Optimal order sizing for the newsvendor model with discrete demand

Spyros Reveliotis

Consider a newsvendor problem with unit overage cost \( c_o \), unit shortage cost \( c_s \), and the demand taking the discrete values \( D_i \), for \( i \in \{1,2,\ldots,n\} \), with corresponding probabilities \( p_i \) (the results hold even if \( n = \infty \)). Also, let \( TC(Q) \) denote the expected total cost resulting from some order size \( Q \).

In this document we shall show that an optimal selection for the order size \( Q \) is provided by the smallest demand level \( D_i \) such that \( \text{Prob}(D \leq D_i) \equiv \sum_{k=1}^{i} p_k \) is greater than or equal to the problem critical ratio \( c_s/(c_s + c_o) \).

We shall establish this result in two steps: First we show that there is no advantage in selecting an order size \( Q \) that does not coincide with one of the discrete levels of the demand. Once we have narrowed down the choices of \( Q \) as stated above, then, we shall prove the main result.

**Lemma 1** Consider an order size \( Q \) that belongs in some open interval \((D_i, D_{i+1})\), i.e., \( Q \) is strictly between the demand levels \( D_i \) and \( D_{i+1} \). Then,

\[
TC(Q) \geq \min\{TC(D_i), TC(D_{i+1})\}.
\]

**Proof:** Since \( Q \in (D_i, D_{i+1}) \), we can write \( Q = D_i + \alpha \), where \( 0 < \alpha < D_{i+1} - D_i \). Also, we set \( \beta = D_{i+1} - (D_i + \alpha) \). From the above definitions, it is also clear that \( \beta > 0 \). Furthermore, we can express \( TC(Q) \) as follows:

\[
TC(Q) = c_o \sum_{k=1}^{i} p_k (Q - D_k) + c_s \sum_{k=i+1}^{n} p_k (D_k - Q)
\]

\[
= c_o \sum_{k=1}^{i} p_k (D_i + \alpha - D_k) + c_s \sum_{k=i+1}^{n} p_k (D_k - D_i - \alpha)
\]

\[
= c_o \sum_{k=1}^{i} p_k (D_i - D_k) + c_o \sum_{k=1}^{i} p_k \alpha
\]

\[
+ c_s \sum_{k=i+1}^{n} p_k (D_k - D_i) - c_s \sum_{k=i+1}^{n} p_k \alpha
\]

We also have:

\[
TC(D_i) = c_o \sum_{k=1}^{i} p_k (D_i - D_k) + c_s \sum_{k=i+1}^{n} p_k (D_k - D_i)
\]

1
From Eqs 1 and 2, we have:

\[ TC(Q) - TC(D_i) = \alpha [ c_o \sum_{k=1}^{i} p_k - c_s \sum_{k=i+1}^{n} p_k ] \]  

(3)

So, if \([ c_o \sum_{k=1}^{i} p_k - c_s \sum_{k=i+1}^{n} p_k ] \geq 0\), we have \( TC(Q) \geq TC(D_i) \), and the lemma holds true.

In the opposite case (i.e., if \([ c_o \sum_{k=1}^{i} p_k - c_s \sum_{k=i+1}^{n} p_k ] < 0\)), consider \( TC(D_{i+1}) \). We have:

\[ TC(D_{i+1}) = c_o \sum_{k=1}^{i} p_k (D_{i+1} - D_k) + c_s \sum_{k=i+1}^{n} p_k (D_k - D_{i+1}) \]  

(4)

Also, from Eqs 1 and 4:

\[
\begin{align*}
TC(Q) - TC(D_{i+1}) &= c_o \sum_{k=1}^{i} p_k (D_i - D_{i+1}) + c_o \sum_{k=1}^{i} p_k \alpha \\
&\quad - c_s \sum_{k=i+1}^{n} p_k \alpha - c_s \sum_{k=i+1}^{n} p_k (D_i - D_{i+1}) \\
&= c_o \sum_{k=1}^{i} p_k (D_i + \alpha - D_{i+1}) \\
&\quad - c_s \sum_{k=i+1}^{n} p_k (D_i + \alpha - D_{i+1}) \\
&= -c_o \sum_{k=1}^{i} p_k \beta + c_s \sum_{k=i+1}^{n} p_k \beta \\
&= -\beta [c_o \sum_{k=1}^{i} p_k - c_s \sum_{k=i+1}^{n} p_k] > 0
\end{align*}
\]  

(5)

The last inequality in Eq. 5 results from the working hypothesis that \([ c_o \sum_{k=1}^{i} p_k - c_s \sum_{k=i+1}^{n} p_k ] < 0\). Eq. 5 implies that in this second case \( TC(Q) > TC(D_{i+1}) \), and once again, Lemma 1 is true. Furthermore, since the two considered cases for the sign of the difference \( TC(Q) - TC(D_i) \) exhaust all the possibilities, Lemma 1 must hold true. □

Next we state and prove the main result of this document.

**Theorem 1** In the considered newsvendor problem, an optimal selection for the order size \( Q \) is the smallest demand level \( D_i \) such that

\[ \text{Prob}(D \leq D_i) \equiv \sum_{k=1}^{i} p_k \geq \frac{c_s}{c_s + c_o}. \]
Proof: From Lemma 1, we know that we can restrict our search for an optimal selection of $Q$ over the set of the discrete demand levels $D_i$, $i = 1, 2, \ldots, n$. Next, we consider the difference $TC(D_{i+1}) - TC(D_i)$. From Eqs 2 and 4, we have:

$$
TC(D_{i+1}) - TC(D_i) = c_o \sum_{k=1}^{i} p_k (D_{i+1} - D_k) + c_s \sum_{k=i+1}^{n} p_k (D_k - D_{i+1})
$$

$$
- c_o \sum_{k=1}^{i} p_k (D_i - D_k) - c_s \sum_{k=i+1}^{n} p_k (D_k - D_i)
$$

$$
= c_o \sum_{k=1}^{i} p_k (D_{i+1} - D_i) - c_s \sum_{k=i+1}^{n} p_k (D_{i+1} - D_i)
$$

$$
= (D_{i+1} - D_i) \left[ c_o \sum_{k=1}^{i} p_k - c_s \sum_{k=i+1}^{n} p_k \right]
$$

(6)

Eq. 6 further implies that

$$
TC(D_{i+1}) - TC(D_i) < 0 \iff c_o \sum_{k=1}^{i} p_k - c_s \sum_{k=i+1}^{n} p_k < 0
$$

$$
\iff \frac{\sum_{k=1}^{i} p_k}{\sum_{k=i+1}^{n} p_k} < \frac{c_s}{c_o}
$$

$$
\iff \frac{\sum_{k=1}^{i} p_k}{\sum_{k=1}^{i} p_k + \sum_{k=i+1}^{n} p_k} < \frac{c_s}{c_s + c_o}
$$

$$
\iff \sum_{k=1}^{i} p_k < \frac{c_s}{c_s + c_o}
$$

(7)

In the last derivation of Eq. 7 we have taken into consideration the fact that $\sum_{k=1}^{i} p_k + \sum_{k=i+1}^{n} p_k = \sum_{k=1}^{n} p_k = 1.0$. In plain terms, Eq. 7 implies that the expected total cost can decrease as we go from some demand level $D_i$ to the next level $D_{i+1}$ if and only if $\sum_{k=1}^{i} p_k < \frac{c_s}{c_s + c_o}$. Hence, we can stop as soon as we reach a demand level $D_j$ such that $\sum_{k=1}^{j} p_k \geq \frac{c_s}{c_s + c_o}$. At that point, we set $Q = D_j$. □