events happen, I am to receive N, that the probability both will happen is $\frac{\mathrm{P}}{\mathrm{N}}$, that the 1st will is $\frac{a}{\mathrm{N}}$ (and consequently that the 1st will not is $\frac{\mathrm{N}-a}{\mathrm{N}}$) and that the 2d will happen upon supposition that the 1st does is $\frac{b}{\mathrm{N}}$. Then (by definition 5) P will be the value of my expectation, which will become b if the 1st happens. Consequently if the 1st happens, my gain by it is b – P, and if it fails my loss is P. Wherefore, by the foregoing proposition, $\frac{a}{\text{N}}$ is to $\frac{\text{N}-\text{a}}{\text{N}}$ i.e. a is to N – a as P is to b – P. Wherefore (componendo inversè) a is to N as P is to b. But the ratio of P to N is compounded of the ratio of P to b, and that of b to N. Wherefore the *(page 379)* same ratio of P to N is compounded of the ratio of a to N and that of b to N, i.e. the probability that the two subsequent events both happen is compounded of the probability of the 1st and the probability of the 2d on supposition the 1st happens.

Corollary. Hence if of two subsequent events the probability of the 1st be $\frac{a}{\text{N}}$, and the probability of both together be $\frac{\text{P}}{\text{N}}$, then the probability of the 2d on supposition the 1st happens is $\frac{\text{P}}{a}$.

PROP. 4.

If there be two subsequent events to be determined every day, and each day the probability of the 2d is $\frac{b}{\text{N}}$ and the probability of both $\frac{\text{P}}{\text{N}}$, and I am to receive N if both the events happen the 1st day on which the 2d does; I say, according to these conditions, the probability of my obtaining N is $\frac{\text{P}}{b}$. For if not, let the probability of my obtaining N be $\frac{x}{\text{N}}$ and let y be to x as N – b to N. Then since $\frac{x}{\text{N}}$ is the probability of my obtaining N (by definition 1) x is the value of my expectation. And again, because according to the foregoing conditions the 1st day I have an expectation of obtaining N depending on the happening of both the events together, the probability of which is $\frac{\text{P}}{\text{N}}$, the value of this expectation is P. Likewise, if this coincident should not happen I have an expectation of being reinstated in my former circumstances, i.e. of receiving that which in value is x depending *(page 380)* on the failure of the 2d event the probability of which (by cor. prop. 1) is $\frac{\text{N}-b}{\text{N}}$ or $\frac{y}{x}$, because y is to x as N – b to N. Wherefore since x is the thing expected and $\frac{y}{x}$ the probability of obtaining it, the value of this expectation is y. But these two last expectations together are evidently the same with my original expectation, the value of which is x, and therefore P + y = x. But y is to x as N – b is to

N. Wherefore x is to P as N is to b and $\frac{x}{\mathrm{N}}$ (the probability of my obtaining N) is $\frac{\mathrm{P}}{b}$.

Cor. Suppose after the expectation given me in the foregoing proposition, and before it is at all known whether the 1st event has happened or not, I should find that the 2d event has happened; from hence I can only infer that the event is determined on which my expectation depended, and have no reason to esteem the value of my expectation either greater or less than it was before. For if I have reason to think it less, it would be reasonable for me to give something to be reinstated in my former circumstances, and this over and over again as often as I should be informed that the 2d event had happened, which is evidently absurd. And the like absurdity plainly follows if you say I ought to set a greater value on my expectation than before, for then it would be reasonable for me to refuse something if offered me upon condition I would relinquish it, and be reinstated in my former circumstances; and this likewise over and over again as often as (nothing being known concerning the 1st event) it should appear that the 2d had happened. Notwithstanding therefore this discovery that the 2d *(page 381)* event has happened, my expectation ought to be esteemed the same in value as before, i.e. x, and consequently the probability of my obtaining N is (by definition 5) still $\frac{x}{\mathrm{N}}$ or $\frac{\mathrm{P}}{b}$[1]. But after this discovery the probability of my obtaining N is the probability that the 1st of two subsequent events has happened upon the supposition that the 2d has, whose probabilities were as before specified. But the probability that an event has happened is the same as the probability I have to guess right if I guess it has happened. Wherefore the following proposition is evident.

