

ISyE 6761 – Fall 2012

Homework #6 Solutions

1. The time T required to repair a machine is $\text{Exp}(2)$.
 - (a) What's the probability that a repair time exceeds 1/2 hour?

Solution: Since $T \sim \text{Exp}(2)$, we have

$$P(T > 1/2) = \int_{1/2}^{\infty} 2e^{-2t} dt = e^{-1}. \quad \square$$

- (b) What's the probability that a repair time exceeds 12.5 hours given that it's greater than 12?

Solution: Using the memoryless property, we have

$$P(T > 12.5 | T > 12) = P(T > 1/2) = e^{-1}. \quad \square$$

2. Suppose you arrive at a single-teller bank to find 5 customers (one being served and 4 waiting in line). You join at the end of the line. If the service times are i.i.d. $\text{Exp}(\mu)$, what's the expected amount of time you'll spend in the bank?

Solution: Let T denote the time that you're in the bank; let S_i be the service time of the i th person in line ($i = 1, 2, 3, 4$); let R be the remaining service time of the guy in service when you show up; and let S be your service time. Obviously, the S_i 's and S are i.i.d. $\text{Exp}(\mu)$. By the memoryless property, so is R . Therefore,

$$E[T] = E[R + S_1 + S_2 + S_3 + S_4 + S] = 6/\mu. \quad \square$$

3. Suppose X is an exponential RV. Without any computation, state which one of the following is correct. Explain.
 - (a) $E[X^2 | X > 1] = E[(X + 1)^2]$
 - (b) $E[X^2 | X > 1] = E[X^2] + 1$
 - (c) $E[X^2 | X > 1] = (1 + E[X])^2$

Solution: By the memoryless property, the conditional distribution of X , given that $X > 1$, is the same as the unconditional distribution of $X + 1$. Therefore, (a) is correct. Since the right-hand sides of (b) and (c) are obviously different than that of (a), we see that (b) and (c) are false. $\square$

4. Consider a post office with two clerks. Three people, A , B , and C , enter at the same time. A and B go directly to the clerks, and C waits until either A or B leaves before he begins service. What's the probability that A is still in the post office after the other two have left, i.e., $P(A > B + C)$, when...?

- (a) The service time for each clerk is always 10 minutes.

Solution: Here, $A = B = C = 10$ implies that $P(A > B + C) = 0$. $\square$

- (b) The service times are 1, 2, or 3, each w.p. $1/3$.

Solution: Now, A, B, C are i.i.d. discrete uniform $\{1, 2, 3\}$. This implies that

$$P(A > B + C) = P(A = 3, B = 1, C = 1) = 1/27. \quad \square$$

- (c) The service times are i.i.d. $\text{Exp}(\mu)$.

Solution: Now, A, B, C are i.i.d. $\text{Exp}(\mu)$. First of all, note that

$$\begin{aligned} P(A > B + x) &= \int_0^\infty P(A > B + x | B = y) f_B(y) dy \\ &= \int_0^\infty P(A > x + y) \mu e^{-\mu y} dy \\ &= \int_0^\infty e^{-\mu(x+y)} \mu e^{-\mu y} dy \\ &= \frac{1}{2} e^{-\mu x}. \end{aligned}$$

This implies that

$$\begin{aligned}
 P(A > B + C) &= \int_0^\infty P(A > B + C | C = x) f_C(x) dx \\
 &= \int_0^\infty P(A > B + x) \mu e^{-\mu x} dx \\
 &= \int_0^\infty \frac{1}{2} e^{-\mu x} \mu e^{-\mu x} dx \\
 &= \frac{1}{4}. \quad \square
 \end{aligned}$$

5. The life of a radio is exponential with a mean of 10 years. If Tom buys a 10-year-old radio, what's the probability that it'll be working after another 10 years?

