ISyE 3044 — Fall 2014 — Test #2 Solutions

This test is 85 minutes. You are allowed two cheat sheets.

1. Customers arrive at a store according to a Poisson process at the rate of 10 customers an hour. They receive FIFO service by a clerk who can process customers in i.i.d. Exponential time at the rate of 6 customers an hour. This is actually pretty slow, so management has imposed a limit of 12 customers in the system (in line + in service). In other words, if there are 12 customers in the system, an arriving customer gets kicked out before he can enter. (a) What is the steady-state probability that an arriving customer will be allowed to enter the system? (b) Describe in words how you would implement in Arena the limit of 12 customers.

Solution: (a) This is an $M/M/1/N$ queue with $\lambda = 10$, $\mu = 6$, and $N = 12$. Going to the tables, we have $a = \lambda / \mu = 1.667$, 

$$P_N = \frac{(1-a)a^N}{1-a^{N+1}} = \frac{(1-a)a^{12}}{1-a^{13}} = 0.4005.$$ 

This is the probability that an arriving customer will find the system full and then be turned away. So the desired answer is $1 - P_N = 0.5995$. ♦

(b) You could use a QUEUE block with capacity 11; or you could use a DECIDE block that incorporates an NQ query on the queue size. Many other ways. ♦

2. Short answer questions — Just write your answers.

(a) Consider the linear congruential generator $X_{n+1} = (27X_n + 3) \text{mod}(128)$. Using $X_0 = 9$, calculate the first pseudo-random number $U_1$.

Solution: $X_1 = (27X_0 + 3) \text{mod}(128) = 118$, so $U_1 = 118/128 = 0.9219$. ♦

(b) Suppose that $U_1 = 0.35$ and $U_2 = 0.75$ are realizations of two i.i.d. $\text{U}(0,1)$’s. Use the Box-Muller method to generate a $\chi^2$ random variate with two degrees of freedom. [Hint: If $Z_1$ and $Z_2$ are i.i.d. normal(0,1), then $Z_1^2 + Z_2^2 \sim \chi^2(2).$]
Solution: We have

\[ Z_1 = \sqrt{-2\ln(U_1)} \cos(2\pi U_2) = 0 \quad \text{and} \quad Z_2 = \sqrt{-2\ln(U_1)} \sin(2\pi U_2) = -1.449. \]

Then by the Hint, \( Z_1^2 + Z_2^2 = 2.100. \) ♦

(c) Using the two \( U_i \)'s from Question 2b, generate a realization from an \( \text{Erlang}_{k=2}(\lambda = 1) \) distribution.

Solution: We have

\[ X = -\frac{1}{\lambda} \ln \left( \prod_{i=1}^{k} U_i \right) = -\ln(U_1 U_2) = 1.338. \] ♦

Other answers possible.

(d) On Assignment #4, you analyzed 100,000 \( U(0,1) \) pseudo-random numbers. Name the two statistical tests for independence that I asked you to perform.

Solution: Runs up-and-down; runs above-and-below-the-mean. ♦

(e) Suppose \( U_1 = 0.33 \) is a \( U(0,1) \) random number. Use this number to generate a \( \text{Geometric}(0.4) \) random variate.

Solution: We have

\[ X = \left\lceil \frac{\ln(U_1)}{\ln(1 - p)} \right\rceil = \left\lceil \frac{\ln(0.33)}{\ln(0.6)} \right\rceil = 3. \] ♦

Other answers possible.

(f) BONUS: The Zombies’ lead singer, Colin Blunstone, recorded a solo album in 1978. Name the album and the record company’s owner.

Solution: “It’s His Time Of The Season”; Elton John. ♦

3. Suppose we observe 1000 pseudo-random numbers to obtain the following data.

<table>
<thead>
<tr>
<th>interval</th>
<th>[0.0, 0.2)</th>
<th>[0.2, 0.4)</th>
<th>[0.4, 0.6)</th>
<th>[0.6, 0.8)</th>
<th>[0.8, 1.0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>number observed</td>
<td>185</td>
<td>210</td>
<td>203</td>
<td>189</td>
<td>213</td>
</tr>
</tbody>
</table>
Conduct a $\chi^2$ goodness-of-fit test to see if these numbers are approximately $U(0,1)$. Use level of significance $\alpha = 0.05$. Here are some table entries that you may need: $\chi^2_{0.05,3} = 7.81$, $\chi^2_{0.05,4} = 9.49$, and $\chi^2_{0.05,5} = 11.1$.

**Solution:** Let’s re-write the table:

<table>
<thead>
<tr>
<th>interval</th>
<th>[0.0,0.2)</th>
<th>[0.2,0.4)</th>
<th>[0.4,0.6)</th>
<th>[0.6,0.8)</th>
<th>[0.8,1.0]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$O_i$</td>
<td>185</td>
<td>210</td>
<td>203</td>
<td>189</td>
<td>213</td>
</tr>
<tr>
<td>$E_i$</td>
<td>200</td>
<td>200</td>
<td>200</td>
<td>200</td>
<td>200</td>
</tr>
</tbody>
</table>

Then

$$\chi^2_0 = \sum_{i=1}^{5} \frac{(O_i - E_i)^2}{E_i} = 3.12.$$ 

Note that we have $k - 1 = 4$ degrees of freedom. Then since $\chi^2_0 < \chi^2_{0.05,4}$, we fail to reject uniformity. 

4. Suppose the random variable $X$ has the following c.d.f.

$$F(x) = \begin{cases} \frac{1}{2}e^{3x} & \text{if } x \leq 0 \\ 1 - \frac{1}{2}e^{-3x} & \text{if } x > 0 \end{cases} .$$

(a) Give an inverse transform method for generating realizations of $X$.

