1. Three white and three black balls are distributed in two urns in such a way that each urn contains 3 balls. The system is in state \( i, \ i = 0,1,2,3, \) if the first urn contains \( i \) white balls. At each step, we swap a ball from each urn. Let \( X_n \) be the state of the system after the \( n \)th step. Is \( \{X_n\} \) a Markov chain? If so, calculate its transition probability matrix.

**Solution:** Yes, this is a MC (since the next state only depends on current information). The transition matrix is easily found to be

\[
P = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1/9 & 4/9 & 4/9 & 0 \\
0 & 4/9 & 4/9 & 1/9 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

\( \square \)

2. Suppose that whether or not it rains today depends on previous weather conditions over the last three days.

(a) Show how this system can be analyzed by using a MC.

(b) Suppose that if it had rained for the past three days, then it will rain today with probability 0.8; if it didn’t rain for any of the past three days, then it will rain today w.p. 0.2; and in any other case, the weather today will be the same as the weather yesterday with probability 0.6. Determine \( P \).

**Solution:** You need 8 states: \( RRR, RRD, \ldots, DDD \), where, e.g., \( RDD \) means “dry today, dry yesterday, rain the day before yesterday.”

\[
P = \begin{pmatrix}
RRR & RRD & RDR & RDD & DRR & DRD & DDR & DDD \\
0.8 & 0.2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.4 & 0.6 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.6 & 0.4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0.4 & 0.6 & 0 \\
0.6 & 0.4 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.4 & 0.6 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.6 & 0.4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.2 & 0.8
\end{pmatrix}.
\]

\( \square \)
3. A MC with states 0,1,2 has

\[
P = \begin{pmatrix}
\frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\
0 & \frac{1}{3} & \frac{2}{3} \\
\frac{1}{2} & 0 & \frac{1}{2}
\end{pmatrix}.
\]

If \(P(X_0 = 0) = P(X_0 = 1) = P(X_0 = 2) = 1/3\), find \(E[X_3]\).

**Solution:** After a little algebra, we obtain

\[
P^3 = \frac{1}{108} \begin{pmatrix}
39 & 22 & 47 \\
48 & 16 & 44 \\
45 & 24 & 39
\end{pmatrix}.
\]

Then since \(\alpha = (1/3,1/3,1/3)\), we get

\[
P(X_3 = j) = \sum_{i=0}^{\infty} P^{(3)}_{ij} \alpha_i = \frac{1}{3} \sum_{i=0}^{2} P^{(3)}_{ij}, \quad \text{for } j = 0, 1, 2.
\]

In particular, we find that \(P(X_3 = 0) = 132/324\), \(P(X_3 = 1) = 62/324\), and \(P(X_3 = 2) = 130/324\); and this yields \(E[X_3] = 322/324 = 0.9938\). □

4. Suppose

\[
P = \begin{pmatrix}
p & 1-p \\
1-p & p
\end{pmatrix}.
\]

Show that

\[
P^{(n)} = \frac{1}{2} \begin{pmatrix}
1 + (2p - 1)^n & 1 - (2p - 1)^n \\
1 - (2p - 1)^n & 1 + (2p - 1)^n
\end{pmatrix}.
\]

(1)

**Solution:** Easy induction. The \(n = 1\) case is trivial. Now, assume it’s true for general \(n\), as in Equation (1). What remains is to prove that the result holds for the \(n + 1\) case. To do so, note that

\[
P^{(n+1)} = P^{(n)}P = \begin{pmatrix}
P^{(n)}_{00} & P^{(n)}_{01} \\
P^{(n)}_{10} & P^{(n)}_{11}
\end{pmatrix} \begin{pmatrix}
P_{00} & P_{01} \\
P_{10} & P_{11}
\end{pmatrix}.
\]

Then the induction hypothesis implies that

\[
P^{(n+1)}_{00} = P^{(n)}_{00}P_{00} + P^{(n)}_{01}P_{10}
\]

\[
= \left[\frac{1}{2} + \frac{1}{2}(2p - 1)^n\right]p + \left[\frac{1}{2} - \frac{1}{2}(2p - 1)^n\right](1-p)
\]

\[
= \frac{1}{2} + \frac{1}{2}(2p - 1)^{n+1},
\]
which is what we want.

