3. Renewal Theory

3.1 Introduction

Def: A renewal process \((\text{RP})\) is a counting process \(w/\text{iid interarrivals}\).

Eq: Pois \((\lambda)\) process

Eq: A bulb is observed until burnout \& is immediately replaced. Failure times are \(\{S_n, n \geq 0\}\). This \(\text{is a pure RP if the first bulb is fresh.} \text{ If the first bulb has a different lifetime than the others, \(it's a delayed RP}\).}

Eq: \(\mathbb{P}\{X_n, n \geq 0\}\) is a Markov chain \(w/\text{a finite state space.} \text{ Define successive return times to state} i\) by:
\[
T_0(i) = \min \{n : X_n = i\}
\]
\[
\text{and } T_{n+1}(i) = \min \{m \geq T_n(i) : X_m = i\}, \ n \geq 0.
\]

If we start in \(i\), \(\{T_n(i), n \geq 0\}\) \(\text{is a pure RP.}\)

Eq: "On/off process." A machine operates, shuts down for repairs, operates, etc... \(\mathbb{P}\) all periods are iid. Then there are two RPs:
* Times when machine becomes inoperative
* Times when service is completed \& \(M/C\) works again.
Not'n: Let $X_1, X_2, \ldots$ be a seq of nonneg iid RV's w/ cdf $F(x)$ such that $F(0) < 1$.

$X_n =$ "time between the $(n-1)\text{st} + n\text{th} \text{ arrival}"

* Resnick also considers an initial arrival, $X_0$, which we'll sometimes deal with.

Let $\mu = \mathbb{E}X_n, \ n \geq 1$

$S_n = \sum_{i=1}^{n} X_i, \ n \geq 1$ $\leftarrow \text{n}\text{th arrival time}$

$N(t) = \max \{ n : S_n \leq t \}$ $\leftarrow \# \text{ of arrivals by time } t$

$\{N(t), t \geq 0\}$ is the renewal process

$\{S_n : n \geq 0\}$ is the renewal seq.
Aside #1 - Integration

Consider the equivalent notations,

\[ \int_0^\infty g(x) \, dU(x) \quad \text{or} \quad \int_0^\infty g(x) \, U(dx) \]
monotone fn

* If \( U \) is absolutely cts, then there is a "density" (not nec'ly a pdf) \( u(x) \geq 0 \) s.t.

\[ \int_0^T u(x) \, dx < \infty \quad \forall T > 0 \quad \text{and} \quad \int_a^b u(x) \, dx = U(a) - U(b) \]

Then \( \int_0^\infty g(x) \, dU(x) = \int_0^\infty g(x) \, u(x) \, dx \)

* If \( U \) is discrete, then there are pt. masses (weights) \( w_i \) at particular pts \( a_i \). Then

\[ U(x) = \sum_{i: \text{occ}} w_i \quad \text{and} \]

\[ \int_0^\infty g(x) \, dU(x) = \sum_i g(a_i) \, w_i \]

* Mixture: \( U(x) = \alpha U_{AC}(x) + \beta U_d(x) \), \( \alpha, \beta > 0 \)

\[ = \alpha \int g(x) \, U_{AC}(x) \, dx + \beta \sum_i g(a_i) \, w_i \]

Eq encountered in renewal theory:

Resnick counts arrivals at time 0.

\[ U(x) = 1 + \int_0^x u(t) \, dt \]
Aside #2 - Convolution

Def: $f$ and $g$ is a locally bounded (LB) fn and $F$ is a cdf. Then the convolution of $F$ and $g$ is the fn

$$F * g(t) = \int_0^t g(t-x) \, dF(x), \quad t \geq 0$$

Easy Properties:

1. $F * g \geq 0$ if $g(t) \geq 0 \quad \forall t \geq 0$

2. $F * g$ is LB.

Proof: \[
|F * g(t)| = \left| \int_0^t g(t-x) \, dF(x) \right| \\
\leq \int_0^t |g(t-x)| \, dF(x) \\
\leq C \int_0^t \, dF(x) = CF(t) \cdot \infty
\]

3. If $g$ is bdd + cts, then $F * g$ is cts.

4. Convolution can be repeated.

$$F^0*(x) \equiv 1_{(0,\infty)}(x) = \begin{cases} 1 \text{ if } x \in (0,\infty) \\ 0 \text{ otherwise} \end{cases}$$

$$F^1*(x) \equiv F(x)$$

$$F^{(n+1)}*(x) \equiv F^n * F(x)$$
(5) Associative Prop:
\[ F \ast (F \ast g) = (F \ast F) \ast g = F^{2 \ast} \ast g, \]
and \( F^{2 \ast} \) is a cdf.