**Solution:** $F(X) = \frac{1}{2}e^{3X} = U \Rightarrow X = \frac{1}{3}\ln(2U)$ if $0 \leq U \leq 0.5$.

And $F(X) = 1 - \frac{1}{2}e^{3X} = U \Rightarrow X = -\frac{1}{3}\ln(2(1-U))$ if $0.5 \leq U \leq 1$.

(b) Use the $U(0,1)$ random number 0.55 to generate a realization of $X$.

**Solution:** $X = -\frac{1}{3}\ln(2(1-0.55)) = -\frac{1}{3}\ln(0.9) = 0.0351$.

5. Questions on Poisson random variables.

(a) Use the acceptance-rejection technique to generate one Poisson(3.5) random variate. Use as many of the following $U(0,1)$ numbers as is necessary.

0.89 0.75 0.83 0.02 0.11 0.23 0.06

**Solution:** Define $p_n \equiv \prod_{i=1}^{n+1} U_i$. We’ll stop as soon as $p_n < e^{-3.5} = 0.0302$. Let’s make the following convenient table.
\[
\begin{array}{ccc}
 n & U_{n+1} & p_n & \text{Stop?} \\
 0 & 0.89 & 0.89 & \text{nope} \\
 1 & 0.75 & 0.6675 & \text{nope} \\
 2 & 0.83 & 0.5540 & \text{nope} \\
 3 & 0.02 & 0.0111 & \text{yup} \\
\end{array}
\]

So we take \( N = 3 \). \( \diamond \)

(b) How many \( U(0, 1) \)’s would you have \textit{expected} to use in Question 5a?

\textbf{Solution:} From class notes, \( \mathbb{E}[N + 1] = 4.5 \). \( \diamond \)

(c) What would you have had to do if I had asked you to generate a Poisson(500) random variate?

\textbf{Solution:} Use the normal approximation.

\[
N = \lceil \lambda + \sqrt{\lambda Z} - 0.5 \rceil = \lceil 500 + \sqrt{500} Z - 0.5 \rceil. \quad \diamond
\]

6. Joey owns “Dogs ‘R Us,” a fancy hot dog stand in Smyrna. Customers arrive to buy hot dogs according to a Poisson process at the rate of 10/hour. Two servers work for Joey. Each customer buys \( X \) hot dogs, where \( X \sim \text{Bin}(3, 0.4) \), i.e.,

\[
P(X = k) = \frac{3!}{k!(3-k)!} (0.4)^k (0.6)^{3-k}, \quad k = 0, 1, 2, 3.
\]

It takes exactly 4 minutes to serve each hot dog. After a customer is served, there is a 20% chance that he will chat for \( U(4, 8) \) minutes with his server.

The servers take a 30-minute break every 4 hours (although they will finish serving any customer currently in progress).

Write a \textbf{VERY, VERY NEAT} Arena program to simulate this system for 40 hours. In particular,

(a) Draw an appropriate block diagram. I know that you are not an artist, so label each block indicating what kind of block it is; and indicate any necessary information about the block, such as parameters like arrival rates, or commands such as Seize-Delay-Release.

(b) Write out the precise command that you would need to implement the Binomial random variable.

\textbf{Solution:} It’s easy to show that
<table>
<thead>
<tr>
<th>$k$</th>
<th>$P(X = k)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.216</td>
</tr>
<tr>
<td>1</td>
<td>0.432</td>
</tr>
<tr>
<td>2</td>
<td>0.288</td>
</tr>
<tr>
<td>3</td>
<td>0.064</td>
</tr>
</tbody>
</table>

So you can use \texttt{DISC(0.216,0,0.648,1,0.936,2,1.0,3)} in Arena. Other solutions possible.

(c) Describe in words how you might schedule the breaks.

\textbf{Solution:} Define a resource schedule and fill in the table or chart.

(d) Describe in words how you would run the simulation for 40 hours.

\textbf{Solution:} Go to Run Setup > Replication Parameters and stop the run at 40 hours.

7. More short answer questions.

(a) If the number of arrivals to a parking lot follows a nonhomogeneous Poisson process with intensity function $\lambda(t) = 2t$, where $t \in [0, 10]$ denotes hours, find the expected number of customers to show up between $t = 4$ and 6.

\textbf{Solution:} $E[N(6) - N(4)] = \int_4^6 \lambda(t) \, dt = 20$.

(b) What is the name of the method we discussed in class to generate NHPP arrivals?

\textbf{Solution:} Thinning.

(c) Suppose that $W_{i+1}^Q$ is the $(i + 1)$st customer’s waiting time. Write down a simple expression for this time in terms of the previous customer $i$’s waiting time and service time, and customer $(i + 1)$’s interarrival time.

\textbf{Solution:} $W_{i+1}^Q = \max \{0, W_i^Q + S_i - I_{i+1}\}$
(d) If $W(t)$ is a Brownian motion process, find $\text{Cov}(W(0.5), W(1))$.

**Solution:** $\min(0.5, 1) = 0.5$. ◦

(e) Find $\text{Cov}\left(\int_0^1 W(t) \, dt, \, W(1)\right)$.

**Solution:** $\int_0^1 \text{Cov}(W(t), \, W(1)) \, dt = \int_0^1 t \, dt = 1/2$. ◦

(f) If $X_1, X_2, \ldots$ follow a stationary AR(1) process, $X_{i+1} = \phi X_i + \epsilon_{i+1}$, with the $\epsilon_i$’s being i.i.d. $\text{Nor}(0, 1 - \phi^2)$, with $\phi = 0.9$, find the correlation between $X_3$ and $X_7$.

**Solution:** $\phi^4 = 0.6561$. ◦