Since \( P_{00}^{(n+1)} + P_{01}^{(n+1)} = 1 \), the result is also true for \( P_{01}^{(n+1)} \). Similarly, the result holds for the second row of \( P^{(n+1)} \), and we are done. □

5. Recall the example from class in which the weather today depends on the weather for the previous two days. For that example, state 0 is RR, state 1 is SR, state 2 is RS, and state 3 is SS, and we had the following probability transition matrix:

\[
P = \begin{pmatrix}
0.7 & 0 & 0.3 & 0 \\
0.5 & 0 & 0.5 & 0 \\
0 & 0.4 & 0 & 0.6 \\
0 & 0.2 & 0 & 0.8
\end{pmatrix}.
\]

Now suppose that it was sunny both yesterday and the day before yesterday. What’s the probability that it will rain tomorrow?

**Solution:**

\[
P^{(2)} = \begin{pmatrix}
0.49 & 0.12 & 0.21 & 0.18 \\
0.35 & 0.20 & 0.15 & 0.30 \\
0.20 & 0.12 & 0.20 & 0.48 \\
0.10 & 0.16 & 0.10 & 0.64
\end{pmatrix}.
\]

(Actually, you don’t really have to calculate every single entry of \( P^{(2)} \).) You want \( P_{30}^{(2)} + P_{31}^{(2)} = 0.26 \). □

6. Suppose coin 1 has probability 0.7 of coming up H, and coin 2 has probability 0.6 of coming up H. If the coin flipped today comes up H, then we select coin 1 to flip tomorrow; and if the coin we flip today comes up T, then we flip coin 2 tomorrow. (a) If the coin initially flipped is equally likely to be 1 or 2, then what’s the probability that the coin flipped on the third day after the initial flip is coin 1? (b) Suppose the coin flipped on Monday comes up H. What’s the probability that the coin flipped on Friday also comes up H?

**Solution:**

\[
P^{(3)} = \begin{pmatrix}
0.7 & 0.3 \\
0.6 & 0.4
\end{pmatrix}^3 = \begin{pmatrix}
0.667 & 0.333 \\
0.666 & 0.334
\end{pmatrix}.
\]
(a) \((P_{11}^{(3)} + P_{21}^{(3)})/2 = 0.6665\). □
(b) Similarly, \(P_{11}^{(4)} = 0.6667\). □

7. Specify the states of the following MC’s, and determine whether they are transient or recurrent.

(a) 
\[
P_1 = \begin{pmatrix}
0 & 1/2 & 1/2 \\
1/2 & 0 & 1/2 \\
1/2 & 1/2 & 0
\end{pmatrix}.
\]

Solution \(P_1: \{0, 1, 2\} \) recurrent. □

(b) 
\[
P_2 = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
1/2 & 1/2 & 0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}.
\]

Solution \(P_2: \{0, 1, 2, 3\} \) recurrent (since 0 → 3 → 2 → 1 → 3 → 2 → 0). □

(c) 
\[
P_3 = \begin{pmatrix}
1/2 & 0 & 1/2 & 0 & 0 \\
1/4 & 1/2 & 1/4 & 0 & 0 \\
1/2 & 0 & 1/2 & 0 & 0 \\
0 & 0 & 0 & 1/2 & 1/2 \\
0 & 0 & 0 & 1/2 & 1/2
\end{pmatrix}.
\]

Solution \(P_3: \{0, 2\} \) recurrent; \{1\} transient; \{3, 4\} recurrent; □

(d) 
\[
P_4 = \begin{pmatrix}
1/4 & 3/4 & 0 & 0 \\
1/2 & 1/2 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1/3 & 2/3 & 0 \\
1 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

Solution \(P_4: \{0, 1\} \) recurrent; \{2\} recurrent; \{3\} transient; \{4\} transient; □
8. Prove that if the number of states in a MC is \( M \), and if state \( j \) can be reached from \( i \), then it can be reached in \( M \) steps or less.

**Solution** Suppose not, i.e., suppose that the shortest path from \( i \) to \( j \) takes \( N > M \) steps. Since \( N > M \) and since there are only \( M \) states, it must be the case that we go through some state more than once. Eliminate this loop, contradicting the assumption that the shortest path is of length \( N \). □

9. Suppose \( P \) is doubly stochastic (and irreducible) with states \( 0, 1, \ldots, M \). What are the limiting probabilities \( \pi_i \)?