Solution: By the memoryless property,

$$P(T > 20 | T > 10) = P(T > 10) = \int_{10}^\infty \frac{1}{10} e^{-t/10} dt = e^{-1}. \quad \square$$

6. A post office is run by two clerks. When Smith enters the system, he finds that Jones is being served by one of the clerks, and Brown is being served by the other. Also suppose that Smith is told that his service will begin as soon as either Jones or Brown leaves.

- (a) If the amount of time that a clerk spends with a customer is $\text{Exp}(\lambda)$, what's the probability that, of the three customers, Smith is the last to leave?

Solution: Consider the time at which Smith first finds a free clerk. At this point, either Jones or Brown would have just left, and the other one would still be in service. However, by memoryless, we know that the amount of time that the other guy (either Jones or Brown) would still have to spend in service is $\text{Exp}(\lambda)$, i.e., the same as if he were just starting his service at this point. Thus, by symmetry, the probability that he finishes before Smith is $1/2$. $\square$.

- (b) What if server i 's times are $\text{Exp}(\lambda_i)$, $i = 1, 2$?

Solution: Suppose Clerk 1 serves Jones and Clerk 2 serves Brown. Let S, J, B denote the respective service times for Smith, Jones, and Brown. Then

$$\begin{aligned}
 & \mathbb{P}(\text{Smith not last} | J < B) \\
 &= \mathbb{P}(S < B) \\
 &\quad (\text{by the memoryless property}) \\
 &= \frac{\lambda_1}{\lambda_1 + \lambda_2} \\
 &\quad (\text{we've done this before}).
 \end{aligned}$$

Similarly, the law of total probability gives

$$\begin{aligned}
 & \mathbb{P}(\text{Smith not last}) \\
 &= \mathbb{P}(\text{Smith not last} | J < B)\mathbb{P}(J < B) + \mathbb{P}(\text{Smith not last} | J \geq B)\mathbb{P}(J \geq B) \\
 &= \frac{\lambda_1}{\lambda_1 + \lambda_2} \cdot \frac{\lambda_1}{\lambda_1 + \lambda_2} + \frac{\lambda_2}{\lambda_1 + \lambda_2} \cdot \frac{\lambda_2}{\lambda_1 + \lambda_2}. \quad \square
 \end{aligned}$$

7. Suppose that X_1 and X_2 are independent continuous RV's. Show that

$$\mathbb{P}(X_1 < X_2 | \min(X_1, X_2) = t) = \frac{r_1(t)}{r_1(t) + r_2(t)},$$

where $R_i(t)$ is the failure rate function of X_i .

Solution: Using the obvious notation for p.d.f.'s and c.d.f.'s, and a little abuse of

rigor which I'll take about at the end of the problem, we have

$$\begin{aligned}
 & \mathbb{P}(X_1 < X_2 | \min(X_1, X_2) = t) \\
 &= \frac{\mathbb{P}(X_1 < X_2, \min(X_1, X_2) = t)}{\mathbb{P}(\min(X_1, X_2) = t)} \\
 &= \frac{\mathbb{P}(X_1 = t, X_2 > t)}{\mathbb{P}(\min(X_1, X_2) = t)} \\
 &= \frac{\mathbb{P}(X_1 = t, X_2 > t)}{\mathbb{P}(X_1 = t, X_2 > t) + \mathbb{P}(X_1 > t, X_2 = t)} \\
 &= \frac{f_1(t)(1 - F_2(t))}{f_1(t)(1 - F_2(t)) + f_2(t)(1 - F_1(t))} \\
 & \quad \text{(by independence)} \\
 &= \frac{\frac{f_1(t)}{1 - F_1(t)}}{\frac{f_1(t)}{1 - F_1(t)} + \frac{f_2(t)}{1 - F_2(t)}} \\
 &= \frac{r_1(t)}{r_1(t) + r_2(t)}.
 \end{aligned}$$

