4. **Markov Chains** (9/23/12, cf. Ross)

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4.1. Introduction

Definition: A *stochastic process* (SP) \( \{X(t) : t \in T\} \) is a collection of RV’s. Each \( X(t) \) is a RV; \( t \) is usually regarded as “time.”

Example: \( X(t) = \) the number of customers in line at the post office at time \( t \).

Example: \( X(t) = \) the price of IBM stock at time \( t \).
$T$ is the *index set* of the process. If $T$ is countable, then \( \{X(t) : t \in T\} \) is a *discrete-time* SP. If $T$ is some continuum, then \( \{X(t) : t \in T\} \) is a *continuous-time* SP.

Example: \( \{X_n : n = 0, 1, 2, \ldots\} \) (index set of non-negative integers)

Example: \( \{X(t) : t \geq 0\} \) (index set is \( \mathbb{R}_+ \))
The *state space* of the SP is the set of all possible values that the RV’s $X(t)$ can take.

Example: If $X_n = j$, then the process is in *state* $j$ at time $n$.

Any realization of $\{X(t)\}$ is a *sample path*. 
Definition: A *Markov chain* (MC) is a SP such that whenever the process is in state $i$, there is a fixed *transition probability* $P_{ij}$ that its next state will be $j$.

Denote the “current” state (at time $n$) by $X_n = i$.

Let the event $A = \{X_0 = i_0, X_1 = i_1, \ldots X_{n-1} = i_{n-1}\}$ be the previous history of the MC (before time $n$).
\{X_n\} has the **Markov property** if it forgets about its past, i.e.,

\[
\Pr(X_{n+1} = j | A \cap X_n = i) = \Pr(X_{n+1} = j | X_n = i).
\]

\{X_n\} is **time homogeneous** if

\[
\Pr(X_{n+1} = j | X_n = i) = \Pr(X_1 = j | X_0 = i) = P_{ij},
\]
i.e., if the transition probabilities are independent of \(n\).
Recap: A Markov chain is a SP such that

$$\Pr(X_{n+1} = j | A \cap X_n = i) = P_{ij},$$

i.e., the next state depends only on the current state (and is indep of the time).
Since $P_{ij}$ is a probability, $0 \leq P_{ij} \leq 1$ for all $i, j$.

Since the process has to go from $i$ to some state, we must have $\sum_{j=0}^{\infty} P_{ij} = 1$, for all $i$. Note that it may be possible to go from $i$ to $i$ (i.e., “stay” at $i$).

Definition: The one-step transition matrix is

$$
P = \begin{pmatrix}
P_{00} & P_{01} & P_{02} & \cdots \\
P_{10} & P_{11} & P_{12} & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}
$$
Example: A frog lives in a pond with three lily pads (1,2,3). He sits on one of the pads and periodically rolls a die. If he rolls a 1, he jumps to the lower numbered of the two unoccupied pads. Otherwise, he jumps to the higher numbered pad. Let $X_0$ be the initial pad and let $X_n$ be his location just after the $n$th jump. This is a MC since his position only depends on the current position, and the $P_{ij}$'s are independent of $n$.

$$
P = \begin{pmatrix}
0 & 1/6 & 5/6 \\
1/6 & 0 & 5/6 \\
1/6 & 5/6 & 0
\end{pmatrix}.
$$
Example: Let $X_i$ denote the weather (rain or sun) on day $i$. We’ll think of $X_{i-1}$ as yesterday, $X_i$ as today, and $X_{i+1}$ as tomorrow. Suppose that

\[
\begin{align*}
\Pr(X_{i+1} = R \mid X_{i-1} = R, X_i = R) &= 0.7 \\
\Pr(X_{i+1} = R \mid X_{i-1} = S, X_i = R) &= 0.5 \\
\Pr(X_{i+1} = R \mid X_{i-1} = R, X_i = S) &= 0.4 \\
\Pr(X_{i+1} = R \mid X_{i-1} = S, X_i = S) &= 0.2
\end{align*}
\]
$X_0, X_1, \ldots$ isn’t quite a MC, since the probability that it’ll rain tomorrow depends on $X_i$ and $X_{i-1}$.

We’ll transform the process into a MC by defining the following states in terms of today and yesterday.