(6) If \( X_1 \) and \( X_2 \) are independent RV's w/ cdf's \( F_1, F_2 \),
Then \( X_1 + X_2 \) has cdf \( F_1 \ast F_2 \):
\[
P(X_1 + X_2 \leq t) = \int_0^t F_2(t-x) \, dF_1(x)
= \int_0^t F_1(t-x) \, dF_2(x),
\]
which shows that \( F_1 \ast F_2 = F_2 \ast F_1 \) as well.

Similarly, if \( X_1, X_2, \ldots, X_n \) are iid w/ cdf \( F \),
then \( \sum_{i=1}^n X_i \) has cdf \( F^{\ast n} \).

Similarly if \( F_1, F_2 \) are absolutely continuous w/ densities \( f_1, f_2 \),
we have
\[
f_1 \ast f_2 (t) = \int_0^t f_1(t-x) f_2(x) \, dx
= \int_0^t f_2(t-x) f_1(x) \, dx
= f_2 \ast f_1 (t).
\]
etc.
Aside #3 - Laplace Transforms

$X$ is a nonneg $RV$ w/cdf $F$. The $LT$ of $X$ (or $F$) is:

$$\hat{F}(\lambda) = \mathbb{E}(e^{-\lambda X}) = \int_0^\infty e^{-\lambda x} dF(x), \quad \lambda \geq 0$$

Note: Since $e^{-\lambda x} \leq 1$, then $\hat{F}(\lambda) < 1$ \forall $\lambda \geq 0$.

1. Distinct distributions have distinct $LT$s.

2. If $X_1, X_2$ are indep w/cdf's $F_1, F_2$. Then

$$\hat{F_1 * F_2}(\lambda) = \hat{F_1}(\lambda) \hat{F_2}(\lambda).$$

Proof:

$$\hat{F_1 * F_2}(\lambda) = \mathbb{E}e^{-\lambda(X_1 + X_2)} = \mathbb{E}e^{-\lambda X_1} \mathbb{E}e^{-\lambda X_2}$$

$LT$ of convolution = Product of $LT$'s. (sum of RV's)

3. $e.g.$,

$$\hat{(F_1 * F_2)}(\lambda) = (\hat{F}(\lambda))^n$$

$e.g.$: $X \sim U(0,1) \Rightarrow \hat{F}(\lambda) = \mathbb{E}(e^{-\lambda X}) = \int_0^1 e^{-\lambda x} dx = \frac{1-e^{-\lambda}}{\lambda}$

$e.g.$: $X \sim \text{Exp}(\alpha) \Rightarrow \hat{F}(\lambda) = \frac{\alpha}{\alpha + \lambda}$
Eq: \( \sum X_i, \ldots, X_n \overset{iid}{\sim} \text{Exp}(\lambda) \). Then
\[
Y = \sum_{i=1}^{n} X_i \text{ has LT } \mathcal{LT} \left( e^{-\lambda y} \right) = \left( e^{-\lambda x_i} \right)^n = \left( \frac{\lambda}{\lambda + \lambda} \right)^n
\]

Meanwhile, \( Z \sim \text{Gamma}(\eta, \alpha) \). Then
\[
E(e^{-\lambda z}) = \int_{0}^{\infty} e^{-\lambda z} z^{(\alpha-1)} e^{-\alpha z} \frac{1}{\Gamma(\alpha)} \, dz = \left( \frac{\lambda}{\alpha + \lambda} \right)^n
\]

Then by uniqueness, \( Y \sim F \Leftrightarrow \text{Exponential is Gamma} \)

\( \text{(3)} \) Similar to mgfs,
\[
(-1)^n \frac{d^n}{d\lambda^n} \hat{F}(\lambda) = \int_{0}^{\infty} e^{-\lambda x} x^n \, dF(x)
\]

Letting \( \lambda \downarrow 0 \), we have (via mono. convergence)
\[
(-1)^n \left. \frac{d^n}{d\lambda^n} \hat{F}(\lambda) \right|_{\lambda=0} = EX^n
\]

So, \( EX = -\hat{F}'(0) \), \( EX^2 = \hat{F}''(0) \), ...