(Careful! To keep everything legal when calculating some of the probabilities above, we really should look at events like $\min(X_1, X_2) \in (t, t + \delta)$, and then let $\delta \rightarrow 0$.) $\square$

8. Suppose X has failure rate function $r(t)$. Find a nice expression for $\mathbb{E}[1/r(X)]$.

Solution: Assume that X is a positive random variable. By the Law of the Unconscious Statistician, we have

$$\begin{aligned}
 \mathbb{E}[1/r(X)] &= \int_{\mathbb{R}} \frac{1 - F(x)}{f(x)} f(x) dx \\
 &= \int_0^\infty (1 - F(x)) dx \\
 &= \int_0^\infty \int_x^\infty f(t) dt dx \\
 &= \int_0^\infty \int_0^t f(t) dx dt \\
 &= \int_0^\infty t f(t) dt \\
 &= \mathbb{E}[X]. \quad \square
 \end{aligned}$$

9. Suppose that X_1, X_2, X_3 are independent exponential RV's with rate $\lambda_1, \lambda_2, \lambda_3$. Find

- (a) $P(X_1 < X_2 < X_3)$.
- (b) $P[X_1 < X_2 \mid \max(X_1, X_2, X_3) = X_3]$.
- (c) $E[\max(X_i) \mid X_1 < X_2 < X_3]$.
- (d) $E[\max(X_i)]$.

Solutions:

- (a) We have

$$\begin{aligned} P(X_1 < X_2 < X_3) &= P(X_1 = \min(X_1, X_2, X_3))P(X_2 < X_3 \mid X_1 = \min(X_1, X_2, X_3)) \\ &= \frac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_3} \cdot \frac{\lambda_2}{\lambda_2 + \lambda_3}, \end{aligned}$$

where the first term follows as usual, and the second follows from the memoryless property. $\square$

- (b) Now result (a) gives us

$$\begin{aligned} P(X_1 < X_2 \mid X_3 = \max(X_1, X_2, X_3)) &= \frac{P(X_1 < X_2 < X_3)}{P(X_1 < X_2 < X_3) + P(X_2 < X_1 < X_3)} \\ &= \frac{\frac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_3} \cdot \frac{\lambda_2}{\lambda_2 + \lambda_3}}{\frac{\lambda_1}{\lambda_1 + \lambda_2 + \lambda_3} \cdot \frac{\lambda_2}{\lambda_2 + \lambda_3} + \frac{\lambda_2}{\lambda_1 + \lambda_2 + \lambda_3} \cdot \frac{\lambda_1}{\lambda_1 + \lambda_3}} \\ &= \frac{\frac{1}{\lambda_2 + \lambda_3}}{\frac{1}{\lambda_2 + \lambda_3} + \frac{1}{\lambda_1 + \lambda_3}}. \quad \square \end{aligned}$$

- (c) Let's do some preliminary stuff first. To begin,

$$\begin{aligned} P(X_1 < X_2 < X_3 < w) &= \int_0^w \int_0^{x_3} \int_0^{x_2} \lambda_1 \lambda_2 \lambda_3 \exp\{-(\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3)\} dx_1 dx_2 dx_3 \\ &= \frac{\lambda_2 \lambda_3 e^{-(\lambda_1 + \lambda_2 + \lambda_3)w}}{(\lambda_1 + \lambda_2 + \lambda_3)(\lambda_1 + \lambda_2)} + \frac{\lambda_3 e^{-(\lambda_2 + \lambda_3)w}}{\lambda_2 + \lambda_3} - \frac{\lambda_1 e^{-\lambda_3 w}}{\lambda_1 + \lambda_2} + \frac{\lambda_1 \lambda_2}{(\lambda_1 + \lambda_2 + \lambda_3)(\lambda_2 + \lambda_3)}. \end{aligned}$$

And, of course, as a special case of the above, we have

$$P(X_1 < X_2 < X_3) = \frac{\lambda_1 \lambda_2}{(\lambda_1 + \lambda_2 + \lambda_3)(\lambda_2 + \lambda_3)}.$$