0: $X_{i-1} = R, X_i = R$

1: $X_{i-1} = S, X_i = R$

2: $X_{i-1} = R, X_i = S$

3: $X_{i-1} = S, X_i = S$
Thus, we have, e.g.,

\[
\Pr(X_{i+1} = R \mid X_{i-1} = R, X_i = R) = P_{00} = 0.7
\]
\[
\Pr(X_{i+1} = S \mid X_{i-1} = R, X_i = R) = P_{02} = 0.3
\]

Using similar reasoning, we get

\[
P = \begin{pmatrix}
0.7 & 0 & 0.3 & 0 \\
0.5 & 0 & 0.5 & 0 \\
0 & 0.4 & 0 & 0.6 \\
0 & 0.2 & 0 & 0.8
\end{pmatrix}.
\]

\[\diamondsuit\]
4. Markov Chains

Example: A MC whose state space is given by the integers is called a *random walk* if $P_{i,i+1} = p$ and $P_{i,i-1} = 1 - p$.

\[
P = \begin{pmatrix}
& & & & & \\
& \vdots & \vdots & \vdots & \vdots & \\
& \cdots & 1-p & 0 & p & 0 & 0 & \cdots \\
& \cdots & 0 & 1-p & 0 & p & 0 & \cdots \\
& \cdots & 0 & 0 & 1-p & 0 & p & \cdots \\
& \cdots & 0 & 0 & 0 & 1-p & 0 & \cdots \\
& & & & & \\
& \vdots & \vdots & \vdots & \vdots & \\
\end{pmatrix}.
\]
Example (Gambler’s Ruin): Every time a gambler plays a game, he wins $1 w.p. p$, and he loses $1 w.p. 1 - p$. He stops playing as soon as his fortune is either $0$ or $N$. The gambler’s fortune is a MC with the following $P_{ij}$’s:

\[
P_{i,i+1} = p, \quad i = 1, 2, \ldots, N - 1
\]

\[
P_{i,i-1} = 1 - p, \quad i = 1, 2, \ldots, N - 1
\]

\[
P_{0,0} = P_{N,N} = 1
\]

0 and $N$ are absorbing states — once the process enters one of these states, it can’t leave. ✷
Example (Ehrenfest Model): A random walk on a finite set of states with “reflecting” boundaries. Set of states is \( \{1, 2, \ldots, a\} \).

\[
P_{ij} = \begin{cases} 
\frac{a-i}{a} & \text{if } j = i + 1 \\
\frac{i}{a} & \text{if } j = i - 1 \\
0 & \text{otherwise}
\end{cases}
\]

Idea: Suppose A has \( i \) marbles, B has \( a - i \). Select a marble at random, and put it in the other container.
4.2 Chapman-Kolmogorov Equations

Definition: The \( n \)-step transition probability that a process currently in state \( i \) will be in state \( j \) after \( n \) additional transitions is

\[
P_{ij}^{(n)} \equiv \Pr(X_n = j | X_0 = i), \quad n, i, j \geq 0.
\]

Note that \( P_{ij}^{(1)} = P_{ij} \), and

\[
P_{ij}^{(0)} = \begin{cases} 
1 & \text{if } i = j \\
0 & \text{otherwise}
\end{cases}.
\]
Theorem (C-K Equations):

\[ P_{ij}^{(n+m)} = \sum_{k=0}^{\infty} P_{ik}^{(n)} P_{kj}^{(m)}. \]

Think of going from \( i \) to \( j \) in \( n + m \) steps with an intermediate stop in state \( k \) after \( n \) steps; then sum over all possible \( k \) values.
Proof: By definition,

\[ P_{ij}^{(n+m)} = \Pr(X_{n+m} = j | X_0 = i) \]

\[ = \sum_{k=0}^{\infty} \Pr(X_{n+m} = j \cap X_n = k | X_0 = i) \quad \text{(total prob)} \]

\[ = \sum_{k=0}^{\infty} \Pr(X_{n+m} = j | X_0 = i \cap X_n = k) \Pr(X_n = k | X_0 = i) \]

(since \( \Pr(A \cap C | B) = \Pr(A | B \cap C) \Pr(C | B) \))

\[ = \sum_{k=0}^{\infty} \Pr(X_{n+m} = j | X_n = k) \Pr(X_n = k | X_0 = i) \]

(Markov property). ♦
Definition: The \( n \)-step transition matrix is

\[
P^{(n)} = \begin{pmatrix}
P^{(n)}_{00} & P^{(n)}_{01} & P^{(n)}_{02} & \cdots \\
P^{(n)}_{10} & P^{(n)}_{11} & P^{(n)}_{12} & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

The C-K equations imply \( P^{(n+m)} = P^{(n)}P^{(m)} \).

In particular, \( P^{(2)} = P^{(1)}P^{(1)} = PP = P^2 \).

