Term Structure Estimation

1 Introduction

Consider an 8% coupon bond with 3 years to maturity with semiannual coupon payments and with a face value of 100. Assume that the next coupon payment occurs exactly 0.5 years from now. (We’ll relax this later.) The sequence of future cash flows may be represented as (0.5, 4), (1.0, 4), (1.5, 4), (2.0, 4), (2.5, 4), (3.0, 104), where the first number in each pair refers to the time (measured in years) at which the cash flow occurs, and the second number is the dollar amount paid. Keep in mind that the face value of 100 is repaid at time of maturity. A “fair” price $P$ to pay for this cash flow stream (and hence the bond itself) is given by

$$P = \sum_i C(t_i)e^{-r(t_i)t_i},$$

which represents the present value when the term structure of interest rates is given by $(r(t), t \geq 0)$. Here, $r(t)$ is called a spot rate, which is the rate of interest, expressed in yearly terms, charged for money held from the present time ($t = 0$) until time $t$. Both the interest and original principal are paid at time $t$.

The question we address here is how to estimate the term structure $(r(t), t \geq 0)$. Conceptually the best way to estimate the term structure is find one that is “consistent” with the current prevailing prices in the marketplace. Actually, there are separate term structures for various classes of bonds depending on risk and maturity. The difference in the basis points (number of .01 percents) as compared to the term structure for the (riskless) Treasuries is called the risk premia.

2 Data

It is possible to download data on bonds (see, for example, www.bondsonline.com). Bond classes include corporate bonds, municipal bonds, treasuries/government bonds, and by various sectors (e.g. utilities). For each bond the following data are known:

- S&P rating and Moody’s rating. These ratings are intended to describe the default risk. A separate term structure should be determined for each class of bonds.

- Coupon rate. Coupons are paid every 6 months until time of maturity (when the face value of 100 is repaid).

- Maturity date. The exact date when the bond terminates.

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1To simplify expressions in what follows we shall assume continuous compounding. Replace $e^{r(t)t}$ with $(1 + r(t)/2)^{2t}$ to obtain the corresponding discrete (semitannual) compounding.
• Yield to maturity. This simply is the constant discount rate $y$ for which (1) holds.

• Option features. Some bonds have embedded options that entitle the issuer, for example, to repurchase the bond if the interest rates drop to a certain level, or for the borrower to extend the maturity. Each option has certain terms associated with it. Bonds with an ‘NC’ rating do not include embedded options.

• Price. Quoted price.

Here are some examples.

• Consider a triple A rated General Electric bond whose coupon rate is 8.125%, maturity date of 5/15/12 (May 15, 2012), a yield to maturity of 6.402, no call option features, and a price of 113.801.

• Consider two Shell Oil bonds each with a coupon rate of 6.7%, no call option features, and one with a price of 101.349 and the other with a price of 101.466. One bond represents ‘senior’ debt, the other ‘junior’ debt. Senior debt means that the bond has been collateralized with some type of asset. Hence, its price will be more.

It is important to point out that the quoted price does not consider the interest owed on the first coupon payment, and so does not equal the cash price. An example will explain this point. Suppose a bond was purchased on 12/10/03. The bond has a coupon payment of 4 due on 1/15/04 and the last coupon payment occurred on 7/15/03. Thus, the next coupon payment occurs 36 days from the purchase date, and the last coupon payment occurred exactly 148 days from the purchase date. Thus, the accrued interest on 12/10/03 is the pro-rated share of the 1/15/01 coupon accruing to the bondholder, and is calculated as $(184 - 36)/184 = 0.80435$ of the length of the semiannual period. The convention is that a purchaser of the bond pays a cash price, which equals the quoted price plus the accrued interest. So, in our example, the purchaser here would pay the quoted price plus the accrued interest of $4 \times 0.80435 = 3.22$. When estimating the term structure you must use the cash price.