PROP. 5.

If there be two subsequent events, the probability of the 2d $\frac{b}{\mathrm{N}}$ and the probability of both together $\frac{\mathrm{P}}{\mathrm{N}}$, and it being 1st discovered that the 2d event

[1] What is here said may perhaps be a little illustrated by considering that all that can be lost by the happening of the 2d event is the chance I should have had of being reinstated in my former circumstances, if the event on which my expectation depended had been determined in the manner expressed in the proposition. But this chance is always as much *against* me as it is *for* me. If the 1st event happens, it is *against* me, and equal to the chance for the 2d event's failing. If the 1st event does not happen, it is *for* me, and equal also to the chance for the 2d event's failing. The loss of it, therefore, can be no disadvantage.

has happened, from hence I guess that the 1st event has also happened, the probability I am in the right is $\frac{P}{b}$[1].

(*page 382*) PROP. 6.

The probability that several independent events shall all happen is a ratio compounded of the probabilities of each.

For from the nature of independent events, the probability that any one happens is not altered by the happening or failing of any of the rest, and consequently the probability that the 2d event happens on supposition the 1st does is the same with its original probability; but the probability that any two events happen is a ratio compounded of the probability of the 1st event, and the probability of the 2d on supposition the 1st happens by prop. 3. Wherefore the probability that any two independent events both happen is a ratio compounded of the probability of the 1st and the probability of the 2d. And in like manner considering the 1st and 2d event together as one event; the probability that three independent events all happen is a ratio compounded of the probability that the two 1st both happen and the probability of the 3d. And thus you (*page 383*) may proceed if there be ever so many such events; from whence the proposition is manifest.

Cor. 1. If there be several independent events, the probability that the 1st happens the 2d fails, the 3d fails and the 4th happens, &c. is a ratio compounded of the probability of the 1st, and the probability of the failure of the 2d, and the probability of the failure of the 3d, and the probability of the 4th, &c. For the failure of an event may always be considered as the happening of its contrary.

Cor. 2. If there be several independent events, and the probability of each one be *a*, and that of its failure be *b*, the probability that the 1st happens and the 2d fails, and the 3d fails and the 4th happens, &c. will be *a b b a,* &c.

[1] What is proved be Mr. Bayes in this and the preceding proposition is the same with the answer to the following question. What is the probability that a certain event, when it happens, will be accompanied with another to be determined at the same time? In this case, as one of the events is given, nothing can be due for the expectation of it; and, consequently, the value of an expectation depending on the happening of both events must be the same with the value of an expectation depending on the happening of one of them. In other words; the probability that, when one of two events happens, the other will, is the same with the probability of this other. Call x then the probability of this other, and if $\frac{b}{N}$ be the probability of the given event, and $\frac{p}{N}$ the probability of both, because $\frac{p}{N} = \frac{b}{N} \times x$, $x = \frac{p}{b}$ = the probability mentioned in these propositions.

For, according to the algebraic way of notation, if *a* denote any ratio and *b* another, *a b b a* denotes the ratio compounded of the ratios *a*, *b*, *b*, *a*. This corollary therefore is only a particular case of the foregoing.

Definition. If in consequence of certain data there arises a probability that a certain event should happen, its happening or failing, in consequence of these data, I call it's happening or failing in the 1st trial. And if the same data be again repeated, the happening or failing of the event in consequence of them I call its happening or failing in the 2d trial; and so on as often as the same data are repeated. And hence it is manifest that the happening or failing of the same event in so many diffe- trials is in reality the happening or failing of so many distinct independent events exactly familiar to each other.

(page 383) PROP. 7.

If the probability of an event be *a*, and that of its failure be *b* in each single trial, the probability of its happening *p* times, and failing *q* times in *p* + *q* trials is E $a^p\ b^q$ if E be the coefficient of the term in which occurs $a^p\ b^q$ when the binomial $\overline{a+b}|^{p+q}$ is expanded.