By induction, \( P^{(n)} = P^n \).
Example: Let $X_i = 0$ if it rains on day $i$; otherwise, $X_i = 1$. Suppose $P_{00} = 0.7$ and $P_{10} = 0.4$. Then

$$P = \begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix}.$$ 

Suppose it rains on Monday. Then the prob that it rains on Friday is $P_{00}^{(4)}$. Note that

$$P^{(4)} = P^4 = \begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix}^4 = \begin{pmatrix} 0.5749 & 0.4251 \\ 0.5668 & 0.4332 \end{pmatrix},$$

so that $P_{00}^{(4)} = 0.5749$. ◊
Unconditional Probabilities

Suppose we know the “initial” probabilities,

\[ \alpha_i \equiv \Pr(X_0 = i), \quad i = 0, 1, \ldots. \]

(Note that \( \sum_i \alpha_i = 1 \).) Then by total probability,

\[
\Pr(X_n = j) = \sum_{i=0}^{\infty} \Pr(X_n = j \cap X_0 = i)
\]

\[
= \sum_{i=0}^{\infty} \Pr(X_n = j \mid X_0 = i) \Pr(X_0 = i)
\]

\[
= \sum_{i=0}^{\infty} P_{ij}^{(n)} \alpha_i.
\]
Example: In the above example, suppose $\alpha_0 = 0.4$ and $\alpha_1 = 0.6$. Find the prob that it will not rain on the 4th day after we start keeping records (assuming nothing about the first day).

$$\Pr(X_4 = 1) = \sum_{i=0}^{\infty} P_{i1}^{(4)} \alpha_i$$

$$= P_{01}^{(4)} \alpha_0 + P_{11}^{(4)} \alpha_1$$

$$= (0.4251)(0.4) + (0.4332)(0.6)$$

$$= 0.4300. \diamondsuit$$
4.3 Types of States

Definition: If $P_{ij}^{(n)} > 0$ for some $n \geq 0$, state $j$ is accessible from $i$.

Notation: $i \rightarrow j$.

Definition: If $i \rightarrow j$ and $j \rightarrow i$, then $i$ and $j$ communicate.

Notation: $i \leftrightarrow j$. 
Theorem: Communication is an equivalent relation:

(i) $i \leftrightarrow i$ for all $i$ (reflexive).

(ii) $i \leftrightarrow j$ implies $j \leftrightarrow i$ (symmetric).

(iii) $i \leftrightarrow j$ and $j \leftrightarrow k$ imply $i \leftrightarrow k$ (transitive).

Proof: (i) and (ii) are trivial, so we’ll only do (iii). To do so, suppose $i \leftrightarrow j$ and $j \leftrightarrow k$. Then there are $n, m$ such that $P_{ij}^{(n)} > 0$ and $P_{jk}^{(m)} > 0$. So by C-K,

$$P_{ik}^{(n+m)} = \sum_{r=0}^{\infty} P_{ir}^{(n)} P_{rk}^{(m)} \geq P_{ij}^{(n)} P_{jk}^{(m)} > 0.$$ 

Thus, $i \rightarrow k$. Similarly, $k \rightarrow i$. \diamond
Definition: An *equivalence class* consists of all states that communicate with each other.

Remark: Easy to see that two equiv classes are disjoint.

Example: The following $P$ has equiv classes $\{0, 1\}$ and $\{2, 3\}$.

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{3}{4} & \frac{1}{4} \\ 0 & 0 & \frac{1}{4} & \frac{3}{4} \end{pmatrix}.$$
Example: \( P \) again has equiv classes \( \{0, 1\} \) and \( \{2, 3\} \) — note that 1 isn’t accessible from 2.

\[
P = \begin{pmatrix}
\frac{1}{2} & \frac{1}{2} & 0 & 0 \\
\frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 \\
0 & 0 & \frac{3}{4} & \frac{1}{4} \\
0 & 0 & \frac{1}{4} & \frac{3}{4}
\end{pmatrix}.
\]

Definition: A MC is \emph{irreducible} if there is only one equiv class (i.e., if all states communicate).

Example: The previous two examples are \textbf{not} irreducible.
Example: The following $P$ is irreducible since all states communicate ("loop" technique: $0 \rightarrow 1 \rightarrow 0$).

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}.$$ 

Example: $P$ is irreducible since $0 \rightarrow 2 \rightarrow 1 \rightarrow 0$.

$$P = \begin{pmatrix} \frac{1}{4} & 0 & \frac{3}{4} \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$
Definition: The probability that the MC eventually returns to state $i$ is

$$f_i \equiv \Pr(X_n = i \text{ for some } n \geq 1|X_0 = i).$$

Example: The following MC has equiv classes $\{0, 1\}$, $\{2\}$, and $\{3\}$, the latter of which is absorbing.

$$P = \begin{pmatrix}
\frac{1}{2} & \frac{1}{2} & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 \\
\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\
0 & 0 & 0 & 1
\end{pmatrix}.$$

We have $f_0 = f_1 = 1$, $f_2 = \frac{1}{4}$, $f_3 = 1$. ♦
Remark: The $f_i$'s are usually hard to compute.

Definition: If $f_i = 1$, state $i$ is *recurrent*. If $f_i < 1$, state $i$ is *transient*.