Now let $W = \max_i X_i$. Using the above two results, we can calculate the conditional c.d.f. of $W|X_1 < X_2 < X_3$ by plugging into

$$\begin{aligned} F_{W|X_1 < X_2 < X_3}(w) &= P(W \leq w | X_1 < X_2 < X_3) \\ &= \frac{P(X_1 < X_2 < X_3 < w)}{P(X_1 < X_2 < X_3)}. \end{aligned}$$

Then we have the conditional p.d.f.

$$f_{W|X_1 < X_2 < X_3}(w) = \frac{d}{dw} F_{W|X_1 < X_2 < X_3}(w),$$

and expected value

$$E[W|X_1 < X_2 < X_3] = \int_0^\infty f_{W|X_1 < X_2 < X_3}(w) dw = \frac{1}{\lambda_1 + \lambda_2 + \lambda_3} + \frac{1}{\lambda_2 + \lambda_3} + \frac{1}{\lambda_3},$$

after all of the algebraic smoke clears. $\square$

By the way, could you have gotten this result by inspection?

(d) Summing over all 6 permutations of 1,2,3, and using part (c), we have

$$\begin{aligned} E[\max(X_i)] &= \sum_{i \neq j \neq k} E[\max(X_\ell) | X_i < X_j < X_k] P(X_i < X_j < X_k) \\ &= \sum_{i \neq j \neq k} \frac{\lambda_i}{\lambda_1 + \lambda_2 + \lambda_3} \frac{\lambda_j}{\lambda_j + \lambda_k} \left[\frac{1}{\lambda_1 + \lambda_2 + \lambda_3} + \frac{1}{\lambda_j + \lambda_k} + \frac{1}{\lambda_k} \right]. \quad \square \end{aligned}$$

10. Suppose X is $\text{Exp}(\lambda)$.

(a) Use the definition of conditional expectation to find $E[X|X < c]$.

Solution: The conditional p.d.f. of X given $X < c$ is

$$f_{X|X < c}(x) = \frac{f_X(x)}{P(X < c)} = \frac{\lambda e^{-\lambda x}}{1 - e^{-\lambda c}}, \quad 0 < x < c.$$

This implies that

$$\begin{aligned} E[X|X < c] &= \int_{-\infty}^{\infty} x f_{X|X < c}(x) dx \\ &= \int_0^c x \frac{\lambda e^{-\lambda x}}{1 - e^{-\lambda c}} dx \\ &= \frac{1}{\lambda} - \frac{c e^{-\lambda c}}{1 - e^{-\lambda c}}, \end{aligned}$$

after some calculus. $\square$

(b) Now find $E[X|X < c]$ by using

$$E[X] = E[X|X < c]P(X < c) + E[X|X > c]P(X > c).$$

Solution: Using the identity, we get

$$\frac{1}{\lambda} = E[X|X < c](1 - e^{-\lambda c}) + \left(c + \frac{1}{\lambda}\right)e^{-\lambda c}.$$

This immediately simplifies to the same answer as (a). $\square$

11. Let X_1 and X_2 be i.i.d. $\text{Exp}(\mu)$. Let $X_{(1)} = \min(X_1, X_2)$ and $X_{(2)} = \max(X_1, X_2)$. Find

- (a) $E[X_{(1)}]$.
- (b) $\text{Var}(X_{(1)})$.
- (c) $E[X_{(2)}]$.
- (d) $\text{Var}(X_{(2)})$.

Solutions:

(a) We've shown in class that $X_{(1)} \sim \text{Exp}(2\mu)$. This implies that $E[X_{(1)}] = 1/(2\mu)$. $\square$

(b) Similarly, $\text{Var}(X_{(1)}) = 1/(2\mu)^2$. $\square$

(c) There are lots of ways to do this. One is simply to derive the p.d.f. of $X_{(2)}$ and then do the usual calculus. But I'll show you here a method that I stole from Ross's answer book, which is a nice, intuitive technique.