For the happening or failing of an event in different trials are so many independent events. Wherefore (by cor. 2. prop. 6.) the probability that the event happens the 1st trial, fails the 2d and 3d, and happens the 4th, fails the 5th, &c. (thus happening and failing till the number of times it happens be *p* and the number it fails be *q* is *a b b a b* &c. till the number of *a*'s be *p* and the number of *b*'s be *q*, that is; 'tis $a^p\ b^q$. In like manner if you consider the event as happening *p* times and failing *q* times in any other particular order, the probability for it is $a^p\ b^q$; but the number of different orders according to which an event may happen or fail, so as in all to happen *p* times and fail *q*, in *p* + *q* trials is equal to the number of permutations that *a a a a b b b* admit of when the number of *a*'s is *p*, and the number of *b*'s is *q*. And this number is equal to E, the coefficient of the term in which occurs $a^p\ b^q$ when $\overline{a+b}|^{p+q}$ is expanded. The event therefore may happen *p* times and fail *q* in *p* + *q* trials E different ways and no more, and its happening and failing these several different ways are so many inconsistent events, the probability for each of which is $a^p\ b^q$, and therefore by *(page 385)* prop. 1. the probability that some way or other it happens *p* times and fails *q* times in $p+q$ trials is E $a^p\ b^q$.

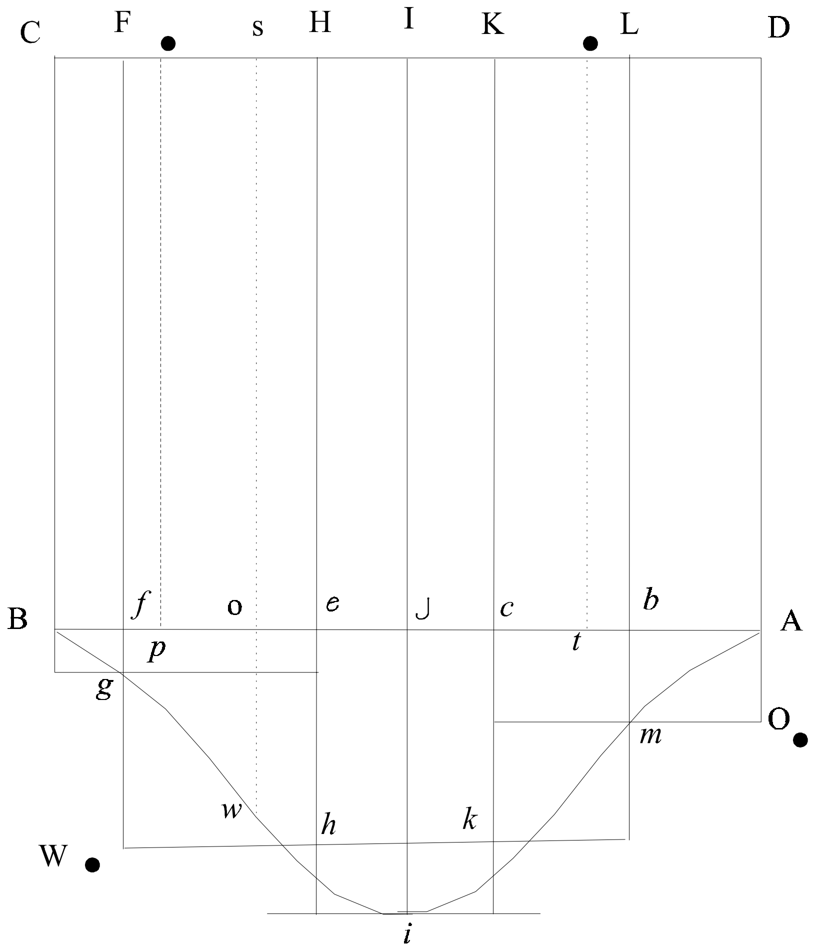
SECTION II.

Postulate. 1. I Suppose the square table or plane A B C D to be so made and levelled, that if either of the balls *o* or W be thrown upon it, there shall be the same probability that it rests upon any one equal part of the plane as another, and that it must necessarily rest somewhere upon it.

2. I suppose that the ball W shall be 1st thrown, and through the point where it rests a line *o s* shall be drawn parallel to A D, and meeting C D and A B in *s* and *o*; and that afterwards the ball O shall be thrown *p* + *q* or *n* times, and that its resting between A D and *o s* after a single throw be called the happening of the event M in a single trial. These things supposed,

Lem. 1. The probability that the point *o* will fall between any two points in the line A B is the ratio of the distance between the two points to the whole line A B.

