1. Given a finite aperiodic irreducible MC, prove that for some $n$, all terms of $P^n$ are positive.

**Solution:** Since the MC is aperiodic, for every state $i$, there exists $N(i)$ such that $P^{(n)}_{ii} > 0$ whenever $n \geq N(i)$. Since the MC is irreducible, for every pair $i, j$, there exists $N(i, j)$ such that $P^{(n)}_{ij} > 0$ whenever $n \geq N(i, j)$. There are only a finite number of states, so we can set $N = \max_i N(i) + \max_{i,j} N(i, j) < \infty$. Now take $n \geq N$, so that

$$P^{(n)}_{ij} \geq P^{(N(i,j))}_{ij} P^{(n-N(i,j))}_{ii} > 0,$$

for all $i, j$. □

Here’s an alternative argument. Note that for a finite aperiodic irreducible MC, we have $\lim_{n \to \infty} P^{(n)}_{ij} = \pi_j > 0$. So there exists $N(j)$ such that every entry of column $j$ of $P^n$ is positive whenever $n \geq N(j)$. Now let $n \geq \max_j N(j)$. □

2. Suppose the probability transition matrix of an irreducible MC is idempotent, i.e., $P = P^2$. Prove that $P_{ij} = P_{jj}$ for all $i$ and $j$ and that the MC is aperiodic.

**Solution:** Since the MC is irreducible, there exists $n$ such that $P^{(n)}_{ij} > 0$. Since the matrix is idempotent, $P_{ij} = P^{(n)}_{ij} > 0$ for all $i \neq j$. Then by idempotency and Chapman-Kolmogorov, $P_{ii} = P^{(2)}_{ii} > P_{ij} P_{ji} > 0$, for all $i$. So every state has period 1; and thus the MC is aperiodic. □

Since the MC is aperiodic and irreducible (and hence ergodic), $\pi_j = \lim_{n \to \infty} P^{(n)}_{ij} = \lim_{n \to \infty} P_{ij} = P_{jj}$, where the penultimate step follows by idempotency. Similarly, $\pi_j = \lim_{n \to \infty} P^{(n)}_{jj} = P_{jj}$. □

3. It follows from the Kolmogorov Criterion that for a time-reversible MC,

$$P_{ij} P_{jk} P_{ki} = P_{ik} P_{kj} P_{ji}, \text{ for all } i, j, k. \quad (1)$$

Prove that if the state space is finite and $P_{ij} > 0$ for all $i, j$, then (1) is also a sufficient condition for time reversibility. (Thus, in this case, you only have to
check Kolmogorov for paths going from \( i \) to \( i \) with two intermediate states.) Hint: Fix \( i \) and show that the equations \( \pi_j P_{jk} = \pi_k P_{kj} \) are satisfied by \( \pi_j = cP_{ij}/P_{ji} \), where \( c \) is chosen so that \( \sum_j \pi_j = 1 \).

**Solution:** Following the hint, suppose \( \pi_j = cP_{ij}/P_{ji} \) (which is well-defined since we have assumed that \( P_{ij} > 0 \) for all \( i,j \)), where \( c \) must be chosen so that \( \sum_j \pi_j = 1 \). Then \( \pi_j P_{jk} = \pi_k P_{kj} \) if and only if
\[
\frac{cP_{ij}P_{jk}}{P_{ji}} = \frac{cP_{ik}P_{kj}}{P_{ki}}
\]
if and only if
\[
P_{ij}P_{jk}P_{ki} = P_{ik}P_{kj}P_{ji},
\]
which is true by hypothesis. \( \square \)

4. Show that the MC having the following transition matrix is time reversible.

\[
P = \begin{pmatrix}
0 & 1/2 & 1/2 \\
1/4 & 1/2 & 1/4 \\
1/4 & 1/4 & 1/2 \\
\end{pmatrix}
\]

**Solution:** Solving \( \pi = \pi P \), we eventually get \( \pi = (1/5, 2/5, 2/5) \).
All we need to do now is to verify that \( \pi_i P_{ij} = \pi_j P_{ji} \), for all \( i,j \). For example,
\[
\pi_0 P_{01} = (1/5)(1/2) = (2/5)(1/4) = \pi_1 P_{10}.
\]
I’ll leave it to you to verify that the reversibility equality holds for all other \( i,j \). \( \square \)

5. Resnick 3.3. Find the renewal function for the following. Do the drill without Laplace transforms. In both cases, verify the Elementary Renewal Theorem directly.