Define $A = X_{(2)} - X_{(1)}$. By the memoryless property, $A \sim \text{Exp}(\mu)$ and is independent of $X_{(1)}$. Thus,

$$E[X_{(2)}] = E[X_{(1)} + A] = \frac{1}{2\mu} + \frac{1}{\mu} = \frac{3}{2\mu}. \quad \square$$

(d) Similarly,

$$\text{Var}(X_{(2)}) = \text{Var}(X_{(1)} + A) = \frac{1}{4\mu^2} + \frac{1}{\mu^2} = \frac{5}{4\mu^2}. \quad \square$$

12. Repeat Exercise 11, but this time suppose that X_1, X_2 are independent exponentials with rates μ_1, μ_2 .

Solutions:

Let $Y = X_{(1)} = \min(X_1, X_2)$ and $W = X_{(2)} = \max(X_1, X_2)$.

(a) and (b) The c.d.f. of Y is

$$\begin{aligned} \mathbf{P}(Y \leq y) &= \mathbf{P}(\min(X_1, X_2) \leq y) \\ &= 1 - \mathbf{P}(\min(X_1, X_2) > y) \\ &= 1 - \mathbf{P}(X_1 > y, X_2 > y) \\ &= 1 - \mathbf{P}(X_1 > y)\mathbf{P}(X_2 > y) \\ &= 1 - e^{-(\mu_1 + \mu_2)y}, \end{aligned}$$

implying that $Y \sim \text{Exp}(\mu_1 + \mu_2)$. This immediately implies that $\mathbf{E}[Y] = 1/(\mu_1 + \mu_2)$ and $\text{Var}(Y) = 1/(\mu_1 + \mu_2)^2$. $\square$

(c) and (d) The c.d.f. of W is

$$\begin{aligned} \mathbf{P}(W \leq w) &= \mathbf{P}(\max(X_1, X_2) \leq w) \\ &= \mathbf{P}(X_1 \leq w, X_2 \leq w) \\ &= \mathbf{P}(X_1 \leq w)\mathbf{P}(X_2 \leq w) \\ &= (1 - e^{-\mu_1 w})(1 - e^{-\mu_2 w}) \\ &= 1 - e^{-\mu_1 w} - e^{-\mu_2 w} + e^{-(\mu_1 + \mu_2)w}. \end{aligned}$$

Thus, the p.d.f. is

$$f_W(w) = \mu_1 e^{-\mu_1 w} + \mu_2 e^{-\mu_2 w} - (\mu_1 + \mu_2) e^{-(\mu_1 + \mu_2)w}, \quad w \geq 0.$$

After a little algebra, we get

$$\mathbf{E}[W] = \int_0^\infty w f_W(w) dw = \frac{\mu_1^2 + \mu_1 \mu_2 + \mu_2^2}{\mu_1 \mu_2 (\mu_1 + \mu_2)}$$

and

$$\mathbb{E}[W^2] = \int_0^\infty w^2 f_W(w) dw = \frac{2(\mu_1^4 + 2\mu_1^3\mu_2 + \mu_1^2\mu_2^2 + 2\mu_1\mu_2^3 + \mu_2^4)}{\mu_1^2\mu_2^2(\mu_1 + \mu_2)^2}$$

so that

$$\text{Var}(W) = \mathbb{E}[W^2] - (\mathbb{E}[W])^2 = \frac{\mu_1^4 + 2\mu_1^3\mu_2 - \mu_1^2\mu_2^2 + 2\mu_1\mu_2^3 + \mu_2^4}{\mu_1^2\mu_2^2(\mu_1 + \mu_2)^2}. \quad \square$$

13. Two people, A and B, need kidney transplants. If A doesn't receive a new kidney, she'll die after an $\text{Exp}(\mu_A)$ amount of time, and similarly, B will die after an $\text{Exp}(\mu_B)$ amount of time. new kidneys arrive according to a $\text{PP}(\lambda)$. It's been decided that the first kidney will go to A (or to B if B is alive and A is not at that time), and the next one will go to B (if still living).