Let any two points be named, as *f* and *b* in the line A B, and through them parallel to A D draw *f F*, *b* L meeting C D in F and L. Then if the rectangles C *f*, F *b*, L A are *(page 386)* commensurable to each other, they may each be divided into the same equal parts, which being done, and the ball W thrown, the probability it will rest somewhere upon any number of these equal parts will be the sum of the probabilities it has to rest upon each one of them, because its resting upon any different parts of the plane A C are so many inconsistent events; and this sum, because the probability it should rest upon any one equal part as another is the same, is the probability it should rest upon any one equal part multiplied by the number of parts. Consequently, the probability there is that the ball W should rest somewhere upon *(page 387)* F *b* is the probability it has to rest upon one equal part multiplied by the number of equal parts in F *b*; and the probability it rests somewhere upon C *f* or L A, i.e. that it dont rest upon F *b* (because it must rest somewhere upon A C) is the probability it rests upon one equal part multiplied by the number of equal parts in C *f*, L A taken together. Wherefore, the probability it rests upon F *b* is to the probability it dont as the number of equal parts in F *b* is to the number of equal parts in C *f*, L A together, or as F *b* to C *f*, L A together, or as *f b* to B *f* A *b* together. Wherefore the probability it rest upon F *b* is to the probability it dont as *f b* to B *f*, A *b* together. And (*componendo inverse*) the probability it rests upon F *b* is to the probability it rests upon F *b* added to the probability it dont, as *f b* to A B, or as the ratio of *f b* to A B to the ratio of A B to A B. But the probability of any event added to the probability of its failure is the ratio of equality; wherefore, the probability it rest upon F *b* is to the ratio of equality as the ratio of *f b* to A B to the ratio of A B to A B, or the ratio of equality; and therefore the probability it rest upon F *b* is the ratio of *f b* to A B. But *ex hypothesi* according as the ball W falls



upon F *b* or not the point *o* will lie between *f* and *b* or not and therefore the probability the point *o* will lie between *f* and *b* is the ratio of *f b* to A B.

Again; if the rectangles C *f*, F *b*, L A are not commensurable, yet the last mentioned probability can be neither greater nor less than the ratio of *f b* to A B; for, if it be less, let it be the ratio of *f c* to A B, and upon the line *f b* take the points *p* and *t*, so that *p t* shall be greater than *f c*, and the three lines B *p*, *p t*, *t* A commensurable (which it is evident may be always done by dividing A B into equal parts less than half *c b*, and taking *p* and *t* the nearest points of division to *f* and *c* that lie upon *f b*). Then because B *p*, *p t*, *t* A are commensurable, so are the rectangles C *p*, D *t*, and that upon *p t* compleating the square A B. Wherefore, by what has been said, the probability that the point *o* will lie between *p* and *t* is the ratio of *p t* to A B. But if it lies between *p* and *t* it must lie between *f* and *b*. Wherefore, the probability it should lie between *f* and *b* cannot be less than the ratio of *p t* to A B, and therefore must be greater than the ratio of *f c* to A B (since *p t* is greater than *f c*). And after the same manner you may prove that the forementioned probability cannot be greater than the ratio of *f b* to A B, it must therefore be the same.

Lem. 2. The ball W having been thrown, and the line *o s* drawn, the probability of the event M in a single trial is the ratio of A *o* to A B.

For, in the same manner as the foregoing lemma, the probability that the ball *o* being thrown shall *(page 388)* rest somewhere upon D *o* or between A D and *s o* is the ratio of A *o* to A B. But the resting of the ball *o* between A D and *s o* after a single throw is the happening of the event M in a single trial. Wherefore the lemma is manifest.

PROP. 8.