(a) \( f(x) = \alpha e^{-\alpha x}, x > 0 \).
Solution: The $Y_i$’s are i.i.d. Exp($\alpha$). Thus, $F^{n*}$ will be the c.d.f. for the Erlang$_n(\alpha)$ distribution. So

$$U(x) = \sum_{n=0}^{\infty} F^{n*}(x)$$

$$= 1 + \sum_{n=1}^{\infty} F^{n*}(x)$$

$$= 1 + \sum_{n=1}^{\infty} \int_0^x \frac{\alpha^n t^{n-1} e^{-\alpha t}}{(n-1)!} dt$$

$$= 1 + \alpha \int_0^x e^{-\alpha t} \sum_{n=1}^{\infty} \frac{\alpha^{n-1} t^{n-1}}{(n-1)!} dt$$

$$= 1 + \alpha \int_0^x e^{-\alpha t} e^{-\alpha t} dt$$

$$= 1 + \alpha x. \quad \Box$$

Further, $\lim_{t \to \infty} \frac{U(t)}{t} = \alpha = 1/E[Y]$, showing that the ERT holds. \Box

(b) $f(x) = \alpha^2 x e^{-\alpha x}$, $x > 0$.

Solution: The $Y_i$’s are i.i.d. Erlang$_2(\alpha)$. Thus, $F^{n*}$ will be the c.d.f. for the Erlang$_{2n}(\alpha)$ distribution. So

$$U(x) = 1 + \sum_{n=1}^{\infty} F^{n*}(x)$$

$$= 1 + \sum_{n=1}^{\infty} \int_0^x \frac{\alpha^2 t^{2n-1} e^{-\alpha t}}{(2n-1)!} dt$$

$$= 1 + \alpha \int_0^x e^{-\alpha t} \sum_{n=1}^{\infty} \frac{(at)^{2n-1}}{(2n-1)!} dt$$

$$= 1 + \alpha \int_0^x e^{-\alpha t} \sinh(\alpha t) dt$$

$$= 1 + \alpha \int_0^x e^{-\alpha t} \frac{(e^{\alpha t} - e^{-\alpha t})}{2} dt$$

$$= 1 + \frac{\alpha}{2} \int_0^x (1 - e^{-2\alpha t}) dt$$

$$= \frac{3}{4} \alpha x + \frac{e^{-2\alpha x}}{4}. \quad \Box$$
Further, \( \lim_{t \to \infty} \frac{U(t)}{t} = \alpha/2 = 1/E[Y] \), showing that the ERT holds.

6. Resnick 3.4. (I’ve re-written this problem to make a little more sense of it.) Suppose that \( X_1, X_2, \ldots \) are i.i.d. positive random variables with c.d.f. \( F \), and \( N \) is a non-negative integer random variable such that \( p_k = P(N = k) \), \( k = 0, 1, 2, \ldots \). Further suppose that the \( X_i \)’s and \( N \) are independent. Let \( G = \sum_{k=0}^{\infty} p_k F^{k*} \).

Check that \( G \) is the distribution of \( \sum_{i=1}^{N} X_i \). Then find the Laplace transform of \( G \).