Theorem: Suppose $X_0 = i$. Let $N$ denote the number of times that the MC is in state $i$ (before leaving $i$ forever). Note that $N \geq 1$ since $X_0 = i$. Then $i$ is recurrent iff $E[N] = \infty$ (and $i$ is transient iff $E[N] < \infty$).
Proof: If $i$ is recurrent, it’s easy to see that the MC returns to $i$ an infinite number of times; so $E[N] = \infty$. Otherwise, suppose $i$ is transient. Then

$$
\Pr(N = 1) = 1 - f_i \quad \text{(never returns)}
$$

$$
\Pr(N = 2) = f_i(1 - f_i) \quad \text{(returns exactly once)}
$$

$$
\vdots
$$

$$
\Pr(N = k) = f_i^{k-1}(1 - f_i) \quad \text{(returns $k - 1$ times)}
$$

So $N \sim \text{Geom}(1 - f_i)$. Finally, since $f_i < 1$, we have $E[N] = \frac{1}{1-f_i} < \infty$. \diamondsuit
Theorem: $i$ is recurrent iff $\sum_{n=1}^{\infty} P_{ii}^{(n)} = \infty$. (So $i$ is transient iff $\sum_{n=1}^{\infty} P_{ii}^{(n)} < \infty$.)

Proof: Define the event

$$A_n \equiv \begin{cases} 1 & \text{if } X_n = i, \\ 0 & \text{if } X_n \neq i \end{cases}.$$ 

Note that $N \equiv \sum_{n=1}^{\infty} A_n$ is the number of returns to $i$.

Then by the trick that allows us to treat the expected value of an indicator function as a probability, we have...
\[ \sum_{n=1}^{\infty} P_{ii}^{(n)} = \sum_{n=1}^{\infty} \Pr(X_n = i \mid X_0 = i) \]
\[ = \sum_{n=1}^{\infty} E[A_n \mid X_0 = i] \quad \text{(trick)} \]
\[ = E \left[ \sum_{n=1}^{\infty} A_n \mid X_0 = i \right] \]
\[ = E[N \mid X_0 = i] \quad (N = \text{number of returns}) \]
\[ = \infty \]
\[ \Leftrightarrow \text{i is recur \ (by previous theorem).} \]
Corollary 1: If $i$ is recur and $i \leftrightarrow j$, then $j$ is recur.

Proof: See Ross.  

Corollary 2: In a MC with a finite number of states, not all of the states can be transient.

Proof: Suppose not. Then the MC will run out of states not to go to an infinite number of times. This is a contradiction.
Corollary 3: If one state in an equiv class is transient, then all states are trans.

Proof: Suppose not, i.e., suppose there's a recur state. Since all states in the equiv class communicate, Corollary 1 implies all states are recur. This is a contradiction. ♦

Corollary 4: All states in a finite irreducible MC are recurrent.
Proof: Suppose not, i.e., suppose there’s a trans state. Then Corollary 3 implies all states are trans. But this contradicts Corollary 1.

Definition: By Corollary 1, all states in an equiv class are recur if one state in that class is recur. Such a class is a recurrent equiv class.

By Corollary 3, all states in an equiv class are trans if one state in that class is trans. Such a class is a transient equiv class.
Example: Consider the prob transition matrix

\[ P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}. \]

Clearly, all states communicate. So this is a finite, irreducible MC. So Corollary 4 implies all states are recurrent. ◇
Example: Consider

$$P = \begin{pmatrix} 1/4 & 0 & 0 & 3/4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$ 

Loop: 0 → 3 → 2 → 1 → 0. Thus, all states communicate; so they’re all recurrent. ♦
Example: Consider

\[ P = \begin{pmatrix}
\frac{1}{4} & 0 & \frac{3}{4} & 0 & 0 \\
0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\
\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\
0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\
\frac{1}{5} & \frac{1}{5} & 0 & 0 & \frac{3}{5}
\end{pmatrix}. \]

The equiv classes are \{0, 2\} (recur), \{1, 3\} (recur), and \{4\} (trans).  

\[ \diamond \]
Example: Random Walk: A drunk walks on the integers 0, ±1, ±2, ... with transition probabilities

\[ P_{i,i+1} = p \]
\[ P_{i,i-1} = q = 1 - p \]

(i.e., he steps to the right w.p. \( p \) and to the left w.p. \( 1 - p \)).
The prob transition matrix is

\[ P = \begin{pmatrix}
\vdots \\
q & 0 & p & 0 & 0 \\
0 & q & 0 & p & 0 \\
\ldots & 0 & 0 & q & 0 & p & \ldots \\
0 & 0 & 0 & q & 0 \\
\vdots 
\end{pmatrix}. \]

Are the states recurrent or transient?