If upon B A you erect the figure B *g h i k m* A whose property is this, that (the base B A being divided into any two parts, as A B, and B *b* and at the point of division *b* a perpendicular being erected and terminated by the figure in *m*; and *y*, *x*, *r* representing respectively the ratio of *b m*, A *b*, and B *b* to A B, and E being the coefficient of the term in which occurs $a^p\ b^q$ when the binomial $\overline{a+b}|^{p+q}$ is expanded) $y = \mathrm{E}\ x^p\, r^q$. I say that before the ball W is thrown, the probability the point *o* should fall between *f* and *b*, any two points named in the line A B and withall that the event M should happen *p* times and fail *q* in *p* + *q* trials, is the ratio of *f g h i k m b*, the part of the figure B *g h i k m* A intercepted between the perpendiculars *f g*, *b m* raised upon the line A B, to C A the square upon A B.

DEMONSTRATION

For if not; 1st let it be the ratio of D a figure greater than *f g h i k m b* to C A, and through the points *e d c* draw perpendiculars to *f b* meeting the curve A *m i g* B in *h*, *i*, *k*; the point *d* being so placed that *d i* shall be the longest of the *(page 389)* perpendiculars terminated by the line *f b*, and the curve A *m i g* B; and the points *e*, *d*, *c* being so many and so placed that the rectangles, *b k*, *c i*, *e i*, *f h* taken together shall differ less from *f g h i k m b* than D does; all which may be easily done by the help of the equation of the curve, and the difference between D and the figure *f g h i k m b* given. Then since *d i* is the longest of the perpendicular ordinates that insist upon *f b*, the rest will gradually decrease as they are farther and farther from it on each side, as appears from the construction of the figure, and consequently *e h* is greater than *g f* or any other ordinate that insists upon *e f*.

Now if A *o* were equal to A *e*, then by lem. 2. the probability of the event M in a single trial would be the ratio of A *e* to A B, and consequently by cor. Prop. 1. the probability of it's failure would be the ratio of B *e* to A B. Wherefore, if *x* and *r* be the two forementioned ratios respectively, by Prop. 7. the probability of the event M happening *p* times and failing *q* in *p* + *q* trials would be E $x^p\, r^q$. But *x* and *r* being respectively the ratios of A *e* to A

B and B *e* to A B, if *y* is the ratio of *e h* to A B, then, by construction of the figure A *i* B, $y = E\ x^p\ r^q$. Wherefore, if A *o* were equal to A *e* the probability of the event M happening *p* times and failing *q* in *p* + *q* trials would be *y*, or the ratio of *e h* to A B. And if A *o* were equal to A *f* or were any mean between A *e* and A *f*, the last mentioned probability for the same reasons would be the ratio of *f g* or some other of the ordinates insisting upon *e f*, to A B. But *e h* is the greatest of all the ordinates that insist upon *e f*. Wherefore, upon supposition the point should lie *(page 390)* any where between *f* and *e*, the probability of the event M happens *p* times and fails *q* in *p* + *q* trials can't be greater than the ratio of *e h* to A B. There then being these two subsequent events, the 1st that the point *o* will lie between *e* and *f*, the 2d that the event M will happen *p* times and fail *q* in *p* + *q* trials, and the probability of the 1st (by lemma 1st) is the ratio of *e f* to A B, and upon supposition the 1st happens, by what has been now proved, the probability of the 2d cannot be greater than the ratio of *e h* to A B, it evidently follows (from Prop. 3.) that the probability both together will happen cannot be greater than the ratio compounded of that of *e f* to A B and that of *e h* to A B, which compound ratio is the ratio of *f h* to C A. Wherefore, the probability that the point *o* will lie between *f* and *e,* and the event M happen *p* times and fail *q*, is not greater than the ratio of *f h* to C A. And in like, manner the probability the point *o* will lie between *e* and *d*, and the event M happen and fail as before, cannot be greater than the ratio of *e i* to C A. And again, the probability the point *o* will lie between *d* and *c*, and the event M happen and fail as before, cannot be greater than the ratio of *c i* to C A. And lastly, the probability that the point *o* will lie between *c* and *b*, and the event M happen and fail as before, cannot be greater than the ratio of *b k* to C A. Add now all these several probabilities together, and their sum, (by Prop. 1.) will be the probability that the point will lie somewhere between *f* and *b*, and the event M happen *p* times and fail *q* in *p* + *q* trials. Add likewise the correspondent ratios together, and their sum will be the ratio of the sum of the antecedents *(page 391)* to their common consequent, i.e. the ratio of *f h*, *e i*, *c i*, *b k* together to C A; which ratio is less than that of D to C A, because D is greater than *f h*, *e i*, *c i*, *b k* together. And therefore, the probability that the point *o* will lie between *f* and *b* and withal that the event M will happen *p* times and fail *q* in *p* + *q* trials, is *less* than the ratio of D to C A; but it was supposed the same which is absurd. And in like manner, by inscribing rectangles within the figure, as *e g*, *d h*, *d k*, *c m*, you may prove that the last mentioned probability is *greater* than the ratio of any figure less than *f g h i k m b* to C A.