**Solution:** First of all, let \( S_k = \sum_{i=1}^{k} X_i, \ k = 0, 1, 2, \ldots \). Then

\[
G(x) = P(S_N \leq x)
\]

\[
= \sum_{k=0}^{\infty} P(S_N \leq x | N = k) P(N = k)
\]

\[
= \sum_{k=0}^{\infty} P(S_k \leq x | N = k) P(N = k)
\]

\[
= \sum_{k=0}^{\infty} P(S_k \leq x) P(N = k) \quad \text{(everything’s independent)}
\]

\[
= \sum_{k=0}^{\infty} F^{k*}(x)p_k. \quad \square
\]

Now we’ll get the LT of \( G \).

\[
\hat{G}(\lambda) = E(e^{-\lambda S_N})
\]

\[
= \sum_{k=0}^{\infty} E(e^{-\lambda S_N} | N = k) P(N = k)
\]

\[
= \sum_{k=0}^{\infty} E(e^{-\lambda S_k} | N = k) p_k
\]

\[
= \sum_{k=0}^{\infty} E(e^{-\lambda S_k}) p_k \quad \text{(everything’s independent)}
\]

\[
= \sum_{k=0}^{\infty} (\hat{F}(\lambda))^{k} p_k. \quad \square
\]

7. Resnick 3.5. If \( X > 0 \) and \( Y > 0 \) are independent with c.d.f.’s \( F \) and \( G \), show that \( E[e^{-\lambda XY}] = \int_{0}^{\infty} \hat{F}(\lambda y) \ dG(y) = \int_{0}^{\infty} \hat{G}(\lambda y) \ dF(y) \).
Solution: Let $f(x,y)$ be the joint p.d.f./p.m.f.

$$E[e^{-\lambda XY}] = \int_0^\infty \int_0^\infty e^{-\lambda xy} f(x,y) \, dx \, dy$$

$$= \int_0^\infty \int_0^\infty e^{-\lambda xy} dF(x) \, dG(y) \quad \text{(by independence)}$$

$$= \int_0^\infty \left( \int_0^\infty e^{-\lambda xy} dF(x) \right) \, dG(y)$$

$$= \int_0^\infty \hat{F}(\lambda y) \, dG(y). \quad \square$$

The result $E[e^{-\lambda XY}] = \int_0^\infty \hat{G}(\lambda y) \, dF(y)$ follows in the same way. \square

8. Resnick 3.6. Items of a certain type have a mean lifetime of one day with a standard deviation of two hours. There is a unit cost per item, and items are used successively according to a renewal model. Use of items starts at $t = 0$ and continues for a 10-year period (3600 days). If $C$ is the total cost for the 10-year period, give bounds $\alpha$ and $\beta$ such that $P(\beta \leq C \leq \alpha) \approx 0.95$.

Solution: By the CLT for renewal processes, for large $t$,

$$C = N(t) \approx \text{Nor} \left( t, \frac{t\sigma^2}{\mu^3} \right).$$

Thus, for the current problem with $t = 3600$, $\mu = 1$, and $\sigma^2 = 1/144$ days$^2$, we have

$$N(3600) \approx \text{Nor} \left( 3600, \frac{3600}{144} \right).$$

By the usual manipulations from any baby statistics class,

$$\alpha = E[C] + z_{0.25} \sqrt{\text{Var}(C)} = 3600 + 1.96(5) = 3610$$

and

$$\beta = E[C] - z_{0.25} \sqrt{\text{Var}(C)} = 3600 - 1.96(5) = 3590. \quad \square$$

9. Resnick 3.10. (a) Prove the memoryless property of the exponential distribution, i.e., if $X \sim \text{Exp}(\lambda)$, then $P(X > s + t | X > s) = P(X > t)$ for any $s, t > 0$. 


Solution: This is so standard that I’m not going to do it! □

(b) Waiting times in doctors’ offices in Ithaca are exponential with a mean of $1/\lambda = 75$ min. Given that you’ve already waited an hour, what’s the probability that you’ll have to wait another 45 min. to see the doc?