**Solution:** Let’s see if \( \pi_j = 1/(M + 1) \) for all \( j \) works. We have (since \( P \) is doubly stochastic)

\[
\begin{align*}
(i) \quad \pi_j &= \frac{1}{M + 1} = \frac{1}{M + 1} \sum_{i=0}^{M} P_{ij} = \sum_{i=0}^{M} \pi_i P_{ij} \\
(ii) \quad \sum_{j=0}^{M} \pi_j &= 1.
\end{align*}
\]

Since (i) and (ii) hold, we are done by the fact that the limiting probabilities are unique. □

10. A DNA nucleotide has any of 4 values. A standard MC model for mutational change in the nucleotide supposes that in going from time period to time period, the nucleotide does not change with probability \( 1 - 3\alpha \); and if it does change, then it’s equally likely to change to any of the other 3 states, where \( 0 < \alpha < 1/3 \).

(a) Show that \( P_{11}^{(n)} = \frac{1}{4}[1 + 3(1 - 4\alpha)^n] \).

(b) What’s the long-run proportion of time in each state?

**Solution:** (a) First of all,

\[
\begin{pmatrix}
1 - 3\alpha & \alpha & \alpha & \alpha \\
\alpha & 1 - 3\alpha & \alpha & \alpha \\
\alpha & \alpha & 1 - 3\alpha & \alpha \\
\alpha & \alpha & \alpha & 1 - 3\alpha
\end{pmatrix}.
\]
We’ll do this problem via induction, assuming that we have states 1, 2, 3, 4. The 
\( n = 0 \) case is trivial. We’ll assume the expression holds for the \( n \) case and try to 
prove the analogous expression for the \( n + 1 \) case. Thus, we will assume that

\[
P_{11}^{(n)} = \frac{1}{4}[1 + 3(1 - 4\alpha)^n] \quad \text{and} \quad P_{ij}^{(n)} = \frac{1}{3}\left(1 - \frac{1}{4}[1 + 3(1 - 4\alpha)^n]\right), \quad \text{for } j = 2, 3, 4,
\]

where the latter \( P_{ij}^{(n)} \)’s follow from symmetry. So we have

\[
P_{11}^{(n+1)} = \sum_{k=1}^{4} P_{1k}^{(n)} P_{k1}
\]

\[
= \frac{1}{4}[1 + 3(1 - 4\alpha)^n](1 - 3\alpha) + \frac{1}{3}\left(1 - \frac{1}{4}[1 + 3(1 - 4\alpha)^n]\right)3\alpha
\]

\[
= \frac{1}{4}[1 + 3(1 - 4\alpha)^{n+1}] \quad \text{(after some algebra)},
\]

and we are done with the induction. □

(b) Since the columns of \( P \) each sum to 1, the matrix is doubly stochastic. So 
\( \pi_j = 1/4 \) for all \( j \). □

11. Each morning, Joe leaves his house and goes for a run. He’s equally likely to leave 
either from the front or back door. Upon leaving the house, he chooses a pair

of running shoes (or goes barefoot if there are no shoes at the door from which 
he departed). On his return, Joe is equally likely to enter, and leave his running 
shoes, either by the front or back door. If he owns a total of \( k \) pairs of shoes, 
what’s the long-run proportion of time that he runs barefooted?

**Solution:** Let \( X_n \) denote the number of pairs of shoes at the door from which Joe 
leaves on day \( n \). (Without loss of generality, suppose he leaves from door A; call 
the other door B.)

To get things going, first suppose that \( X_n = 0 \). Then \( X_{n+1} \) can only equal 0 or \( k \),
each with probability 1/2 (Why?)

Similarly, if \( X_n = i \) \((i = 1, 2, \ldots, k)\), then

\[
X_{n+1} \equiv \begin{cases} 
    i & \text{leave from A, return to A, leave from A the next day} \\
    i-1 & \text{leave from A, return to B, leave from A the next day} \\
    k-i & \text{leave from A, return to A, leave from B the next day} \\
    k-i+1 & \text{leave from A, return to B, leave from B the next day}
\end{cases}
\]
each with probability 1/4 (Why?)