Clearly, all states communicate. So Corollary 1 implies that if one of the states are recur, then they all are. Otherwise, all states will be transient.
Consider a typical state 0. If 0 is recurrent [transient], then all states will be recurrent [transient]. We’ll find out which is the case by calculating \( \sum_{n=1}^{\infty} P_{00}^{(n)} \).

Suppose the drunk starts at 0. Since it’s impossible for him to return to 0 in an odd number of steps, we see that \( P_{00}^{(2n+1)} = 0 \) for all \( n \geq 0 \).
So the only chance he has of returning to 0 is if he’s taken an even number of steps, say $2n$. Of these steps, $n$ must be taken to the left, and $n$ to the right. So, thinking binomial, we have

\[
P_{00}^{(2n)} = \binom{2n}{n} p^n q^n = \frac{(2n)!}{n!n!} p^n q^n, \quad n \geq 1.
\]

Aside: For large $n$, Stirling’s approximation says that

\[
n! \approx \sqrt{2\pi n} n^{n+\frac{1}{2}} e^{-n}.
\]
After the smoke clears,

\[ P_{00}^{(2n)} \approx \frac{[4p(1 - p)]^n}{\sqrt{\pi n}}, \]

so that

\[
\sum_{n=1}^{\infty} P_{00}^{(n)} = \sum_{n=1}^{\infty} P_{00}^{(2n)} \\
= \sum_{n=1}^{\infty} \frac{[4p(1 - p)]^n}{\sqrt{\pi n}} \left\{ \begin{array}{ll}
= \infty & \text{if } p = 1/2 \\
< \infty & \text{if } p \neq 1/2 \end{array} \right. .
\]

So the MC is recur if \( p = 1/2 \) and trans otherwise.

\[\diamondsuit\]
4. Markov Chains

Definition: If \( p = 1/2 \), the random walk is *symmetric*.

Remark: A 2-dimensional r.w. with probability 1/4 of going each way yields a recurrent MC.

A 3-dimensional r.w. with probability 1/6 of going each way (N, S, E, W, up, down) yields a transient MC.
4.4 Limiting Probabilities

Example: Note that the following matrices appear to be converging.

\[ P = \begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix}, \quad P^{(2)} = \begin{pmatrix} 0.61 & 0.39 \\ 0.52 & 0.48 \end{pmatrix}, \]

\[ P^{(4)} = \begin{pmatrix} 0.575 & 0.425 \\ 0.567 & 0.433 \end{pmatrix}, \quad P^{(8)} = \begin{pmatrix} 0.572 & 0.428 \\ 0.570 & 0.430 \end{pmatrix}, \ldots \]
Definition: Suppose that $P_{ii}^{(n)} = 0$ whenever $n$ is not divisible by $d$, and suppose that $d$ is the largest integer with this property. Then state $i$ has period $d$. Think of $d$ as the greatest common divisor of all $n$ values for which $P_{ii}^{(n)} > 0$.

Example: All states have period 3.

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$ 

\diamondsuit
Definition: A state with period 1 is *aperiodic*.

Example: 

\[
P = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
1/4 & 1/4 & 1/4 & 1/4 \\
0 & 0 & 1/2 & 1/2 \\
\end{pmatrix}.
\]

Here, states 0 and 1 have period 2, while states 2 and 3 are aperiodic. 

\diamondsuit
Definition: Suppose state $i$ is recurrent and $X_0 = i$. If the expected time until the process returns to $i$ is finite, then $i$ is positive recurrent.

Remark: It turns out that...

(1) In a finite MC, all recur states are positive recur.
(2) In an $\infty$-state MC, there may be some recur states that are not positive recur. Such states are null recur.

Definition: Pos recur, aperiodic states are ergodic.
Theorem: For an irreducible, ergodic MC,

(1) $\pi_j \equiv \lim_{n \to \infty} P_{ij}^{(n)}$ exists and is independent of $i$.
(The $\pi_j$'s are called limiting probabilities.)

(2) $\pi_j$ is the unique, nonnegative solution of

\[
\begin{cases}
\pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij}, \quad j \geq 0 \\
1 = \sum_{j=0}^{\infty} \pi_j
\end{cases}
\]

In vector notation, this can be written as $\pi = \pi P$.

“Heuristics ‘proof’: see Ross. ♦
Remarks: (1) $\pi_j$ is also the long-run proportion of time that the MC will be in state $j$. The $\pi_j$’s are often called *stationary* probs — since if $\Pr(X_0 = j) = \pi_j$, then $\Pr(X_n = j) = \pi_j$ for all $n$.

(2) In the irred, pos recur, *periodic* case, $\pi_j$ can only be interpreted as the long-run proportion of time in $j$.