Solution: By memoryless,
\[ P(X > 60 + 45 | X > 60) = P(X > 45) = e^{-\lambda t} = e^{-45/75} = 0.549. \] □

(c) Dana, Tali, and Sheli go to the post office, where there are two servers. The service requirements $D$, $T$, and $S$ of the three girls are i.i.d. exponential RV’s. Because Sheli is the youngest, Dana and Tali get served before her. Sheli can only start service when one of the older girls finishes (at time $D \land T$). What’s the probability that Sheli is not the last to leave service?

Solution: WLOG, let’s suppose that $D$ finishes first. Now it’s $S$ against $T$. But by memoryless, it’s as if $T$ is starting over. Since $S$ and $T$ are i.i.d., $S$ has a 50% chance of winning. □

10. Resnick 3.11. Calls to the fire department occur according to a PP with a rate of 3/day.

(a) A fireman gets a raise after the 300th call. How many days should he expect to wait for the raise?

Solution: Let $Y_1, Y_2, \ldots$ be the i.i.d. Exp(3) interarrivals. Let $S_n = \sum_{i=1}^n Y_i$ be the time of the $n$th call. $E[S_n] = nE[Y_i] = 300/3 = 100$. □

(b) It turns out that on average, 1/3 of the calls are false alarms. In a single day, what’s the probability of two false alarms?

Solution: Before doing this problem, let’s consider an important (and obvious) auxiliary result. Namely, suppose that arrivals occur according to a PP($\lambda$). Further suppose that each arrival has probability $p$ of being Type I and probability $1-p$ of Type II. Assume everything is independent. Then Type I and II arrivals form independent PP’s with rates $\lambda p$ and $\lambda(1-p)$, respectively. I’ll dispense with a formal
proof of this result now, but let’s just say that it follows from the definition of a PP.

Now on to the problem at hand. Let’s suppose that the false alarms are the Type I process discussed in the auxiliary result. Then the false arrivals are a PP with rate \( \lambda^* = \lambda p = 3/3 = 1 \) per day. So

\[
P(N(1) = 2) = \frac{e^{-\lambda^* t}(\lambda^* t)^k}{k!} = \frac{e^{-1(1)}(1(1))^2}{2!} = 0.184.
\]

\[\square\]

(c) Firemen are paid $100/day. A new pay plan is proposed where they will be paid a random amount of money for each fire they actually fight. In the new scheme, the expected pay per fire fought is $40/fire. What’s the long run reward rate in the new scheme? Is the new scheme better than the old one?

**Solution:** Suppose \( R_1, R_2, \ldots \) are the i.i.d. rewards for the fires they actually fight, and let \( \mu \) be the expected time between renewals (i.e., interarrival times of actual fires). By the auxiliary result above, the actual fires form a PP with rate 2/day. Thus, the mean interarrival time is \( \mu = 1/2 \). If we denote the cumulative reward as \( R(t) \), then we know by the Elementary Renewal Theorem that

\[
\lim_{t \to \infty} \frac{R(t)}{t} = \frac{E[R_1]}{\mu} = \frac{\$40}{1/2} = \$80.
\]

\[\square\]

(d) If we consider only calls which occur after midnight January 1, what’s the probability that a false alarm occurs before a real alarm is phoned in?

**Solution:** 1/3 (trivially). \[\square\]

11. Suppose that the interarrivals for a renewal process are i.i.d. Pois(\( \lambda \)).

(a) Find the distribution of \( S_n \).

**Solution:** Since the \( X_i \)'s are i.i.d. Pois(\( \lambda \)), we have

\[
S_n = \sum_{i=1}^{n} X_i \sim \text{Pois}(n\lambda).
\]

\[\square\]
(b) Calculate $P(N(t) = n)$.

**Solution:** We have

$$P(N(t) = n) = P(N(t) \geq n) - P(N(t) \geq n + 1)$$

$$= P(S_n \leq t) - P(S_{n+1} \leq t)$$

$$= \sum_{k=0}^{[t]} \frac{e^{-n\lambda}(n\lambda)^k}{k!} - \sum_{k=0}^{[t]} \frac{e^{-(n+1)\lambda}((n+1)\lambda)^k}{k!},$$

where $[t]$ is the greatest integer $\leq t$. □

12. If the mean-value function of the renewal process $\{N(t)\}$ is $U(t) = t/2$ (don’t count $Y_0$), find $P(N(5) = 0)$.