Using this result, we can obtain the transition matrix. For example, for \( k = 5 \), we have

\[
P \equiv \begin{pmatrix}
\frac{1}{2} & 0 & 0 & 0 & 0 & \frac{1}{2} \\
\frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{4} & \frac{1}{4} \\
0 & \frac{1}{4} & \frac{1}{4} & 1/4 & 1/4 & 0 \\
0 & 0 & 1/2 & 1/2 & 0 & 0 \\
0 & \frac{1}{4} & \frac{1}{4} & 1/4 & 1/4 & 0 \\
\frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{4} & \frac{1}{4}
\end{pmatrix}
\]

Since \( P \) is doubly stochastic, we have that \( \pi_j = 1/6 \) for all \( j \). □

Note that there are probably a couple of other interesting ways to solve this problem. I'll eventually post them when I get around to writing them up.

12. Every time a team wins a game, it wins the next game w.p. 0.8; every time it loses a game, it wins the next w.p. 0.3. If the team wins a game, then it has dinner together w.p. 0.7; but if it loses, then it has dinner w.p. 0.2. What proportion of games result in a team dinner?

**Solution:** Let the two states be denoted \( W \) and \( L \). We have

\[
P \equiv \begin{pmatrix}
0.8 & 0.2 \\
0.3 & 0.7
\end{pmatrix}
\]

Solving \( \pi = \pi P \), we obtain

\[
\pi_W = \pi_W(0.8) + \pi_L(0.3) \quad \text{and} \quad \pi_W + \pi_L = 1,
\]

which yields \( \pi_W = 0.6 \) and \( \pi_L = 0.4 \). Thus,

\[
P(\text{dinner}) = \pi_W P(\text{dinner} | W) + \pi_L P(\text{dinner} | L) = (0.6)(0.7) + (0.4)(0.2) = 0.5. \quad \Box
\]

13. A company has \( N \) employees (where \( N \) is large). Each person has one of three possible jobs and changes jobs (independently) according to a MC with

\[
P = \begin{pmatrix}
0.7 & 0.2 & 0.1 \\
0.2 & 0.6 & 0.2 \\
0.1 & 0.4 & 0.5
\end{pmatrix}
\]
What’s the long-run percentage of employees in each job?

**Solution:** Must solve

\[
\begin{align*}
\pi_0 &= 0.7\pi_0 + 0.2\pi_1 + 0.1\pi_2 \\
\pi_1 &= 0.2\pi_0 + 0.6\pi_1 + 0.4\pi_2 \\
\pi_2 &= 0.1\pi_0 + 0.2\pi_1 + 0.5\pi_2
\end{align*}
\]

and \(\pi_0 + \pi_1 + \pi_2 = 1\). After some algebra, we eventually get

\[\pi = \left(\frac{6}{17}, \frac{7}{17}, \frac{4}{17}\right). \quad \square\]

14. A taxi provides service in two zones of a city. Fares picked up in zone A will have destinations in zone A w.p. 0.6 and in zone B w.p. 0.4. Fares picked up in B go to A w.p. 0.3 and to B w.p. 0.7. The driver’s expected profit per trip entirely in A is 6; a trip entirely B is 8; and a trip that switches zones is 12. Assuming that the taxi only moves when it has a passenger, find the driver’s average profit.

**Solution:** Let the states be the successive pickup zones. Then we have \(P_{AA} = 0.6,\) \(P_{BA} = 0.3\). Therefore, the long-run proportions of pickups from each zone can be calculated from

\[
\pi_A = 0.6\pi_A + 0.3\pi_B = 0.6\pi_A + 0.3(1 - \pi_A).
\]

This immediately yields \(\pi_A = 3/7\) and \(\pi_B = 4/7\).

Now let \(X\) be the profit from a trip. Conditioning on the zone gives

\[
\begin{align*}
E[X] &= \frac{3}{7}E[X|A] + \frac{4}{7}E[X|B] \\
&= \frac{3}{7}[0.6(6) + 0.4(12)] + \frac{4}{7}[0.3(12) + 0.7(8)] \\
&= 62/7. \quad \square
\end{align*}
\]