(3) Let $m_{jj}$ ≡ expected number of transitions needed to go from $j$ to $j$. Since, on average, the MC spends 1 time unit in state $j$ for every $m_{jj}$ time units, we have $m_{jj} = 1/\pi_j$. 
Example: Find the limiting probabilities of

\[
P = \begin{pmatrix}
0.5 & 0.4 & 0.1 \\
0.3 & 0.4 & 0.3 \\
0.2 & 0.3 & 0.5
\end{pmatrix}.
\]

Solve \( \pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij} \) (\( \pi = \pi P \)), i.e.,

\[
\begin{align*}
\pi_0 &= \pi_0 P_{00} + \pi_1 P_{10} + \pi_2 P_{20} = 0.5\pi_0 + 0.3\pi_1 + 0.2\pi_2, \\
\pi_1 &= \pi_0 P_{01} + \pi_1 P_{11} + \pi_2 P_{21} = 0.4\pi_0 + 0.4\pi_1 + 0.3\pi_2, \\
\pi_2 &= \pi_0 P_{02} + \pi_1 P_{12} + \pi_2 P_{22} = 0.1\pi_0 + 0.3\pi_1 + 0.5\pi_2,
\end{align*}
\]

and \( \pi_0 + \pi_1 + \pi_2 = 1 \). Get \( \pi = \{\frac{21}{62}, \frac{23}{62}, \frac{18}{62}\} \). \( \diamond \)
Definition: A transition matrix $P$ is *doubly stochastic* if each column (and row) sums to 1.

Theorem: If, in addition to the conditions of the previous theorem, $P$ is a doubly stochastic $n \times n$ matrix, then $\pi_j = 1/n$ for all $j$.

Proof: Just plug in $\pi_j = 1/n$ for all $j$ into $\pi = \pi P$ to verify that it works. Since this solution must be unique, we’re done. ♦
Example: Find the limiting probabilities of

\[ P = \begin{pmatrix} 0.5 & 0.4 & 0.1 \\ 0.3 & 0.3 & 0.4 \\ 0.2 & 0.3 & 0.5 \end{pmatrix}. \]

This is a doubly stochastic matrix, so we immediately have that \( \pi_0 = \pi_1 = \pi_2 = 1/3. \)
4.5 Gambler’s Ruin Problem

Each time a gambler plays, he wins $1 w.p. \ p$ and loses $1 w.p. \ 1 - p = q$. Each play is independent. Suppose he starts with $i$. Find the probability that his fortune will hit $N$ (i.e., he breaks the bank) before it hits $0$ (i.e., he is ruined).
Let $X_n$ denote his fortune at time $n$. Clearly, $\{X_n\}$ is a MC.

Note $P_{i,i+1} = p$ and $P_{i,i-1} = q$ for $i = 1, 2, \ldots N - 1$.

Further, $P_{00} = 1 = P_{NN}$.

We have 3 equiv classes: $\{0\}$ (recur), $\{1, 2, \ldots, N - 1\}$ (trans), and $\{N\}$ (recur).
By a standard one-step conditioning argument,

\[ P_i \equiv \Pr(\text{Eventually hit } N|X_0 = i) \]
\[ = \Pr(\text{Event. hit } N|X_1 = i + 1 \text{ and } X_0 = i) \times \Pr(X_1 = i + 1|X_0 = i) \]
\[ + \Pr(\text{Event. hit } N|X_1 = i - 1 \text{ and } X_0 = i) \times \Pr(X_1 = i - 1|X_0 = i) \]
\[ = \Pr(\text{Event. hit } N|X_1 = i + 1)p \]
\[ + \Pr(\text{Event. hit } N|X_1 = i - 1)q \]
\[ = pP_{i+1} + qP_{i-1}, \quad i = 1, 2, \ldots, N - 1. \]
Since $p + q = 1$, we have

$$pP_i + qP_i = pP_{i+1} + qP_{i-1}$$

iff

$$p(P_{i+1} - P_i) = q(P_i - P_{i-1})$$

iff

$$P_{i+1} - P_i = \frac{q}{p}(P_i - P_{i-1}), \quad i = 1, 2, \ldots, N - 1.$$
Since \( P_0 = 0 \), we have

\[
P_2 - P_1 = \frac{q}{p} P_1
\]

\[
P_3 - P_2 = \frac{q}{p} (P_2 - P_1) = \left( \frac{q}{p} \right)^2 P_1
\]

\[\vdots\]

\[
P_i - P_{i-1} = \frac{q}{p} (P_{i-1} - P_{i-2}) = \left( \frac{q}{p} \right)^{i-1} P_1.
\]

Summing up the LHS terms and the RHS terms,

\[
\sum_{j=2}^{i} (P_j - P_{j-1}) = P_i - P_1 = \sum_{j=1}^{i-1} \left( \frac{q}{p} \right)^j P_1.
\]
This implies that

\[ P_i = P_1 \sum_{j=0}^{i-1} \left( \frac{q}{p} \right)^j = \begin{cases} \frac{1-(q/p)^{i}}{1-(q/p)} P_1 & \text{if } q \neq p \ (p \neq 1/2) \\ iP_1 & \text{if } q = p \ (p = 1/2) \end{cases} \]