\( \text{(4)} \) Convenience Functions
\[
\int_{0}^{\infty} e^{-\lambda x} F(x) \, dx = \frac{1}{\lambda} \hat{F}(\lambda)
\]

Pf: \[
\int_{0}^{\infty} e^{-\lambda x} F(x) \, dx = \int_{0}^{\infty} e^{-\lambda x} \int_{0}^{x} dF(y) \, dx = \int_{0}^{\infty} \int_{y}^{\infty} e^{-\lambda x} \, dx \, dF(y) = \int_{0}^{\infty} \frac{1}{\lambda} e^{-\lambda y} \, dF(y)
\]

In add'n, \( \int_{0}^{\infty} e^{-\lambda x}(1-F(x)) \, dx = \frac{1}{\lambda} (1-\hat{F}(\lambda)) \).
Can easily extend LTs to $U(x)$ fn's that aren't cdf's. If $U(x)$ is nondecreasing on $[0, \infty)$, but that $U(\infty)$ may be $> 1$.

Def: If there is a $> 0$ s.t.

$$\hat{U}(\lambda) = \int_0^\infty e^{-\lambda x} \, dU(x) < \infty \text{ for } \lambda > \alpha,$$

then $\hat{U}(\lambda)$ is the LT of $U$.

Now we can think of $U(t) = EN(t)$ with interpreting $U(t)$ as a cdf.

**Ex:** $dU(x) = e^{ax} \, dx$

$$\hat{U}(\lambda) = \int_0^\infty e^{-\lambda x} e^{ax} \, dx = \left\{ \begin{array}{cc}
\frac{1}{\lambda - a}, & \lambda > a \\
\infty, & \lambda \leq a
\end{array} \right.$$

**Ex:** $dU(x) = e^{x^2} \, dx$

$$\hat{U}(\lambda) = \int_0^\infty e^{-\lambda x} e^{x^2} \, dx = \infty \forall \lambda > 0, \text{ so the LT doesn't exist}$$

**Ex:** $dU(x) = (1 - F(x)) \, dx$ (where $F$ is proper on $R_+$.)

$$\hat{U}(\lambda) = \frac{1}{\lambda} (1 - \hat{F}(\lambda))$$

**Ex:** $dU(x) = x^n \, dx$

$$\hat{U}(\lambda) = \int_0^\infty e^{-\lambda x} x^n \, dx = \frac{n!}{\lambda^{n+1}}$$
a good trick you can occasionally use

\[ \mathbb{E}(\text{Inversion}) \quad \mathbb{E} X_1, \ldots, X_n \sim U(0,1). \text{ Find } \mathbb{E} \sum_{i=1}^n X_i. \quad (\text{Note: } X_1 + X_2 \sim \text{Triang}(0,1,2)). \]

The LT of \( \sum_{i=1}^n X_i \) is

\[
\mathbb{E} e^{-\lambda \sum X_i} = (\mathbb{E} e^{-\lambda X_1})^n = \left( \frac{1 - e^{-\lambda}}{\lambda} \right)^n
\]

\[
= \sum_{k=0}^n \binom{n}{k} (-1)^k e^{-\lambda k} \frac{1}{\lambda^n} \quad (\ast)
\]

Meanwhile, let's write

\[
e^{-\lambda k} = \int_0^\infty e^{-\lambda x} dF_k(x), \text{ where } F_k \text{ puts an atom of weight 1 at } x=k.
\]

(\text{So } e^{-\lambda k} \text{ is the LT of } F_k.)

Also, think of the gamma distribution and note that

\[
\frac{1}{\lambda^n} = \int_0^\infty e^{-\lambda x} \frac{x^{n-1}}{(n-1)!} \, dx = \int_0^\infty \frac{\Gamma(n)}{(n-1)!} x^{n-1} \, dx = \Gamma(n)
\]

(\text{So } \frac{1}{\lambda^n} \text{ is the LT of } \Gamma(n).)