15. In the gambler’s ruin problem, suppose that he initially has a fortune of \(i\). Suppose that we know that the fortune hits \(N\) before it hits 0. Show that the probability that he wins the next gamble is

\[
\frac{p[1 - (q/p)^{i+1}]}{1 - (q/p)^i}, \quad \text{if } p \neq 1/2 \quad \text{and} \quad \frac{i + 1}{2i}, \quad \text{if } p = 1/2.
\]
**Solution:** We have

\[
P(X_{n+1} = i + 1 | X_n = i, \lim_{m \to \infty} X_m = N) = \frac{P(X_{n+1} = i + 1, \lim_{m \to \infty} X_m = N | X_n = i)}{P(\lim_{m \to \infty} X_m = N | X_n = i)}
\]

\[
= \frac{P(\lim_{m \to \infty} X_m = N | X_{n+1} = i + 1, X_n = i)P(X_{n+1} = i + 1 | X_n = i)}{P(\lim_{m \to \infty} X_m = N | X_n = i)}
\]

(\text{since } X_n \text{ is a Markov chain})

\[
= \frac{P_{i+1}p}{P_i}.
\]

We get the result by applying the following fact from the class notes:

\[
P_i = \begin{cases} 
\frac{1-(q/p)^i}{1-(q/p)} & \text{if } p \neq 1/2 \\
\frac{i}{N} & \text{if } p = 1/2 
\end{cases}
\]

\[\square\]

16. For the gambler’s ruin problem, let \(M_i\) be the mean number of games until the fortune hits 0 or \(N\), given that the fortune is initially \(i\), \(i = 0, 1, \ldots, N\). Show that

\[
M_0 = M_N = 0; \quad M_i = 1 + pM_{i+1} + qM_{i-1}, \quad i = 1, \ldots, N - 1.
\]

Can you also find \(M_i\)?

**Solution:** Well, this result probably follows by inspection (from a trivial one-step argument). But let’s carry the analysis out carefully. Let \(G\) be the number of games required. Clearly, \(G = 0\) if \(X_0\) starts off at 0 or \(N\). Suppose now that
\( i = 1, 2, \ldots, N - 1 \). Then, using a first-step analysis, we have

\[
M_i = \mathbb{E}[G|X_0 = i] = \sum_{j=0}^{N} \mathbb{E}[G|X_0 = i, X_1 = j] \mathbb{P}(X_1 = j|X_0 = i) \\
= \sum_{j=0}^{N} \mathbb{E}[G|X_1 = j] P_{ij} \\
= 1 + \sum_{j=0}^{N} \mathbb{E}[G|X_0 = j] P_{ij} \\
= 1 + \mathbb{E}[G|X_0 = i - 1] q + \mathbb{E}[G|X_0 = i + 1] p \\
= 1 + q M_{i-1} + p M_{i+1}. \quad \square
\]

Now, in order to find \( M_i \), you should really solve the above via difference equations. Nevertheless, you can cheat (like I did) and merely verify via direct substitution that the following works:

\[
M_i = \begin{cases} 
\frac{i}{q-p} - \frac{N}{q-p} \frac{1-(q/p)^i}{1-(q/p)^N} & \text{if } p \neq 1/2 \\
i(N - i) & \text{if } p = 1/2
\end{cases}. \quad \square
\]

**17.** Consider a branching process with \( \mu < 1 \). Show that if \( X_0 = 1 \), then the expected number of individuals that ever exist in the population is \( 1/(1-\mu) \). What if \( X_0 = n \)?

**Solution:** Let \( X_k \) denote the number of individuals from the \( k \)th generation. Then by the class notes,

\[
\mathbb{E}\left[ \sum_{k=0}^{\infty} X_k | X_0 = 1 \right] = \sum_{k=0}^{\infty} \mathbb{E}[X_k | X_0 = 1] = \sum_{k=0}^{\infty} \mu^k = \frac{1}{1-\mu}.
\]

This immediately implies that

\[
\mathbb{E}\left[ \sum_{k=0}^{\infty} X_k | X_0 = n \right] = \frac{n}{1-\mu}. \quad \square
\]

**18.** For a branching process, calculate \( \pi_0 \) when
(a) \( P_0 = 1/4, P_2 = 3/4. \)

(b) \( P_0 = 1/4, P_1 = 1/2, P_2 = 1/4. \)

(c) \( P_0 = 1/6, P_1 = 1/2, P_2 = 1/3. \)

**Solution:** Solve \( \pi_0 = \sum_{j=0}^{\infty} \pi_0^j P_j. \) Then we get (a) \( \pi_0 = 1/3; \) (b) \( \pi_0 = 1; \) (c) \( \pi_0 = 1/2. \) \( \square \)