In particular, note that

\[ 1 = P_N = \begin{cases} \frac{1-(q/p)^{N}}{1-(q/p)} P_1 & \text{if } p \neq 1/2 \\ NP_1 & \text{if } p = 1/2 \end{cases} \]
Thus,

\[ P_1 = \begin{cases} 
\frac{1-(q/p)^i}{1-(q/p)^N} & \text{if } p \neq 1/2 \\
1/N & \text{if } p = 1/2 
\end{cases} \]

so that

\[ P_i = \begin{cases} 
\frac{1-(q/p)^i}{1-(q/p)^N} & \text{if } p \neq 1/2 \\
i/N & \text{if } p = 1/2 
\end{cases}. \]

By the way, as \( N \to \infty \),

\[ P_i \to \begin{cases} 
1 - (q/p)^i & \text{if } p > 1/2 \\
0 & \text{if } p \leq 1/2 
\end{cases}. \]
Example: A guy can somehow win any blackjack hand w.p. 0.6. If he wins, he fortune increases by $100; a loss costs him $100. Suppose he starts out with $500, and that he’ll quit playing as soon as his fortune hits $0 or $1500. What’s the probability that he’ll eventually hit $1500?

$$P_5 = \frac{1 - (0.4/0.6)^5}{1 - (0.4/0.6)^{15}} = 0.870.$$
4.6 First Passage Time from State 0 to State $N$

$$P_{ij}^{(n)} \equiv P(X_n = j | X_0 = i)$$

Definition: The probability that the first passage time from $i$ to $j$ is $n$ is

$$f_{ij}^{(n)} \equiv P(X_n = j | X_0 = i, X_k \neq j, k = 0, 1, \ldots, n - 1).$$

This is the probability that the MC goes from $i$ to $j$ in exactly $n$ steps (without passing thru $j$ along the way).
Remarks:

(1) By definition, \( f_{ij}^{(1)} = P_{ij}^{(1)} = P_{ij} \)

(2) \( f_{ij}^{(n)} = P_{ij}^{(n)} - \sum_{k=1}^{n-1} f_{ij}^{(k)} P_{jj}^{(n-k)} \)

\( P_{ij}^{(n)} \) = Prob. of going from \( i \) to \( j \) in \( n \) steps
\( f_{ij}^{(k)} \) = Prob. of \( i \) to \( j \) for first time in \( k \) steps
\( P_{jj}^{(n-k)} \) = Prob. of \( j \) to \( j \) in remaining \( n - k \) steps
Special Case: Start in state 0 and state $N$ is an absorbing ("trapping") state.

\[
\begin{align*}
  f_{0N}^{(1)} &= P_{0N}^{(1)} = P_{0N} \\
  f_{0N}^{(2)} &= P_{0N}^{(2)} - f_{0N}^{(1)} P_{NN}^{(1)} = P_{0N}^{(2)} - P_{0N}^{(1)} \\
  f_{0N}^{(3)} &= P_{0N}^{(3)} - f_{0N}^{(1)} - f_{0N}^{(2)} \\
  &= P_{0N}^{(3)} - P_{0N}^{(1)} - (P_{0N}^{(2)} - P_{0N}^{(1)}) = P_{0N}^{(3)} - P_{0N}^{(2)} \\
  &\vdots \\
  f_{0N}^{(n)} &= P_{0N}^{(n)} - P_{0N}^{(n-1)}
\end{align*}
\]

$f_{0N}^{(n)}$'s can be calculated iteratively starting at $f_{0N}^{(1)}$. 
Define $T \equiv$ first passage time from 0 to $N$

$$E(T^k) = \sum_{n=1}^{\infty} n^k \Pr(T = n) = \sum_{n=1}^{\infty} n^k f_{0N}^{(n)}$$

$$= \sum_{n=1}^{\infty} n^k (P_{0N}^{(n)} - P_{0N}^{(n-1)})$$

Usually use a computer to calculate this.

(WARNING! Don’t break this up into 2 separate $\infty$ summations!) Stop calculating when $f_{0N}^{(n)} \approx 0.$
2nd Special Case: 2 absorbing states $N, N'$

Same procedure as before but divide each $f_{0N}^{(n)}, f_{0N'}^{(n)}$ by the probs. of being trapped. So probs. of first passage times to $N, N'$ in $n$ steps are

$$\frac{f_{0N}^{(n)}}{\sum_{k=1}^{\infty} f_{0N}^{(k)}} \quad \text{and} \quad \frac{f_{0N'}^{(n)}}{\sum_{k=1}^{\infty} f_{0N'}^{(k)}}.$$
4.7 Branching Processes $\leftarrow$ Special class of MC’s