Thus, \( e^{-\lambda k} \frac{1}{\lambda^n} \) is the LT of \( F_k \ast g. \)

\[
F_k \ast g = \int_0^x g(x-y) \, dF_k(y) = \begin{cases} 0 & \text{if } x < k \\
\frac{(x-k)^{n-1}}{(n-1)!} & \text{if } x \geq k
\end{cases}
\]

\[
\equiv \frac{(x-k)^{n-1}}{(n-1)!}.
\]
Thus, \( e^{-\lambda k} \frac{1}{\lambda^n} \) is the LT of \( \frac{(x-k)^{n-1}}{(n-1)!} \).

\[ \Rightarrow e^{-\lambda k} \frac{1}{\lambda^n} = \int_0^\infty e^{-\lambda x} \frac{(x-k)^{n-1}}{(n-1)!} \, dx \]

\[ \Rightarrow \text{LT of } \sum_{i=1}^n X_i \text{ is (from (*) )} \]

\[ E(e^{-\lambda \sum X_i}) = \sum_{k=0}^n \frac{n!}{k!} (-1)^k e^{-\lambda k} \frac{1}{\lambda^n} \]

\[ = \sum_{k=0}^n \frac{n!}{k!} (-1)^k \int_0^\infty e^{-\lambda x} \frac{(x-k)^{n-1}}{(n-1)!} \, dx \]

\[ = \int_0^\infty e^{-\lambda x} \sum_{k=0}^n \frac{n!}{k!} (-1)^k \frac{(x-k)^{n-1}}{(n-1)!} \, dx \]

\[ \text{pdf of } \sum X_i \quad (\text{by inversion}) \]
3.2 Properties of \( N(t) \)

*Special Note*: Resnike often has the first renewal occur at \( Y_0 = 0 \) or \( Y_0 \sim G \), where \( G \) has a different distribution than \( Y_1, Y_2, \ldots \) i.i.d. \( F \).

I'll try to be sort of consistent about this, but in any case, I'll point out any issues.

Now let \( S_n = Y_0 + Y_1 + \ldots + Y_n \), though I'll occasionally revert to \( S_n = Y_1 + \ldots + Y_n \).

\[ N(t) \equiv \# \text{ of renewals by time } t = \sum_{n=0}^{\infty} 1_{(0,t)}(S_n) \]

The renewal fn is

\[ U(t) = E[N(t)] = \sum_{n=0}^{\infty} E 1_{(0,t)}(S_n) = \sum_{n=0}^{\infty} F^{n*}(t) \]

We have the foll. relationships between \( S_n \) and \( N(t) \):

\[ N(t) \leq n \iff S_n > t, \quad n \geq 0 \]

# arr'rs by time \( t \) is \( S_n \)

(counts \( Y_0 \))

\[ S_{N(t) - 1} \leq t < S_{N(t)} \] when \( N(t) \geq 1 \)

Time of \( N(t)^{th} \) arr'rs is \( \leq t \)

Time of \( N(t) + 1 \)^{th} arr'rs occurs after \( t \)

arr'rs is \( \text{AFTER } t \)
\[ N(t) = n \iff S_{n-1} \leq t < S_n, \quad n \geq 1 \]

\[ \text{Remark: If we don't count that } \]
\[ \text{annoying } \gamma \text{ obset, we can write } \]
\[ N(t) \geq n \iff S_n \leq t \]

\[ \text{# arrivals by time } t \text{ is at least } n \text{ by time } t \]

Can get moments of \( N(t) \)... 

**Thm 3.3.1**: For any \( t \geq 0 \),

\[ \sum_{n=0}^{\infty} \gamma^n F^{*n}(t) < \infty \quad \text{for } \gamma < 1/F(0) \]

\[ (2) \text{ The mgf of } N(t) < \infty (\implies \text{all moments are finite, in particular, } |U(t)| < \infty). \]