Wherefore that probability must be the ratio of *f g h i k m b* to C A.

Cor. Before the ball W is thrown the probability that the point *o* will lie somewhere between A and B, or somewhere upon the line A B, and withal

Version zh25a.doc at 18:36 on 13 July, 2001

that the event M will happen p times, and fail q in $p + q$ trials is the ratio of the whole figure A i B to C A. But it is certain that the point o will lie somewhere upon A B. Wherefore, before the ball W is thrown the probability the event M will happen p times and fail q in $p + q$ trials is the ratio of A i B to C A.

PROP. 9.

If before any thing is discovered concerning the place of the point o, it should appear that the event M had happened p times and failed q in $p + q$ trials, and from hence I guess that the point o lies between any two points in the line A B, as f and b, and consequently that the probability of the event M in a single trial was somewhere between the ratio of A b to A B and that of A f to A B: the probability I am in the right is the ratio of that part of the figure A i B described as before which is intercepted between perpendiculars erected upon A B at the points f and b to the whole figure A i B.

For, there being these two subsequent events, the first that the point o will lie between f and b, the second that the event M should happen p times and fail q in $p + q$ trials and (by cor. Prop. 8.) the original probability of the second is the ratio of A i B to C A, and (by prop. 8.) the probability of both is the ratio of f g h i m b to C A; wherefore (by prop. 5) it being first discovered that the second has happened, and from hence I guess that the first has happened also, the probability I am in *(page 392)* the right is the ratio of f g h i m b to A i B, the point which was to be proved.

Cor. The same things supposed, if I guess that the probability of the event M lies somewhere between o and the ratio of A b to A B, my chance to be in the right is the ratio of A b m to A i B.

SCHOLIUM.

From the preceding proposition it is plain, that in the case of such an event as I there call M, from the number of times it happens and fails in a certain number of trials, without knowing any thing more concerning it, one may give a guess whereabouts it's probability is, and, by the usual methods computing the magnitudes of the areas there mentioned, see the chance that the guess is right. And that the same rule is the proper one to be used in the case of an event concerning the probability of which *(page 393)* we absolutely know nothing antecedently to any trials made concerning it, seems to appear from the following consideration; viz. that concerning such an event I have no reason to think that, in a certain number of trials, it should rather happen any one possible number of times than another. For, on this account, I may justly reason concerning it as if its probability had been at first unfixed, and then determined in such a manner as to give me no reason to think that, in a certain number of trials, it should rather happen any one possible number of times than another. But this is exactly the case of the event M. For before the ball W is thrown, which determines it's probability in a single trial, (by cor. prop. 8.) the probability it has to happen p times and fail q in $p + q$ or n trials is the ratio of A i B to C A, which ratio is the same when $p + q$ or n is given, whatever number p is; as will appear by computing the magnitude of A i B by the method[1] of fluxions. And consequently before the place of the point o is discovered or the number of times event M has happened in n trials, I can have no reason to think it should rather happen one possible number of times than another.

In what follows therefore I shall take for granted that the rule given concerning the event M in prop. 9. is also the rule to be used in relation to any event concerning the probability of which nothing *(page 394)* at all is known and antecedently to any trials made or observed concerning it. And such an event I shall call an unknown event.

---

[1] It will be proved presently in art. 4. By computing in the method here mentioned that A i B contracted in the ratio of E to 1 is to C A as 1 to $\overline{n + 1} \times E$: from whence it plainly follows that, antecedently to this contraction, A i B must be to C A in the ratio of 1 to $n + 1$, which is a constant ratio when n is given, whatever p is.