Suppose $X_0$ is the number of individuals in a certain population. Suppose the probability that any individual will have exactly $j$ offspring during its lifetime is $P_j$, $j \geq 0$. (Assume that the number of offspring from one individual is independent of the number from any other individual.)
\(X_0 \equiv \text{size of the } 0^{th} \text{ generation}\)

\(X_1 \equiv \text{size of the } 1^{st} \text{ gener' n } = \# \text{ kids produced by individuals from } 0^{th} \text{ gener' n}.\)

\(\vdots\)

\(X_n \equiv \text{size of the } n^{th} \text{ gener' n } = \# \text{ kids produced by indiv.' s from } (n - 1)^{st} \text{ gener' n.}\)

Then \(\{X_n : n \geq 0\}\) is a MC with the non-negative integers as its state space. \(P_{ij} \equiv P(X_{n+1} = j | X_n = i).\)
Remarks:

(1) 0 is recurrent since $P_{00} = 1$.

(2) If $P_0 > 0$, then all other states are transient.

(Proof: If $P_0 > 0$, then $P_{i0} = P_i^0 > 0$. If $i$ is recurrent, we’d eventually go to state 0. Contradiction.)

These two remarks imply that the population either dies out or its size $\to \infty$. 
Denote $\mu \equiv \sum_{j=0}^{\infty} j P_j$, the mean number of offspring of a particular individual.

Denote $\sigma^2 \equiv \sum_{j=0}^{\infty} (j - \mu)^2 P_j$, the variance.

Suppose $X_0 = 1$. In order to calculate $E[X_n]$ and $\text{Var}(X_n)$, note that

$$X_n = \sum_{i=1}^{X_{n-1}} Z_i$$

where $Z_i$ is the # of kids from indiv. $i$ of gener’n $(n - 1)$. 
Since $X_{n-1}$ is indep of the $Z_i$'s,

\[
E[X_n] = E\left[\sum_{i=1}^{X_{n-1}} Z_i\right] = E[X_{n-1}]E[Z_i] = \mu E[X_{n-1}].
\]

Since $X_0 = 1$,

\[
E[X_1] = \mu \\
E[X_2] = \mu E[X_1] = \mu^2 \\
\vdots \\
E[X_n] = \mu^n.
\]
Similarly,

\[ \text{Var}(X_n) = \begin{cases} 
\sigma^2 \mu^{n-1} \left( \frac{\mu^n - 1}{\mu - 1} \right), & \text{if } \mu \neq 1 \\
 n\sigma^2, & \text{if } \mu = 1 
\end{cases} \]
Denote $\pi_0 \equiv \lim_{n \to \infty} \Pr(X_n = 0|X_0 = 1) = \text{prob that the population will eventually die out (given } X_0 = 1).$

**Fact:** If $\mu < 1$, then $\pi_0 = 1$.

**Proof:**

\[
\Pr(X_n \geq 1) = \sum_{j=1}^{\infty} \Pr(X_n = j) \\
\leq \sum_{j=1}^{\infty} j\Pr(X_n = j) \\
= \mathbb{E}[X_n] = \mu^n \to 0 \quad \text{as } n \to \infty.
\]

**Fact:** If $\mu = 1$, then $\pi_0 = 1$. 

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What about the case when $\mu > 1$? Here, it turns out that $\pi_0 < 1$, i.e., the probability population dies out is $< 1$.

$$\pi_0 = \Pr(\text{pop'n dies out})$$

$$= \sum_{j=0}^{\infty} \pi_0^j \frac{\Pr(\text{pop'n dies out}|X_1 = j) \Pr(X_1 = j)}{P_j}$$

where $\pi_0^j$ implies that the families started by the $j$ members of the first generation all die out (indep'ly).
Summary:

\[ \pi_0 = \sum_{j=0}^{\infty} \pi_0^j P_j \quad (\ast) \]

For \( \mu > 1 \), \( \pi_0 \) is the smallest positive number satisfying \((\ast)\).
Example: Suppose \( P_0 = \frac{1}{4}, P_1 = \frac{1}{4}, P_2 = \frac{1}{2} \).

\[
\mu = \sum_{j=0}^{\infty} jP_j = 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{4} + 2 \cdot \frac{1}{2} = \frac{5}{4} > 1
\]

Furthermore, (*) implies

\[
\pi_0 = \pi_0^0 \cdot \frac{1}{4} + \pi_1^0 \cdot \frac{1}{4} + \pi_2^0 \cdot \frac{1}{2} = \frac{1}{4} + \frac{1}{4} \pi_0 + \frac{1}{2} \pi_0^2
\]

\[
\Leftrightarrow 2\pi_0^2 - 3\pi_0 + 1 = 0
\]

Smallest positive sol’n is \( \pi_0 = \frac{1}{2} \).