**Solution:** By the one-to-one correspondence of $U(t)$ and c.d.f. $F(x)$, we have that $N(t)$ is a PP($\lambda = 1/2$) Thus, $P(N(5) = 0) = e^{-5/2}$. □

13. If $P(Y_i = 1) = 1/3$ and $P(Y_i = 2) = 2/3$, find $P(N(1) = k)$, $P(N(2) = k)$, and $P(N(3) = k)$ for all appropriate $k$.

**Solution:** As in a previous problem,

$$P(N(t) = k) = P(N(t) \geq k) - P(N(t) \geq k + 1) = P(S_k \leq t) - P(S_{k+1} \leq t)$$

$$= P(S_k \leq t < S_{k+1}) \text{ for } t \geq 0 \text{ and } k = 1, 2, \ldots,$$

Thus, $P(N(1) = k) = P(S_k \leq 1 < S_{k+1})$.

So

$$P(N(1) = 0) = P(S_0 \leq 1 < S_1) = P(0 \leq 1 < Y_1) = P(Y_1 = 2) = 2/3.$$

Similarly, since $Y_1 + Y_2 \geq 2$, we have

$$P(N(1) = 1) = P(S_1 \leq 1 < S_2) = P(Y_1 \leq 1 < Y_1 + Y_2) = P(Y_1 = 1) = 1/3.$$

Then we see that $P(N(1) = k) = 0$ for all $k \geq 2$. □

Similarly, we can do $P(N(2) = k) = P(S_k \leq 2 < S_{k+1})$ and $P(N(3) = k) = P(S_k \leq 3 < S_{k+1})$. 
But now that we’ve gone through this pretty carefully, it’s actually a little bit easier to argue this problem more succinctly as follows.

\[
\begin{align*}
\mathbb{P}(N(1) = k) & = \mathbb{P}(k \text{ arrivals by time 1}) \\
& = \begin{cases} 
\mathbb{P}(Y_1 = 2) = \frac{2}{3}, & \text{for } k = 0 \\
\mathbb{P}(Y_1 = 1) = \frac{1}{3}, & \text{for } k = 1
\end{cases}.
\end{align*}
\]

\[
\begin{align*}
\mathbb{P}(N(2) = k) & = \mathbb{P}(k \text{ arrivals by time 2}) \\
& = \begin{cases} 
\mathbb{P}(Y_1 = 1, Y_2 = 2) + \mathbb{P}(Y_1 = 2) = \frac{2}{5} + \frac{2}{3} = \frac{8}{15}, & \text{for } k = 1 \\
\mathbb{P}(Y_1 = 1, Y_2 = 1) = \frac{1}{9}, & \text{for } k = 2
\end{cases}.
\end{align*}
\]

\[
\begin{align*}
\mathbb{P}(N(3) = k) & = \mathbb{P}(k \text{ arrivals by time 3}) \\
& = \begin{cases} 
\mathbb{P}(Y_1 = 2, Y_2 = 2) = \frac{4}{9}, & \text{for } k = 1 \\
\mathbb{P}(Y_1 = 1, Y_2 = 2) + \mathbb{P}(Y_1 = 2, Y_2 = 1) \\
+ \mathbb{P}(Y_1 = 1, Y_2 = 1, Y_3 = 2) = \frac{14}{27}, & \text{for } k = 2 \\
\mathbb{P}(Y_1 = 1, Y_2 = 1, Y_3 = 1) = \frac{1}{27}, & \text{for } k = 3
\end{cases}.
\end{align*}
\]

14. A patient arrives at a doctor’s office. W.p. 1/5, he receives service immediately, but w.p. 4/5, his service is deferred an hour. After an hour’s wait again w.p. 1/5 he’s served immediately or another hour of delay is imposed, and so on.