**Pf**: We'll need Markov's inequality:

\[ P( |X| \geq k) \leq \frac{E|X|^r}{k^r} \quad \text{for } r, k > 0. \]

\[ (1) \text{ if } \gamma < 1/F(0). \text{ Then } \]

\[ \lim_{\lambda \to \infty} F(\lambda) = \lim_{\lambda \to \infty} \int_{0}^{\infty} e^{-\lambda x} dF(x) \]

\[ = \lim_{\lambda \to \infty} \left( F(0) + \int_{(0, \infty)} e^{-\lambda x} dF(x) \right) \]

possible atom
\[
\lim_{\lambda \to \infty} \hat{F}(\lambda) = F(0) + \int_{(0,\infty)} \lim_{\lambda \to \infty} e^{-\lambda x} \, dF(x) = F(0)
\]

One can choose \( \lambda \) large enough so that

\[
\hat{F}(\lambda) \gamma \approx F(0) \gamma < 1,
\]

and thus,

\[
\sum_{n=0}^{\infty} \gamma^n F^n(t) = \sum_{n=0}^{\infty} \gamma^n P(S_n \leq t)
\]

\[
= \sum_{n=0}^{\infty} \gamma^n P(e^{-\lambda S_n} \geq e^{-\lambda t})
\]

\[
\leq \sum_{n=0}^{\infty} \gamma^n E e^{-\lambda S_n} / e^{-\lambda t} = e^{\lambda t} \sum_{n=0}^{\infty} (\gamma F(\lambda))^n < \infty
\]

Markov's \( \omega / \lambda = e^{-\lambda S_n}, \lambda = e^{-\lambda t}, \lambda = 1 \)

(2) A pos. RV \( Z \) has an mgf \( \iff \) \( Z \) has a tail that is exponentially bounded (else a higher-order moment will explode). I.e., for some \( K, c > 0 \) and \( \forall x > 0, \quad P(Z > x) \leq Ke^{-cx} \).

\( \text{Pf } \Rightarrow \) \( \iff \) \( Z \) has an mgf in \((0, \infty)\). I.e., the mgf is finite.

Then for \( 0 < \alpha \),

\[
P(Z > x) = P(e^{\alpha Z} > e^{\alpha x}) \leq \frac{E e^{\alpha Z}}{e^{\alpha x}} \text{ if exists}
\]
\[ \Rightarrow \quad P(Z > x) \leq Ke^{-cx} \quad (Z \text{ is exp'ly bdd}). \]

Then for $0 < c$, we have

\[
E(e^{\alpha Z}) < \infty \iff E(e^{\alpha Z - 1}) < \infty
\]

But

\[
E(e^{\alpha Z - 1}) = \int_{0}^{\infty} e^{\alpha u} P(Z > u) \, du
\]

\[
\text{convenience: }\quad \text{eq'n w/ } \lambda \to \alpha
\]

\[
\Rightarrow \int_{0}^{\infty} e^{\alpha u} Ke^{-c u} \, du < \infty.
\]

Thus, to finally prove (2), "all" we have to do is t.s. that $P(N(t) > n)$ is exp'ly bdd...

Take $1/\delta < 1/F(0)$. From the previous half of the pf, we know that

\[
\delta^n F^{n^*}(t) \to 0 \quad \text{as } n \to \infty.
\]

Thus, $\exists n_0 \text{ s.t. for } n \geq n_0$

\[
F^{n^*}(t) \leq \delta^{-n} = e^{-(\log \delta) n}
\]

So, w/ $\gamma = 0$, we have for $n \geq n_0$ that

\[
P(N(t) > n) = P(S_n \leq t) = F^{n^*}(t) \leq e^{-(\log \delta) n}.
\]

For large enough $K$, this can be extended to all $n$, i.e., $P(N(t) > n) \leq K e^{-cn} \forall n$. //
Good news: $U(t)$ is always finite.

Bad news: $U(t)$ can be tough to compute.

\[ X \sim \text{Exp}(\lambda), \quad f(x) = \lambda e^{-\lambda x}, \quad x \geq 0 \]

\[ f^{n*}(x) = \alpha^n x^{n-1} \frac{e^{-\lambda x}}{(n-1)!}, \quad x \geq 0 \quad \text{(gamma)} \]

pdf of $S_n, n \geq 1$

\[ \Rightarrow \sum_{n=1}^{\infty} f^{n*}(x) = \sum_{n=1}^{\infty} \int_{0}^{x} \alpha^n s^{n-1} \frac{e^{-\alpha s}}{(n-1)!} ds \]

\[ = \int_{0}^{x} \sum_{n=1}^{\infty} \alpha^n (\alpha s)^{n-1} \frac{e^{-\alpha s}}{(n-1)!} ds \]

\[ = \int_{0}^{x} \alpha e^{-\alpha s} \sum_{n=0}^{\infty} \frac{(\alpha s)^n}{n!} ds = \alpha x \]

\[ \sum_{n=0}^{\infty} e^{\alpha s} \]

\[ \Rightarrow U(x) = \sum_{n=0}^{\infty} f^{n*}(x) = 1 + \alpha x \quad \text{// Should make intuitive sense!} \]