3 Estimation

Let

$$d(t) := e^{-r(t)t} < 1$$

denote the discount for each time $t$. Determination of $d(t)$ is of course equivalent to estimating $r(t)$. Given our sample of bonds let $t_1 < t_2 < \ldots < t_N$ denote the exact times at which any cash payment is made. Each bond’s cash flows may be represented by a vector $C = (C_1, C_2, \ldots, C_N)$, where $C_i = C(t_i)$ denotes the cash flow paid at time $t_i$. Given a set of $s$ bonds, we let $A$ denote the $s \times N$ matrix of cash flows in which each row corresponds to a particular bond’s cash flow vector. Let $P \in \mathbb{R}^s$ denote the vector of cash prices. Our estimation problem boils down then to finding a vector $d \in \mathbb{R}^N$ for which $Ad = P$. Actually, in theory, it should be the case that
1 > d_1 > d_2 > \ldots > d_N$, namely, the discount rates should be declining; otherwise, there would be a negative forward rate between the two time dates.\footnote{Recall that the forward rate $f(t_1, t_2)$, $t_1 < t_2$, is the one that satisfies the identity $e^{r(t_1)t_1} \cdot e^{f(t_1, t_2)(t_2 - t_1)} = e^{r(t_2)t_2}$, which is equivalent to insisting that $e^{f(t_1, t_2)(t_2 - t_1)} = d(t_1)/d(t_2)$. Nonnegative forward rates imply a monotonically decreasing discount rate curve.} As not all prices are recorded at the same time there is ‘noise’ in the data, and so it is generally not possible to find a $d$ for which $Ad = P$ and $d \geq 0$. Thus, one seeks to find the closest fit, and usually minimizes the norm $||Ad - P||^2$, which is termed a quadratic programming problem. One should also enforce monotonicity constraints on $d$ and to remember to bound $d(t_1)$ by 1.

\section{Implementation}

\subsection{Basics}

One has to transform the raw data into databases suitable for estimation. This would include transforming the coupon dates into time, measured in days, from the current date. (A calendar must be used here!) A naive implementation for say a 20-year horizon would require about 7300 columns, one for each possible date. However, the number of columns required need only match the total number of different payment dates in the sample, which will be far fewer. One should always use an efficient $A$ matrix.

\subsection{Continuous approximations}

It is possible to approximate the discount curve by a continuous approximation. This is usually done in practice. We briefly describe two approaches.

\textit{Cubic splines (Litzenberger and Rolfo).} Let $[0, T]$ denote the interval over which the cash flow payments are made in the sample of bonds. Let $\{\tau_k\}_{k=0}^r \subset [0, T]$ denote a collection of “knot points” for which $0 := \tau_0 < \tau_1 < \ldots < \tau_r := T$. The knots are placed such that an approximately an equal number of payment dates falls into each subinterval. The value for $r$ is set as the integer closest to the square root of the total number of bonds in the sample. The cubic spline approximates $d(t)$ by

\begin{equation}
    d(t) \approx 1 + b_1 t + b_2 t^2 + b_3 t^3 + \sum_{k=1}^r b_{k+3} (t - \tau_k) y_{kt},
\end{equation}

where $y_{kt} = 0$ for $t < \tau_k$ and $y_{kt} = 1$ for $t \geq \tau_k$. For (realized) payment date $t_i$ let $b(t_i)$ denote the vector

\begin{equation}
(1, t_i, t_i^2, t_i^3, (t_i - \tau_1)y_{1t_i}, (t_i - \tau_2)y_{2t_i}, \ldots, (t_i - \tau_r)y_{rt_i}) \in \mathbb{R}^{4+r},
\end{equation}

and let $B$ denote the $N \times (4+r)$ matrix whose $i^{th}$ row is given by $b(t_i)$. (Recall that $N$ denotes the total number of distinct payment dates, which defined the cardinality of the vector $d$.) Our estimation problem now is to minimize the norm

\begin{equation}
||(AB)b - P||^2
\end{equation}
for $b = (1, b_1, b_2, \ldots, b_{3+r}) \geq 0$. Note that the cubic spline (by design) is continuous, and has both continuous first and second derivatives. It will not, however, be guaranteed to be a monotonically decreasing function, which motivates the second approach below.

A related approach is to analyze the general form $d(t) = \sum_j \alpha_j t^j$, which still leads to a quadratic program in $\alpha$, or to model the term structure $r(t) = \sum_j \gamma_j t^j$ directly, which will lead to a convex programming problem.\footnote{The non-monotonicity constraints are of the form $P_i + \epsilon_i \leq \sum_k C_i(t_k)e^{-\sum_j \gamma_j t_k^j}$. The right-hand side is a convex function of $\gamma$, which may be verified by showing that the Hessian $H(x)$ is indeed positive semidefinite. (If $f(x) = e^{-c^T x}$, then $H(x) = f(x)c c^T$, and thus $y^T H(x)y = f(x)y^T (c c^T)y = f(x)(c^T y)^2 \geq 0$.)}

\textit{Beta density (Schaefer).} Here

$$d(t) \approx \sum_{k=0}^{n} \alpha_k \theta_k(t).$$

To adopt this approach, time is measured on the interval $[0, 1]$. To implement simply scale each payment date by the longest payment date in the sample. The choice of component functions is as follows:

$$\theta_0(t) := 1, \quad \alpha_0 = 1,$$

and for each $k = 1, 2, \ldots, n$,

$$\theta_k(t) = -\int_0^t u^{k-1}(1-u)^{n-k} du = \sum_{j=0}^{n-k} (-1)^{j+1} \binom{n-k}{j} \frac{t^{k+j}}{k+j}. \quad (6)$$