(a) What’s the waiting time distribution of the first arrival?

**Solution:** Geom(1/5) − 1 (the “−1” is if he gets served immediately). □

(b) What’s the distribution of the number of patients who receive service over an 8-hour period, assuming the same procedure is followed for every patient and the arrivals follow a Poisson process with rate 1?

**Solution:** The number of tries until the patient gets served is obviously Geom(1/5). If he shows up in hour \( i \), \( i = 1, 2, \ldots, 8 \), he gets \( 9 - i \) potential tries to get served. The probability that he fails all of those tries is \( (4/5)^{9-i} \). So the probability that a patient arriving in hour \( i \) eventually gets served is \( p_i = 1 - (4/5)^{9-i} \).

The number of patients showing up in hour \( i \) is Pois(1). [I guess you could interpret this as Pois(60), but I’ll save ink and work with the smaller number.] Thus, by the auxiliary result mentioned earlier, the number of patients showing up in hour
i who eventually receive service is $N_i \sim \text{Pois}(p_i)$, $i = 1, 2, \ldots, 8$. All hours are independent (by independent increments of a PP), so the total number of people who receive service over the 8-hour period is

$$
\sum_{i=1}^{8} N_i \sim \text{Pois} \left( \sum_{i=1}^{8} p_i \right)
$$

$$
\sim \text{Pois} \left( \sum_{i=1}^{8} (1 - (4/5)^{9-i}) \right)
$$

$$
\sim \text{Pois} \left( 8 - \sum_{i=1}^{8} (4/5)^i \right)
$$

$$
\sim \text{Pois} \left( 8 - \frac{4}{5}(1 - (\frac{4}{5})^8) \right)
$$

$$
\sim \text{Pois}(4.67). \quad \square
$$

15. The random lifetime of an item has c.d.f. $F(x)$. What is the expected remaining life of an item of age $x$?

**Solution:** Let $X$ be the lifetime. Then the remaining life is $X - x$. So the expected remaining life is

$$
\mathbb{E}(X - x \mid \text{item is "alive" at age } x) = \mathbb{E}(X - x \mid X > x)
$$

$$
= \mathbb{E}(X \mid X > x) - x \quad \text{(since } x \text{ is a constant)}
$$

$$
= -x + \int_{x}^{\infty} \mathbb{E}(X \mid X = u) \mathbb{P}(X = u \mid X > x) \, du \quad \text{(excuse probability notation abuse)}
$$

$$
= -x + \int_{x}^{\infty} u \mathbb{P}(X = u \mid X > x) \, du
$$

$$
= -x + \frac{1}{\mathbb{P}(X > x)} \int_{x}^{\infty} u \mathbb{P}(X = u, X > x) \, du
$$

$$
= -x + \frac{1}{\mathbb{P}(X > x)} \int_{x}^{\infty} u \mathbb{P}(X = u) \, du \quad \text{(redundant information)}
$$

$$
= -x + \frac{1}{1 - F(x)} \int_{x}^{\infty} u \, dF(u) \quad \text{(revert to usual notation)}
$$

$$
= -x + \frac{1}{1 - F(x)} \int_{x}^{\infty} dq \, dF(u).
$$
Note that $0 < q < u$ and $x < u < \infty$. We have two possibilities: Either $0 < x < q < u < \infty$ or $0 < q < x < u < \infty$. These lead to the following:

$$E(X - x | X > x) = -x + \frac{1}{1 - F(x)} \left[ \int_x^\infty \int_q^\infty dF(u) dq + \int_0^x \int_x^\infty dF(u) dq \right]$$

$$= -x + \frac{1}{1 - F(x)} \left[ \int_x^\infty (1 - F(q)) dq + \int_0^x (1 - F(x)) dq \right]$$

$$= -x + \frac{\int_x^\infty (1 - F(q)) dq}{1 - F(x)} + x$$

$$= \frac{\int_x^\infty (1 - F(q)) dq}{1 - F(x)}. \quad \square$$