& Don't forget $Y_0$!

\[ \text{Eq: Inversion w/ Gamma (2, 1).} \]

\[ f(x) = xe^{-x}, \quad x \geq 0 \]

Let's find the "density" of \( \sum_{n=1}^{\infty} f^{n*} \) ... not a pdf
The LT of \( \sum_{n=1}^{\infty} F_n^{*} \) is

\[
\left( \sum_{n=1}^{\infty} F_n^{*} \right)(\lambda) = \sum_{n=1}^{\infty} \left( \sum_{k=1}^{\infty} F_k^{*} \right)(\lambda) = \sum_{n=1}^{\infty} \left( \sum_{k=1}^{n} F_k^{*} \right)(\lambda)
\]

\[
= \sum_{n=1}^{\infty} \left( \frac{1}{(1+\lambda)^2} \right)^n = \frac{1}{\lambda(\lambda+2)} = \frac{1}{2} - \frac{1}{2(\lambda+2)}
\]

\( \text{Gam}(2,1) \)

\[
= \int_{0}^{\infty} \frac{1}{2} e^{-\lambda x} \, dx - \int_{0}^{\infty} \frac{1}{2} e^{-\lambda x} e^{-2x} \, dx
\]

\[
= \int_{0}^{\infty} e^{-\lambda x} \left( \frac{1}{2} - \frac{1}{2} e^{-2x} \right) \, dx
\]

and so this must be the "density" of \( \sum_{n=1}^{\infty} F_n^{*} \) (by uniqueness of LT).

\[
\Rightarrow \ U(x) = \sum_{n=0}^{\infty} F_{n+1}^{*}(x) = 1 + \int_{0}^{x} \frac{1}{2} (1 - e^{-2u}) \, du
\]

\[
= \frac{3}{4} + \frac{x}{2} + \frac{1}{4} e^{-2x}
\]

Later, we'll look at renewal eq's that will occasionally allow us to calculate \( U(x) \) explicitly.
Limit Thms for $N(t)$

Thm 3.3.2: \( \mathbb{P} \mu = \text{E}Y_1 \) and \( \sigma^2 = \text{V}Y_1 < \infty \).

(1) If \( \mathbb{P}(Y_0 < \infty) = 1 \), then a.s.,
\[
\frac{N(t)}{t} \xrightarrow{a.s.} \mu \quad \text{as} \quad t \to \infty.
\]

(2) \( N(t) \) is asymptotically normal with

mean \( \frac{t}{\mu} \) and var \( \frac{t \sigma^2}{\mu^3} \).

\( \text{Before type in book...} \quad \text{CDF} \quad \mathcal{N}(0,1) \quad \text{I.e.,} \quad \lim_{t \to \infty} \mathbb{P} \left( \frac{N(t) - t/\mu}{\sqrt{t \sigma^2 / \mu^3}} \leq x \right) = \Phi(x). \)

PF (1) By the SLLN,
\[
\frac{S_n}{n} = \frac{Y_0}{n} + \sum_{i=1}^{n} \frac{Y_i}{n} \xrightarrow{a.s.} 0 + \mu = \mu \quad \text{as} \quad n \to \infty
\]

In addition, note that \( \lim_{t \to \infty} N(t) = \infty. \)

PF: \( \forall \) not, i.e., \( \forall \) \( N(\infty) < \infty \). Then some interarr'l time is \( \infty \). This is a contradiction.