(a) What's the probability that A gets a new kidney?

Solution: Let A and B denote the remaining lifetimes of A and B; let $K_1, K_2, \dots$ denote the i.i.d. $\text{Exp}(\lambda)$ interarrivals of kidneys.

A will get a kidney if $K_1 < A$. As we've seen many times,

$$\mathbb{P}(K_1 < A) = \frac{\lambda}{\lambda + \mu_A}. \quad \square$$

(b) What's the probability that B gets a new kidney?

Solution: B gets a kidney if either of the following events happen:

- $A < K_1 < B$ (A dies, then kidney arrives), or
- $K_1 < A$ and $K_1 < K_2 < B$ (kidney arrives for A, then second kidney arrives for B)

We'll add the probabilities of the two events; the memoryless property and Problem 5–12(a) give us

$$\begin{aligned} \mathbb{P}(\text{B gets kidney}) &= \frac{\mu_A}{\lambda + \mu_A + \mu_B} \cdot \frac{\lambda}{\lambda + \mu_B} + \frac{\lambda}{\lambda + \mu_A + \mu_B} \cdot \frac{\lambda}{\lambda + \mu_B} \\ &= \frac{\mu_A + \lambda}{\lambda + \mu_A + \mu_B} \cdot \frac{\lambda}{\lambda + \mu_B}. \quad \square \end{aligned}$$

14. The lives of Joe's dog and cat are independent exponential RV's with rates λ_d and λ_c . One of them has just died. Find the expected additional lifetime of the other pet.

Solution: Let A be the additional life, and then condition on which pet died. By the memoryless property,

$$\mathbb{E}[A] = \mathbb{E}[A|D]\mathbb{P}(D) + \mathbb{E}[A|C]\mathbb{P}(C) = \frac{1}{\lambda_c} \cdot \frac{\lambda_d}{\lambda_c + \lambda_d} + \frac{1}{\lambda_d} \cdot \frac{\lambda_c}{\lambda_c + \lambda_d} \quad \square$$

15. Consider an n -server parallel queueing system where customers arrive according to a $\text{PP}(\lambda)$, where the service times are $\text{Exp}(\mu)$, and where any arrival finding all servers busy immediately leaves. If an arrival finds all servers busy, find the following.

- (a) The expected number of busy servers found by the next arrival.
- (b) The probability that the next arrival finds all servers free.
- (c) The probability that the next arrival finds exactly i of the servers free.

Solutions:

(a) Let T denote the time until the next arrival, and let N denote the number of servers that are found to be busy by the next arrival. Conditional on $T = t$, the probability that a particular server will be busy when the next arrival occurs is $\mathbb{P}(\text{Exp}(\mu) > t) = e^{-\mu t}$. Thus, $(N|T = t) \sim \text{Bin}(n, e^{-\mu t})$.

Using the usual conditioning argument,

$$\mathbb{E}[N] = \int_0^\infty \mathbb{E}[N|T = t]f_T(t) dt = \int_0^\infty n e^{-\mu t} \lambda e^{-\lambda t} dt = \frac{n\lambda}{\lambda + \mu}. \quad \square$$

(b) and (c). There are a couple of ways to do this. For instance, we can condition on T to eventually get

$$\mathbb{P}(N = 0) = \prod_{j=1}^n \frac{(n - j + 1)\mu}{\lambda + (n - j + 1)\mu}$$

and

$$\mathbb{P}(N = n - i) = \frac{\lambda}{\lambda + (n - i)\mu} \prod_{j=1}^i \frac{(n - j + 1)\mu}{\lambda + (n - j + 1)\mu}. \quad \square$$

16. Events occur according to a PP(2/hour).

(a) What's the probability that no events occur between 8:00 p.m. and 9:00 p.m.?