Each $\theta_k$ is a monotonically decreasing nonpositive function on $[0, 1]$. One constrains each parameter $\alpha_k$ to be nonnegative, which will ensure that $d(t)$ is a monotonically decreasing function. One also adds that

$$\sum_{k=0}^{n} \alpha_k \theta_k(1) \geq 0 \quad (7)$$

to ensure that $d(t) \geq 0$. The transformed problem now estimates the parameter vector $\alpha$. Schaefer uses $n = 25$, but other values may be explored.

### 4.3 Taxes

So far we have ignored the issue of taxes. The term structure so estimated provides an estimate of the before tax interest rates. To determine the after tax term structure one should discount the after tax cash flows.

There is no federal tax on a municipal bond. There is no state tax on a federal bond and there is no state tax on a municipal bond issued within the state; but a state will tax another state’s municipal bond. The highest marginal federal tax rate is currently 35% and state taxes
are low (a few percentage points). Tax rates and the tax code do change frequently. For example, the new capital gains rate has been lowered to 15%.

We describe below two methods for determining the after tax cash flows. The first one is applicable to the Canadian market, and the second one is relevant to the US (and is slightly more cumbersome to handle). In what follows, let $C$ denote the coupon payment, $H$ the accrued interest, $\tau$ the tax on ordinary income, $\tau_G$ the tax on capital gains, $P$ the quoted price, and $M$ the number of payment periods until maturity.

Method 1. There are 2 cases to consider when calculating the after tax cash flows for a bond.\footnote{For purposes of calculating the term structure we assume that one is purchasing the bond.} For a premium bond ($P > 100$) the difference $P - 100$ is treated as a capital loss. Here, $P$ refers to the quoted price. The benefit due to the capital loss is $\tau(P - 100)$, which is taxed as ordinary income and pro-rated over each payment date, as follows. The first after tax cash flow equals

$$C - \tau(C - H) + \tau(P - 100)/M,$$

(8)

all after tax cash flows (except the last) equal

$$C(1 - \tau) + \tau(P - 100)/M,$$

(9)

and the last after cash flow equals

$$C(1 - \tau) + 100 + \tau(P - 100)/M.$$  

(10)

For a discount bond ($P < 100$) the first after tax cash flow equals

$$C - \tau(C - H),$$

(11)

all after tax cash flows (except the last) equal

$$C(1 - \tau),$$

(12)

and the last after cash flow equals

$$C(1 - \tau) + 100 - (100 - P)\tau_G.$$\footnote{On the last payment date you receive $C + 100$ and pay tax in the amount of $\tau C + (100 - P)\tau_G$.}

(13)

Note that one deducts the accrued interest from the first coupon payment. Each interest payment $C$ is taxed as ordinary income, and $(100 - P)$ is taxed as a capital gain.

Method 2. Here one computes the accreted interest, which is taxed as ordinary income, as follows. Let $y$ denote the yield to maturity. A bondholder is taxed as if he receives interest on the outstanding (accreted) principal balance at the rate of $y$, and the coupon payments
are viewed as principal repayments. Let $PB(t)$ denote the outstanding principal balance. In continuous-time (and assuming a continuous coupon stream)

$$\frac{d}{dt}PB(t) = yPB(t) - C(t), \quad (14)$$

which implies that

$$PB(t) = e^{yt}\{PB(0) - \int_0^t C(\tau)e^{-y\tau}d\tau\}. \quad (15)$$

In words, one computes the present value of the cash flow up to time $t$, subtracts it from the initial principal balance (the quoted price) and then computes the future value at time $t$ by multiplying by $e^{yt}$. Note the use of the yield-to-maturity (YTM) for this valuation. The definition of yield-to-maturity ensures that the principal balance equals zero at maturity. The corresponding discrete-time formula is obtained by simply replacing the integral with a sum. Thus, the after tax cash flow at time $t_k$ is

$$C(t_k) - \tau e^{y(t_k-t_{k-1})}PB(t_{k-1}). \quad (16)$$

**Remark.** Consider a fixed-rate mortgage for $T$ years with a continuously compounded interest rate of $y$ and an initial loan (principal) balance of $LB(0)$. Substituting $LB(0)$ and the constant cash flow stream associated with this mortgage into (15) shows that

$$\frac{LB(t)}{LB(0)} = \frac{1 - e^{-r(T-t)}}{1 - e^{-rT}},$$

which we derived via other means in the Fixed Rate Mortgage handout.