PF2: \( P(N(t) > n) = P(S_n \leq t) = G \cdot F^{n*}(t) \to 1 \)

\[
\text{CDF of } Y_0 \text{ as } t \to \infty
\]

Now apply "squeeze" to \( S_{N(t)-1} \leq t < S_{N(t)} \)

\[
\Rightarrow \frac{S_{N(t)-1}}{N(t)-1} \leq \frac{t}{N(t)} < \frac{S_{N(t)}}{N(t)} \quad (\text{SLLN})
\]

As \( t \to \infty \), the 2 extremes converge to \( \mu \) a.s. /
(2) By the usual CLT,

\[ \lim_{n \to \infty} P \left( \frac{S_n - n\mu}{\sqrt{n} \sigma} \leq x \right) = \Phi(x), \text{ uniformly in } x \in \mathbb{R}. \]

So \[ P \left( \frac{N(t) - t\mu}{\sqrt{\sigma^2 t / \mu^3}} \leq x \right) \]

\[ = P \left( N(t) \leq \left[ x \left( \frac{\sigma^2 t}{\mu^3} \right)^{\frac{1}{2}} + \frac{t}{\mu} \right] \right) \]

\[ \uparrow \]

\[ \equiv h(t) \]

I.e., \[ P \left( S_{h(t)} > t \right) = P \left( \frac{S_{h(t)} - \mu h(t)}{\sigma \sqrt{h(t)}} > \frac{t - \mu h(t)}{\sigma \sqrt{h(t)}} \right) \]

\[ \equiv z(t) \]

All we need t.s. is that \( h(t) \to \infty \) and \( z(t) \to -\infty \).

First, \( h(t) \sim \frac{t}{\mu} \to \infty \) \( \checkmark \)

Second, \( h(t) = x \sqrt{\frac{\sigma^2 t}{\mu^3}} + \frac{t}{\mu} + \frac{\varepsilon(t)}{1 + |\varepsilon(t)|} \leq 1 \)

\[ \Rightarrow z(t) = \frac{t - \mu h(t)}{\sigma \sqrt{h(t)}} \]

\[ = \frac{t - \mu x \sqrt{\frac{\sigma^2 t}{\mu^3}} - t - \mu \varepsilon(t)}{\sigma \sqrt{h(t)}} \]

\[ \sim \frac{-\mu x \sqrt{\frac{\sigma^2 t}{\mu^3}} - \mu \varepsilon(t)}{\sigma t \mu} \]

\[ \to -\infty \]
Eg (Ross): 2 machines process jobs forever.

\[ M/c \ 1 \ \text{jobs} \sim \text{Gamma} (n=4, \lambda=2) \]
\[ M/c \ 2 \ \text{jobs} \sim \text{U}(0,4) \]

Find the approx prob that the 2 M/c's can do at least 90 jobs by \( t=100 \).

Solu': \( N_i(t) = \# \text{of jobs on } M/c \ i \text{ by time } t. \)

\[ M/c \ 1: \ \text{Exp} = 2, \ \text{Var} = 1 \]
\[ M/c \ 2: \ \text{Exp} = 2, \ \text{Var} = 16/12 \]

\[ N_1(100) \sim \text{Nor}(50, \frac{100}{8}) \]
\[ N_2(100) \sim \text{Nor}(50, \frac{100}{6}) \]

\[ N_1(100) + N_2(100) \sim N(100, \frac{175}{6}) \]

\[ P(N_1(100) + N_2(100) > 89.5) \approx 0.974 \]

\( \text{continuity correction} \)
Elementary Renewal Thm 3.3.3 : Let \( \mu = \mathbb{E}Y \), and \( \mathbb{P} \{Y < \infty\} \) a.s. Then
\[
\lim_{t \to \infty} \frac{U(t)}{t} = \frac{1}{\mu}.
\]

Thm 3.3.2: \( \frac{N(t)}{t} \to \frac{1}{\mu} \) as \( t \to \infty \) \( \chi \)

Thm 3.3.3 \( \frac{U(t)}{t} \to \frac{1}{\mu} \) as \( t \to \infty \)

The ERT is obviously similar to the previous Thm, but \( 3.3.2 \not\Rightarrow \) ERT.

Eg: Just because a RV chugs to some constant doesn't mean that its exp value chugs to that constant. For instance, if \( U \sim U(0,1) \) and \( Y_n = 0 \) if \( U > \frac{1}{n} \) \( \chi \)

Clearly, \( Y_n \to 0 \) \( \chi \) by \( \mathbb{E}Y_n = 1 \) \( \forall n \) \( \chi \).