Solution: The number of arrivals between 8:00 p.m. and 9:00 p.m. is $\text{Pois}(2)$. This implies

$$P(\text{no arrivals}) = \frac{e^{-2}2^0}{0!} = e^{-2}. \quad \square$$

(b) Starting at noon, what's the expected time at which the fourth event occurs?

Solution: 2:00 p.m. $\square$

(c) What the probability that at least 2 events occur between 6:00 p.m. and 8:00 p.m.?

Solution: $1 - 5e^{-4}$. $\square$

17. Cars arrive to a parking deck according to a PP(3/min). Half of the cars are driven by men. Assume everything is independent. (a) What's the probability that no cars driven by men will show up by time t ? (b) What's the expected number of guy cars that will show up by time t ?

Solutions:

$$X(t) \sim \text{Pois}(\lambda pt) = \text{Pois}(3 \cdot \frac{1}{2} \cdot t) = \text{Pois}(2t).$$

$$(a) P(X(t) = 0) = e^{-2t}. \quad \square$$

$$(b) E[X(t)] = 2t. \quad \square$$

18. There are two types of claims that are made to an insurance company. Let $N_i(t)$ denote the number of type i claims made by time t , and suppose that $\{N_1(t)\}$ and $\{N_2(t)\}$ are independent PP's with rates $\lambda_1 = 10$ and $\lambda_2 = 1$. The amounts of type 1 claims are independent exponential RV's with mean \$1000, and the amounts of type 2's are independent exponentials with mean \$5000. Suppose that a claim for \$4000 has just been received. What's the probability that it's a type 1?

Solution: The unconditional probability that the claim is type 1 is $10/11$. Thus, by Bayes, we have

$$\begin{aligned} P(\text{Type 1}|\$4000) &= \frac{P(\$4000|\text{Type 1})P(\text{Type 1})}{P(\$4000|\text{Type 1})P(\text{Type 1}) + P(\$4000|\text{Type 2})P(\text{Type 2})} \\ &= \frac{e^{-4}(10/11)}{e^{-4}(10/11) + 0.2e^{-0.8}(1/11)}. \quad \square \end{aligned}$$

19. Customers arrive at a bank via a $PP(\lambda/\text{hour})$. Suppose 2 customers arrived during the first hour. What's the probability that

- (a) Both arrived during the first 20 minutes?
- (b) At least one arrived during the first 20 minutes?

Solutions: By the conditional result from class, the two arrivals are i.i.d. $\text{Unif}(0,60 \text{ min})$.

(a) $P(\text{both in first 20 min}) = (20/60)^2 = 1/9. \quad \square$

(b) $P(\geq 1 \text{ in first 20 min}) = 1 - P(0 \text{ in first 20 min}) = 1 - (2/3)^2 = 5/9. \quad \square$

20. From 8A to 10A, customers arrive at a store according to a $PP(4/\text{hour})$. From 10A to 12P, they arrive as a $PP(8/\text{hr})$. From 12P to 2P the arrival rate increases steadily from 8/hour at 12P to 10/hour at 2P. From 2P to 5P the arrival rate drops steadily from 10/hour at 2P to 4/hour at 5P. Find the probability distribution of the number of customers who enter the store on a given day.

Solution: Start at $t = 0$ (8AM) and go until $t = 9$ (5PM). Then the intensity function is

$$\lambda(t) = \begin{cases} 4, & \text{if } 0 < t \leq 2 \\ 8, & \text{if } 2 < t \leq 4 \\ 4 + t, & \text{if } 4 < t \leq 6 \\ 22 - 2t, & \text{if } 6 < t \leq 9 \end{cases}.$$

Then $N(9) \sim \text{Pois}(\int_0^9 \lambda(t) dt) = \text{Pois}(63). \quad \square$